Tensor products of C^*-bundles

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$C_0(X)$-algebras

Definition

Let X be a locally compact Hausdorff space and A a C*-algebra. If there exists a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that $\mu_A(C_0(X)) \cdot A$ is dense in A, we say that the triple (A, X, μ_A) is a *$C_0(X)$-algebra* .

For $f \in C_0(X)$, we will write $f\cdot$ for $\mu_A(f)$.
For $x \in X$, define

- $C_{0,x}(X) = \{f \in C_0(X) : f(x) = 0\}$, and note that $C_{0,x}(X) \cdot A$ is a closed two-sided ideal of A,
- $A_x = \dfrac{A}{C_{0,x}(X) \cdot A}$ the quotient C*-algebra, and
- $\pi_x : A \to A_x$ the quotient homomorphism.

$C_0(X)$-algebras and C*-bundles

We regard A as an algebra of sections of $\coprod_{x \in X} A_x$, identifying each $a \in A$ with $\hat{a} : X \to \coprod_{x \in X} A_x$, where

$$\hat{a}(x) = \pi_x(a)$$

for all $x \in X$.
For all $a \in A$, we have

- $\|a\| = \sup_{x \in X} \|\pi_x(a)\|$,
- the function $X \to \mathbb{R}_+$, $x \mapsto \|\pi_x(a)\|$ is upper-semicontinuous, and vanishes at infinity on X.

Thus, we think of a $C_0(X)$-algebra as the algebra of sections (vanishing at infinity) of a C*-bundle over X.
If for all $a \in A$, the norm functions $x \mapsto \|\pi_x(a)\|$ are continuous on X, then we say that (A, X, μ_A) is a *continuous $C_0(X)$-algebra*.

$C_0(X)$-algebras and C*-bundles

Interest in $C_0(X)$-algebras and C*-bundles: to decompose the study of a given C*-algebra A into that of

- the fibre algebras A_x,
- the behaviour of A as an algebra of sections of $\coprod_{x \in X} A_x$.

e.g. every irreducible representation of a $C_0(X)$-algebra A is lifted from a fibre A_x for some $x \in X$.

Question: For two C*-algebras A and B, let $A \otimes B$ denote their minimal tensor product. Given a $C_0(X)$-algebra structure on A and a $C_0(Y)$-algebra structure on B, what can be said about $A \otimes B$ as a $C_0(X \times Y)$-algebra?
Related question: ideal structure of $A \otimes B$?

Ideals of $A \otimes B$

If $I \triangleleft A$ and $J \triangleleft B$, let $q_I : A \to A/I$ and $q_J : B \to B/J$ be the quotient maps. Then $q_I \odot q_J : A \odot B \to (A/I) \odot (B/J)$ has

$$\ker(q_I \odot q_J) = I \odot B + A \odot J,$$

which, by injectivity, has closure

$$I \otimes B + A \otimes J \triangleleft A \otimes B.$$

Extending $q_I \odot q_J$ to $q_I \otimes q_J : A \otimes B \to (A/I) \otimes (B/J)$ gives a closed two-sided ideal

$$\ker(q_I \otimes q_J) \triangleleft A \otimes B.$$

Clearly

$$\ker(q_I \otimes q_J) \supseteq I \otimes B + A \otimes J.$$

but this inclusion may be strict.

The fibrewise tensor product

- Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, and denote by $\pi_x : A \to A_x$ and $\sigma_y : B \to B_y$ the quotient $*$-homomorphisms, where $x \in X, y \in Y$.
- We get $*$-homomorphisms $\pi_x \otimes \sigma_y : A \otimes B \to A_x \otimes B_y$,
- Hence we may regard $A \otimes B$ as an algebra of sections of $\coprod\{A_x \otimes B_y : (x, y) \in X \times Y\}$, where $c \in A \otimes B$ is identified with

$$\begin{aligned} \hat{c} : X \times Y &\to \coprod\{A_x \otimes B_y : (x, y) \in X \times Y\} \\ \hat{c}((x, y)) &= (\pi_x \otimes \sigma_y)(c). \end{aligned}$$

- If A and B are continuous, this construction gives $A \otimes B$ the structure of a lower-semicontinuous bundle over $X \times Y$.

