Central sequences, dimension, and $\mathcal{Z}$-stability of C^*-algebras

Aaron Tikuisis
(joint work with Leonel Robert)

This talk concerned the potential equivalence of different regularity properties for C^*-algebras. It was shown, by Rørdam [4] and Toms [7], building on work of Villadsen [8], that amenability is not a strong enough condition for classification; certain new regularity properties are being studied, with the aim of developing a new, stronger amenability-type notion. While there are candidates for what this new notion might be, there is no sweeping statement about many definitions being equivalent (as is the case for amenability).

The two candidates that I focused on are finite nuclear dimension [11] and amenability+$\mathcal{Z}$-stability [1, 5]. That these properties are equivalent, for simple, unital, non-type I, separable C^*-algebras, is part of the Toms-Winter conjecture. However, it is worthwhile to conjecture that they are equivalent even without assuming the algebras are simple and unital (as long as we ask that no ideal has type I representations).

Jointly with Wilhelm Winter, I have been involved in showing that $\mathcal{Z}$-stability implies finite nuclear dimension [6], for a special class of C^*-algebras; our result is complemented by a recent result of the same nature by Matui and Sato (but for a very different special class of C^*-algebras) [2].

It is known in considerably more generality that finite nuclear dimension implies $\mathcal{Z}$-stability (although there are known obstructions that we must assume away). Arguments in this direction were pioneered by Winter [9, 10]. As it turns out, these arguments revolve around comparison and divisibility properties of the central sequence algebra (either implicitly or explicitly). (In the nonunital case, one should use Kirchberg's central sequence algebra $\mathbf{F}(A) := (\prod_\omega A \cap A')/\{x \in \prod_\omega A | xA = Ax = 0\}$.) $\mathcal{Z}$-stability, it turns out, is equivalent to M-comparison and N-almost-divisibility of $\mathbf{F}(A)$. The problem of showing that A is $\mathcal{Z}$-stable when it has finite nuclear dimension then comes down, largely, to exploring the extent to which these properties transfer from A to $\mathbf{F}(A)$.

The following lemma, reminiscent of the definition of nuclear dimension, allows certain regularity properties to be pass from A to $\mathbf{F}(A)$ (especially comparison properties):

Lemma. *(Robert-T* [3]*) Let A be a* C**-algebra of finite nuclear dimension n, and let $N := 2n+1$. Then there exist hereditary subalgebras $C^{(0)}, \ldots, C^{(N)}$ of A_∞ and maps making the following diagram commute:*

$$\begin{tikzcd}
A_\infty \arrow[rr, "\subset"] \arrow[dr, "c.p.c.,\ order\ 0"'] & & (A_\infty)_\infty \cap A' \\
& \mathbf{C}^{(0)} \oplus \cdots \oplus \mathbf{C}^{(N)} \arrow[ur, "\sum_{i=0}^{N} c.p.c.,\ order\ 0"'] &
\end{tikzcd}$$

L. Robert and I have proven that, if A has finite nuclear dimension, then it is $\mathcal{Z}$-stable if and only if $\mathbf{F}(A)$ has two full, orthogonal elements [3]. We have moreover shown that finite nuclear dimension implies $\mathcal{Z}$-stability in the following cases: (i) the C*-algebra has no purely infinite subquotients and its primitive ideal space has a basis of compact open sets, (ii) the C*-algebra has no purely infinite quotients and its primitive ideal space is Hausdorff. The stumbling block to going beyond these cases, at present, is producing full orthogonal elements, first in A, and then centrally, in $\mathbf{F}(A)$.

Slides from the talk may be found on my website, `http://homepages.abdn.ac.uk/a.tikuisis/`.

*TODO: remove "revision in progress" and insert bib here

References

[1] Xinhui Jiang and Hongbing Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.

[2] Hiroki Matui and Yasuhiko Sato. Decomposition rank of UHF-absorbing C^*-algebras, 2013. arXiv preprint math.OA/1303.4371.

[3] Leonel Robert and Aaron Tikuisis. Nuclear dimension and $\mathcal{Z}$-stability of non-simple C*-algebras, 2013. arXiv preprint math.OA/1308.2941.

[4] Mikael Rørdam. A simple C^*-algebra with a finite and an infinite projection. *Acta Math.*, 191(1):109–142, 2003.

[5] Mikael Rørdam and Wilhelm Winter. The Jiang-Su algebra revisited. *J. Reine Angew. Math.*, 642:129–155, 2010.

[6] Aaron Tikuisis and Wilhelm Winter. Decomposition rank of $\mathcal{Z}$-stable C^*-algebras. arXiv preprint math.OA/1210.1386, 2012. arXiv preprint math.OA/1210.1386.

[7] Andrew S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. of Math. (2)*, 167(3):1029–1044, 2008.

[8] Jesper Villadsen. Simple C^*-algebras with perforation. *J. Funct. Anal.*, 154(1):110–116, 1998.

[9] Wilhelm Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.

[10] Wilhelm Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. *Invent. Math.*, 187(2):259–342, 2012.

[11] Wilhelm Winter and Joachim Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.