Calculating K-theory of substitution tiling C*-algebras using dual tilings

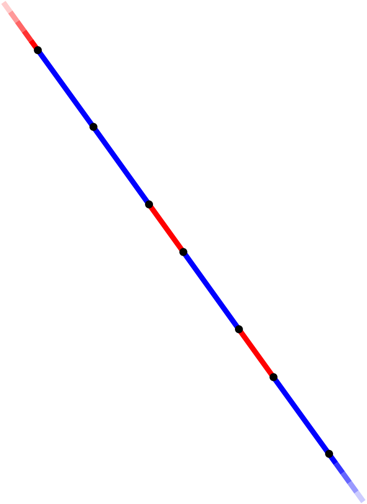
Greg Maloney

Newcastle University

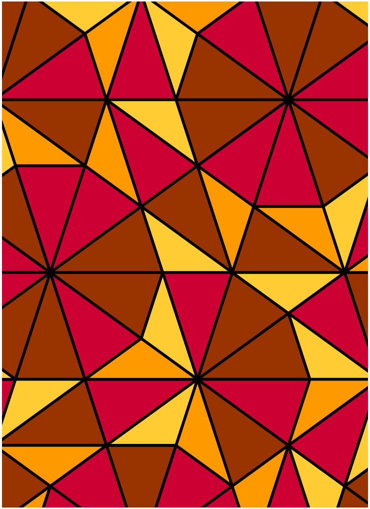
Joint work with
Franz Gähler and John Hunton

Scottish Operator Algebras Seminar, 14 March 2014

Tilings and the tiling metric



A tiling in 1-d



A tiling in 2-d

Tilings and the tiling metric

Definition (Tile)

A *tile* is a subset of $\mathbb{R}^d$ that is homeomorphic to the closed unit ball.

Definition (Partial Tiling, Support)

A *partial tiling* is a set of tiles, any two of which have disjoint interiors. The *support* of a partial tiling is the union of its tiles.

Definition (Tiling)

A *tiling* of $\mathbb{R}^d$ is a partial tiling, the support of which is $\mathbb{R}^d$.

Tilings and the tiling metric

Definition (Tile)

A *tile* is a subset of $\mathbb{R}^d$ that is homeomorphic to the closed unit ball.

Definition (Partial Tiling, Support)

A *partial tiling* is a set of tiles, any two of which have disjoint interiors. The *support* of a partial tiling is the union of its tiles.

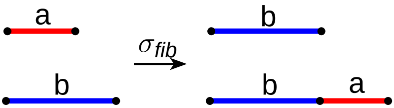
Definition (Tiling)

A *tiling* of $\mathbb{R}^d$ is a partial tiling, the support of which is $\mathbb{R}^d$.

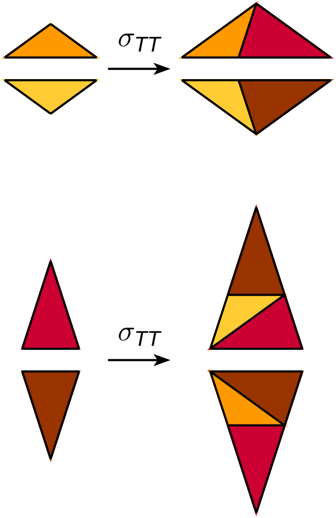
There is a metric on the set of tilings of $\mathbb{R}^d$, in which two tilings are close if, up to a small translation, they agree on a large ball around the origin.

$$d(T, T') = \inf(\{1\} \cup \{\epsilon > 0 : T - u \text{ agrees with } T' - v \text{ on } B_{1/\epsilon}(0) \text{ for some } u, v \in B_\epsilon(0)\})$$

Substitutions

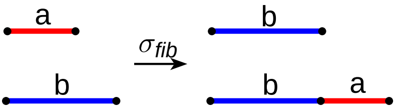


The Fibonacci substitution



The Tübingen triangle substitution

Substitutions

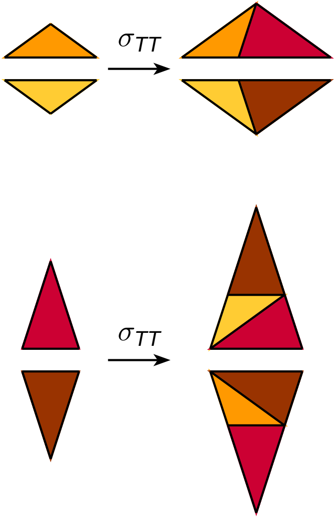


The Fibonacci substitution

Let $P = \{p_1, \ldots, p_k\}$ be a set of tiles, which we will call prototiles. Let $\tilde{\Omega}$ denote the set of all partial tilings containing only translates of tiles from P.

Definition (Substitution)

A *substitution* is a map $\sigma : P \to \tilde{\Omega}$ for which there exists an *inflation constant* $\lambda > 1$ such that the support of $\sigma(p_i)$ is λp_i.