$A \otimes B$ as a $C_0(X \times Y)$-algebra

Since $C_0(X) \otimes C_0(Y) \equiv C_0(X \times Y)$ and
$ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes B)$, we get a $*$-homomorphism

$$\mu_A \otimes \mu_B : C_0(X \times Y) = C_0(X) \otimes C_0(Y) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes B).$$

The triple $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is then a $C_0(X \times Y)$-algebra.
For $(x, y) \in X \times Y$, it can be shown that

$$C_{0,(x,y)}(X \times Y) \cdot (A \otimes B) = (C_{0,x}(X) \cdot A) \otimes B + A \otimes (C_{0,y}(Y) \cdot B)$$

Hence the fibre algebras of $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ are given by

$$\begin{aligned} (A \otimes B)_{(x,y)} &= \frac{A \otimes B}{\ker(\pi_x) \otimes B + A \otimes \ker(\sigma_y)} \\ &\neq A_x \otimes B_y, \end{aligned}$$

in general.

Continuity of the fibrewise tensor product

Clearly we have

$$\begin{aligned} &(A \otimes B)_{(x,y)} \equiv A_x \otimes B_y \\ &\Leftrightarrow \ker(\pi_x) \otimes B + A \otimes \ker(\sigma_y) = \ker(\pi_x \otimes \sigma_y). \end{aligned} \qquad (F_{X,Y})$$

Theorem (Kirchberg & Wassermann)

Let (A, X, μ_A) be a continuous $C_0(X)$-algebra and (B, Y, μ_B) a continuous $C_0(Y)$-algebra. Then the norm functions

$$(x, y) \mapsto \|(\pi_x \otimes \sigma_y)(c)\|$$

are continuous on $X \times Y$ for all $c \in A \otimes B$ if and only if $(F_{X,Y})$ holds.

Note that if $(F_{X,Y})$ holds, then this also implies that the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.

Continuity of the $C_0(X \times Y)$-algebra $A \otimes B$

By contrast, we have shown that there exist (A, X, μ_A) and (B, Y, μ_B), both continuous, such that

- the fibrewise tensor product of (A, X, μ_A) and (B, Y, μ_B) is discontinuous, but
- the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.
- Let $A = \prod_{n \geq 1} M_n(\mathbb{C})$, then A defines a continuous $C(\beta\mathbb{N})$-algebra, with fibres $A_n = M_n(\mathbb{C})$ for $n \in \mathbb{N}$.
- $B = B(H)$ and $Y = \{y\}$ a one-point space, so that B is trivially a continuous $C(Y)$-algebra.
- Then $(A \otimes B, \beta\mathbb{N}, \mu_A \otimes 1)$ is a continuous $C(\beta\mathbb{N})$-algebra, but there is $p \in \beta\mathbb{N} \backslash \mathbb{N}$ such that
  - $(A \otimes B)_p \neq A_p \otimes B$ (i.e. property $(F_{X,Y})$ fails) and
  - $p \mapsto \|(\pi_p \otimes \mathrm{id})(c)\|$ is discontinuous at p for some $c \in A \otimes B$, hence the fibrewise tensor product is a discontinuous C*-bundle.
- In fact this occurs whenever B is an inexact C*-algebra.

Property (F)

- Let A and B be C*-algebras. If for all ideals $I \lhd A$ and $J \lhd B$ we have

$$\ker(q_I \otimes q_J) = I \otimes B + A \otimes J, \tag{F}$$

  then $A \otimes B$ is said to satisfy Tomiyama's property (F).
- Given (A, X, μ_A) and (B, Y, μ_B), $(F) \Rightarrow (F_{X,Y})$, which in turn implies
  - $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ has fibres $A_x \otimes B_y$, and
  - if A and B are continuous then so is $A \otimes B$.
- A is *exact* iff $A \otimes B$ satisfies (F) for all B.
- If (A, X, μ_A) is continuous and A exact, then $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous whenever (B, Y, μ_B) continuous.

Continuity and exactness

Theorem (M.)

Let (A, X, μ_A) be a continuous $C_0(X)$-algebra. TTFAE:

(i) *A is exact,*

(ii) *for every continuous $C_0(Y)$-algebra B, the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.*

Analogous result for the fibrewise tensor product due to Kirchberg and Wassermann: A exact $\Leftrightarrow$ fibrewise tensor product continuous for all B.

$$\ker(\pi_x) \otimes B + A \otimes \ker(\sigma_y) = \ker(\pi_x \otimes \sigma_y). \qquad (F_{X,Y})$$

By contrast, given continuous (A, X, μ_A) and (B, Y, μ_B), we have

- fibrewise tensor product of A and B continuous $\Leftrightarrow$ $(F_{X,Y})$ holds,
- $(F_{X,Y})$ holds $\Rightarrow$ $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ continuous, but
- $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ continuous $\not\Rightarrow$ $(F_{X,Y})$ holds.

Glimm ideals

Let A be a C*-algebra, and $\hat{Z}$ the maximal (primitive) ideal space of $ZM(A)$, so that we have an isomorphism $\theta_A : C(\hat{Z}) \equiv ZM(A)$.

- Note that any C*-algebra A defines a $C(\hat{Z})$-algebra $(A, \hat{Z}, \theta_A)$.
- For $p \in \hat{Z}$, denote by G_p the ideal of A given by

$$G_p = \{f \in C(\hat{Z}) : f(p) = 0\} \cdot A.$$

Define the space of *Glimm ideals* of A via

$$\mathrm{Glimm}(A) = \{G_p : p \in \hat{Z}, G_p \neq A\},$$

with subspace topology inherited from $\hat{Z}$.

- If $\mathrm{Glimm}(A)$ is locally compact then A is a $C_0(\mathrm{Glimm}(A))$-algebra.
- For a locally compact Hausdorff space X, a C*-algebra A is a $C_0(X)$-algebra iff there exists a continuous map $\mathrm{Glimm}(A) \to X$.

Remark: $\mathrm{Glimm}(A)$ may be constructed from the topological space $\mathrm{Prim}(A)$ of primitive ideals of A (with the hull kernel topology) alone; no need for multiplier algebras.

Characterisation of $\mathrm{Glimm}(A \otimes B)$

Theorem (M.)

Let A and B be C^*-algebras. Then the map

$$\begin{array}{rcl} \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) & \to & \mathrm{Glimm}(A \otimes B) \\ (G_p, G_q) & \mapsto & G_p \otimes B + A \otimes G_q \end{array}$$

is an open bijection , which is a homeomorphism if

(i) *A is σ-unital and $\mathrm{Glimm}(A)$ locally compact (in particular if A unital), or*

(ii) *A is a continuous $C_0(\mathrm{Glimm}(A))$-algebra.*

Remark: in general, the topology on $\mathrm{Glimm}(A \otimes B)$ depends only on the product space $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

Exactness and Glimm ideals

Theorem (M.)

For a C^*-algebra A, the following are equivalent:

(i) *A is exact,*

(ii) *For every C^*-algebra B and $q \in \mathrm{Glimm}(B)$, the sequence*

$$0 \longrightarrow A \otimes G_q \xrightarrow{\mathrm{id}\otimes\iota} A \otimes B \xrightarrow{\mathrm{id}\otimes\sigma_q} A \otimes (B/G_q) \longrightarrow 0$$

is exact, where $\sigma_q : B \to B/G_q$ is the quotient map, (i.e. $A \otimes G_q = \ker(\mathrm{id} \otimes \sigma_q)$).