The Tübingen triangle substitution

Substitution tiling spaces

σ extends to a map $\sigma : \tilde{\Omega} \to \tilde{\Omega}$ by setting $\sigma(T) = \bigcup_{p_i + u \in T} (\sigma(p_i) + \lambda u)$.

Definition (Substitution Tiling Space)

The *substitution tiling space* Ω_σ is the set of all tilings $T \in \tilde{\Omega}$ such that for every patch S of T with bounded support there exist $n \in \mathbb{N}$, an index i, and a vector u such that $S \subseteq \sigma^n(p_i + u)$.

Substitution tiling spaces

σ extends to a map $\sigma : \tilde{\Omega} \to \tilde{\Omega}$ by setting $\sigma(T) = \bigcup_{p_i + u \in T} (\sigma(p_i) + \lambda u)$.

Definition (Substitution Tiling Space)

The *substitution tiling space* Ω_σ is the set of all tilings $T \in \tilde{\Omega}$ such that for every patch S of T with bounded support there exist $n \in \mathbb{N}$, an index i, and a vector u such that $S \subseteq \sigma^n(p_i + u)$.

We make four standard assumptions about σ and Ω_σ.

1. σ is *primitive*: there is some $n \in \mathbb{N}$ such that, for all $i, j \leq k$, $\sigma^n(p_i)$ contains a translate of p_j.
2. $\sigma : \Omega_\sigma \to \Omega_\sigma$ is injective.
3. Ω_σ has *finite local complexity*: for any $r > 0$, there are, up to translation, finitely many patches supported in a ball of radius r.
4. Each $T \in \Omega_\sigma$ is a *CW*-complex, in which the tiles are d-cells.

Substitution tiling spaces

σ extends to a map $\sigma : \tilde{\Omega} \to \tilde{\Omega}$ by setting $\sigma(T) = \bigcup_{p_i + u \in T} (\sigma(p_i) + \lambda u)$.

Definition (Substitution Tiling Space)

The *substitution tiling space* Ω_σ is the set of all tilings $T \in \tilde{\Omega}$ such that for every patch S of T with bounded support there exist $n \in \mathbb{N}$, an index i, and a vector u such that $S \subseteq \sigma^n(p_i + u)$.

We make four standard assumptions about σ and Ω_σ.

1. σ is *primitive*: there is some $n \in \mathbb{N}$ such that, for all $i, j \leq k$, $\sigma^n(p_i)$ contains a translate of p_j.
2. $\sigma : \Omega_\sigma \to \Omega_\sigma$ is injective.
3. Ω_σ has *finite local complexity*: for any $r > 0$, there are, up to translation, finitely many patches supported in a ball of radius r.
4. Each $T \in \Omega_\sigma$ is a *CW*-complex, in which the tiles are d-cells.

Primitivity $\implies$ $(\Omega_\sigma, \mathbb{R}^d)$ is *minimal* (Ω_σ is the closure of the translation orbit of any of its points).

Tiling C*-algebras

Consider the groupoid of Ω_σ under the $\mathbb{R}^d$-action by translation.

- As a topological space, this is $\Omega_\sigma \times \mathbb{R}^d$.
- (T, v) and (T', v') are composable if $T' = T + v$, and their composition is $(T, v)(T', v') = (T, v + v')$.

Tiling C*-algebras

Consider the groupoid of Ω_σ under the $\mathbb{R}^d$-action by translation.

- As a topological space, this is $\Omega_\sigma \times \mathbb{R}^d$.
- (T, v) and (T', v') are composable if $T' = T + v$, and their composition is $(T, v)(T', v') = (T, v + v')$.

Let $C^*(\sigma)$ denote the C*-algebra of this groupoid. This has been studied using two different approaches.

Tiling C*-algebras

Consider the groupoid of Ω_σ under the $\mathbb{R}^d$-action by translation.

- As a topological space, this is $\Omega_\sigma \times \mathbb{R}^d$.
- (T, v) and (T', v') are composable if $T' = T + v$, and their composition is $(T, v)(T', v') = (T, v + v')$.

Let $C^*(\sigma)$ denote the C*-algebra of this groupoid. This has been studied using two different approaches.

1. (Kellendonk) Pick a distinguished point, called a *puncture*, in the interior of each prototile; by translation, this defines a puncture in the interior of each tile. Then restrict to groupoid elements (T, v) for which T has a puncture at the origin. This yields a more tractable groupoid, the C*-algebra of which is strongly Morita equivalent to that of the original groupoid.

Tiling C*-algebras

Consider the groupoid of Ω_σ under the $\mathbb{R}^d$-action by translation.

- As a topological space, this is $\Omega_\sigma \times \mathbb{R}^d$.
- (T, v) and (T', v') are composable if $T' = T + v$, and their composition is $(T, v)(T', v') = (T, v + v')$.