(iii) *For every C^*-algebra B and $(p, q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, we have*

$$A \otimes G_q + G_p \otimes B = \ker(\pi_p \otimes \sigma_q),$$

with $\pi_p : A \to A/G_p, \sigma_q : B \to B/G_q$ the quotient maps.

$C_0(\mathrm{Glimm}(A))$-representations

If $\mathrm{Prim}(A)$ is Hausdorff in the hull-kernel topology, then

- $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$ as sets of ideals and topologically, and
- A is canonically a continuous $C_0(\mathrm{Prim}(A))$-algebra, with simple fibres given by the primitive quotients of A.

The converse is true also: given a continuous $C_0(X)$-algebra (A, X, μ_A) with all fibres simple (and nonzero), then $\mathrm{Prim}(A)$ is homeomorphic to X.

More generally, a separable C*-algebra A is called *quasi-standard* if

1. $(A, \mathrm{Glimm}(A), \theta_A)$ is a continuous $C_0(\mathrm{Glimm}(A))$-algebra, and
2. there is a dense subset $D \subseteq \mathrm{Glimm}(A)$ with G_p primitive for all $p \in D$.

General definition: replace 'G_p primitive' with 'G_p primal' (is the kernel of the GNS representation π_ϕ associated with a state ϕ which is a weak-* limit of factorial states on A).

Examples of quasi-standard C*-algebras: all von Neumann algebras, many group C*-algebras, all homogeneous C*-algebras, $\ell^\infty(A)$ for A primitive.

Examples

1. Let A be the C*-algebra of sequences $x = (x_n) \subset M_2(\mathbb{C})$ such that $x_n \to x_\infty \in M_2(\mathbb{C})$. Then $\mathrm{Prim}(A)$ is homeomorphic to $\hat{\mathbb{N}}$ (one-point compactification), where $n \in \hat{\mathbb{N}}$ is identified with the ideal

$$P_n = \{x \in A : x_n = 0\},\ 1 \le n \le \infty.$$

   Thus the Dauns-Hofmann representation of A gives the trivial bundle $C(\hat{\mathbb{N}}, M_2(\mathbb{C}))$.

2. Let $B = \{x \in A : x_\infty = \mathrm{diag}(\lambda(x), \mu(x))\} \subset A$. Then $\mathrm{Prim}(B) = \{P_n : n \in \mathbb{N}\} \cup \{\ker(\lambda)\} \cup \{\ker(\mu)\}$, where the P_n are isolated and $\ker(\lambda) \approx \ker(\mu)$. Thus

$$\mathrm{Glimm}(B) = \{P_n : n \in \mathbb{N}\} \cup \{\ker(\lambda \oplus \mu)\} \equiv \hat{\mathbb{N}},$$

   and the Dauns-Hofmann bundle $(\hat{\mathbb{N}}, B, \pi_n : B \to B_n)$ has fibres

$$B_n = \begin{cases} M_2(\mathbb{C}) & \text{if} \quad n \in \mathbb{N} \\ \mathbb{C} \oplus \mathbb{C} & \text{if} \quad n = \infty, \end{cases}$$

   and B is quasi-standard.

A discontinuous example

Take the C*-algebra C of sequences $x = (x_n) \subset M_2(\mathbb{C})$ such that

$$\begin{aligned} x_{2n} &\to \mathrm{diag}(\lambda_1(x), \lambda_2(x)) \\ x_{2n+1} &\to \mathrm{diag}(\lambda_2(x), \lambda_3(x)). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Prim}(C) &= \{P_n : n \in \mathbb{N}\} \cup \{\ker(\lambda_i) : i = 1, 2, 3\}. \\ \mathrm{Glimm}(C) &= \{P_n : n \in \mathbb{N}\} \cup \{\ker(\oplus_{i=1}^{3} \lambda_i)\} \equiv \hat{\mathbb{N}} \end{aligned}$$

The bundle $(\hat{\mathbb{N}}, C, \pi_n : C \to C_n)$ has fibres $C_n = M_2(\mathbb{C})$ for $n \in \mathbb{N}$, and $C_\infty = \mathbb{C}^3$, where $\pi_\infty(x) = (\lambda_1(x), \lambda_2(x), \lambda_3(x))$.
Upper-semicontinuous, but not continuous: take $x = (x_n) \in C$ with

$$x_{2n} = \mathrm{diag}(1, 0) \text{ and } x_{2n+1} = 0.$$

Group C*-algebras

Let G be the discrete Heisenberg group, then $A = C^*(G)$ is quasi-standard, with

- $\mathrm{Glimm}(A) \equiv \mathbb{T}$,
- For $\lambda \in \mathbb{T}$ irrational, the corresponding Glimm quotient A/G_λ is the irrational rotation algebra A_λ. Since A_λ is simple, G_λ is primitive.
- For $\lambda \in \mathbb{T}$ rational, A/G_λ is non-simple and n-homogeneous, where n is the least positive integer such that $\lambda^n = 1$.

With H the continuous Heisenberg group, $B = C^*(H)$ is also quasi-standard; here $\mathrm{Glimm}(B) \equiv \mathbb{R}$, quotients given by $B/H_t \equiv K(H)$ and $B/H_0 \equiv C_0(\mathbb{R}^2)$.

In fact many group C*-algebras are quasi-standard (E. Kaniuth, G. Schlichting, K. Taylor); $C_r^*(G)$ for every locally compact [SIN]-group G. In particular, $C^*(G)$ is quasi-standard for every discrete, amenable G.

Classes of Dauns-Hofmann representations

We have the following relations:

$$
\begin{aligned}
& \{ \text{C}^*\text{-algebras } A \text{ with } \mathrm{Prim}(A) \text{ Hausdorff } \} \\
= \quad & \{ \text{continuous } C_0(\mathrm{Prim}(A))\text{-algebras } \} \\
\subsetneq \quad & \{ \text{quasi-standard C*-algebras } A \} \\
\subsetneq \quad & \{ \text{continuous } C_0(\mathrm{Glimm}(A))\text{-algebras } \} \\
\subsetneq \quad & \{ C(\hat{Z})\text{-algebras } \} \\
= \quad & \{ \text{all C*-algebras } \}.
\end{aligned}
$$

Question: are these classes closed under tensor products?

Characterisation of exactness

Theorem (M.)

Let A be a C^*-algebra.

(i) *If $(A, \mathrm{Glimm}(A), \theta_A)$ is a continuous $C_0(\mathrm{Glimm}(A))$-algebra, then A is exact $\Leftrightarrow$ for all C^*-algebras B with $(B, \mathrm{Glimm}(B), \theta_B)$ continuous, the $C_0(\mathrm{Glimm}(A \otimes B))$-algebra $(A \otimes B, \mathrm{Glimm}(A \otimes B), \theta_A \otimes \theta_B)$ is continuous,*

(ii) *If A is quasi-standard, then A is exact $\Leftrightarrow$ $A \otimes B$ is quasi standard for all quasi-standard C^*-algebras B,*

(iii) *If $\mathrm{Prim}(A)$ is Hausdorff, then A is exact $\Leftrightarrow$ $\mathrm{Prim}(A \otimes B)$ is Hausdorff for all C^*-algebras B with $\mathrm{Prim}(B)$ Hausdorff.*

Tensor products of C*-bundles

Thus none of the classes

$$
\begin{aligned}
&\{\ \text{C}^*\text{-algebras } A \text{ with } \mathrm{Prim}(A) \text{ Hausdorff }\} \\
= \quad &\{\ \text{continuous C*-bundles over } \mathrm{Prim}(A)\} \\
\subsetneq \quad &\{\ \text{quasi-standard C*-algebras } A\} \\
\subsetneq \quad &\{\ \text{continuous C*-bundles over } \mathrm{Glimm}(A)\}
\end{aligned}
$$

are closed under tensor products. In each case, their intersection with the class of exact C*-algebras is the largest $\otimes$-closed subclass.

Other C*-norms

- Given (A, X, μ_A) and (B, Y, μ_B) with A and B unital and X and Y compact, let $\|\cdot\|_\gamma$ be any C*-norm on $A \odot B$.
- By a result of Archbold $Z(A \otimes_\gamma B) = Z(A) \otimes Z(B)$,
- It follows that $A \otimes_\gamma B$ is a $C(X \times Y)$-algebra, with fibres $(A \otimes_\gamma B)/J_{x,y}$ where

$$J_{x,y} = \overline{(C_x(X) \cdot A) \odot B + A \odot (C_y(Y) \cdot B)}^{A \otimes_\gamma B}$$

- If $\|\cdot\|_\gamma$ is defined via a tensor product functor $(A, B) \mapsto A \otimes_\gamma B$, then the evaluation maps $\{\pi_x : A \to A_x : x \in X\}$ and $\{\sigma_y : B \to B_y : y \in Y\}$ give rise to $*$-homomorphisms

$$\{\pi_x \otimes_\gamma \sigma_y : A \otimes_\gamma B \to A_x \otimes_\gamma B_y\}$$

- Can also study the tensor product fibrewise; identify $c \in A \otimes_\gamma B$ with

$$\hat{c} : X \times Y \to \coprod A_x \otimes_\gamma B_y$$
$$\hat{c}((x, y)) = (\pi_x \otimes_\gamma \sigma_y)(c)$$

Exact tensor product functors

- If the tensor product functor γ is exact (in both variables), then these two representations of $A \otimes_\gamma B$ agree, that is,

$$\frac{A \otimes_\gamma B}{(C_x(X) \cdot A) \otimes_\gamma B + A \otimes_\gamma (C_y(Y) \cdot B)} = A_x \otimes_\gamma B_y$$

- In particular, it follows that the norm functions

$$(x, y) \mapsto \|(\pi_x \otimes_\gamma \sigma_y)(x)\|$$

  are upper-semicontinuous on $X \times Y$ for all $c \in A \otimes_\gamma B$.
- On the other hand, if the tensor product functor γ fails to be injective, one can construct continuous A and B and $c \in A \otimes_\gamma B$ whose norm function fails to be lower semicontinuous on $A \otimes_\gamma B$.

This last fact is also true for *partial tensor product functors*; that is, for a fixed C*-algebra B, the functor $\cdot \otimes_\gamma B : A \mapsto A \otimes_\gamma B$.

The maximal tensor product

The maximal tensor product is an example of an exact tensor product functor which fails to be injective.
In particular, a C*-algebra B is nuclear if and only if the partial tensor product functor $\cdot \otimes_{\max} B$ is injective.
If A and B are unital, then

$$\mathrm{Glimm}(A \otimes_{\max} B) = \{G \otimes_{\max} B + A \otimes_{\max} H : (G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)\}$$

and by exactness of $\otimes_{\max}$

$$\frac{A \otimes_{\max} B}{G \otimes_{\max} B + A \otimes_{\max} H} = (A/G) \otimes_{\max} (B/H).$$

Theorem (M.)

Let A be a unital quasi-standard C-algebra. Then A is nuclear*
$\Leftrightarrow$ $A \otimes_{\max} B$ is quasi-standard for all quasi-standard C-algebras B.*

Questions on other tensor product functors

1. Does there exist a tensor product functor $\otimes_\gamma$ which preserves continuity of $C(X)$-algebras?
   - Clearly, exactness and injectivity are central to this question.
   - By Pisier and Ozawa, there exist uncountably many distinct injective tensor product functors. On the other hand, by Kirchberg, $\otimes_{\max}$ is the unique exact tensor product functor iff Connes' embedding holds.
2. If we fix a continuous $C(Y)$-algebra B, is there a partial tensor product functor $\cdot \otimes_\gamma B$ which preserves continuity for all continuous A? By a result of Kirchberg, for separable B, there always exists $\cdot \otimes_\gamma B$ which is both injective and exact.