Let $C^*(\sigma)$ denote the C*-algebra of this groupoid. This has been studied using two different approaches.

1. (Kellendonk) Pick a distinguished point, called a *puncture*, in the interior of each prototile; by translation, this defines a puncture in the interior of each tile. Then restrict to groupoid elements (T, v) for which T has a puncture at the origin. This yields a more tractable groupoid, the C*-algebra of which is strongly Morita equivalent to that of the original groupoid.
2. (Anderson-Putnam) The groupoid C*-algebra is isomorphic to the crossed product C*-algebra of Ω_σ by $\mathbb{R}^d$, and so by the Connes-Thom isomorphism, the K-theory of the algebra is related to that of the space Ω_σ.

K-theory of tiling C*-algebras

K-theory is an invariant that we expect will yield useful information about Ω_σ, and hence about the tiling T itself. How do we compute K-theory?

Theorem (Anderson-Putnam 1998)

If Ω_σ is a substitution tiling space of tilings with dimension 1, then

$$K_0(C^*(\Omega_\sigma)) \cong H^1(\Omega_\sigma), \quad K_1(C^*(\Omega_\sigma)) \cong H^0(\Omega_\sigma).$$

If Ω_σ is a substitution tiling space of tilings with dimension 2, then

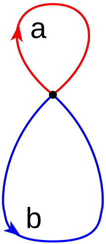
$$K_0(C^*(\Omega_\sigma)) \cong H^2(\Omega_\sigma) \oplus H^0(\Omega_\sigma), \quad K_1(C^*(\Omega_\sigma)) \cong H^1(\Omega_\sigma).$$

(Here H^* denotes Cech cohomology with integer coefficients.)

The Anderson-Putnam complex

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

- Its cells are translation equivalence classes of tiles from tilings in the space.
- Cells meet at a boundary if they have representatives that meet at a boundary.

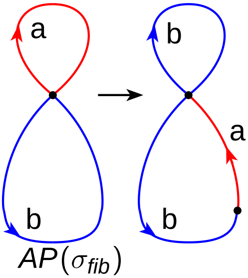


$AP(\sigma_{fib})$

The Anderson-Putnam complex

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

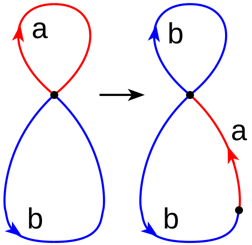
- Its cells are translation equivalence classes of tiles from tilings in the space.
- Cells meet at a boundary if they have representatives that meet at a boundary.
- σ induces a self map on $AP(\sigma)$.



The Anderson-Putnam complex

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

- Its cells are translation equivalence classes of tiles from tilings in the space.
- Cells meet at a boundary if they have representatives that meet at a boundary.
- σ induces a self map on $AP(\sigma)$.
- Consider the inverse limit $\varprojlim AP(\sigma)$ under this map.

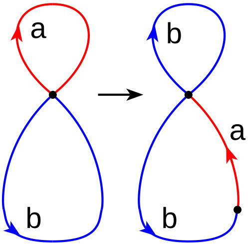


$AP(\sigma_{fib})$

The Anderson-Putnam complex

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

- There is a map from the tiling space to the inverse limit $\varprojlim AP(\sigma)$:

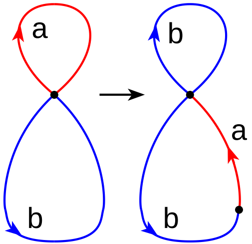


$AP(\sigma_{fib})$

The Anderson-Putnam complex

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

- There is a map from the tiling space to the inverse limit $\varprojlim AP(\sigma)$:
- The origin in T gives a point $x_0 \in AP(\sigma)$.

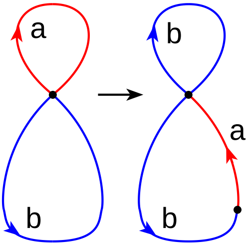


$AP(\sigma_{fib})$

$x_0 \in AP(\sigma_{fib})$  T

To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

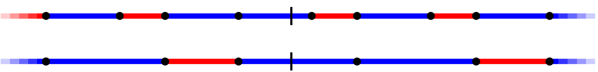
- There is a map from the tiling space to the inverse limit $\varprojlim AP(\sigma)$:
- The origin in T gives a point $x_0 \in AP(\sigma)$.
- The origin in $\sigma^{-1}(T)$ gives a point $x_1 \in AP(\sigma)$ that maps to x_0 under the self map.



$AP(\sigma_{fib})$

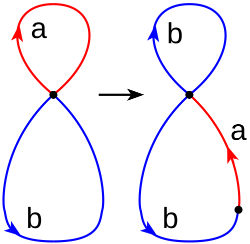
$x_0 \in AP(\sigma_{fib})$ $\qquad$ T

$x_1 \in AP(\sigma_{fib})$ $\qquad$ $\lambda\sigma_{fib}^{-1}(T)$

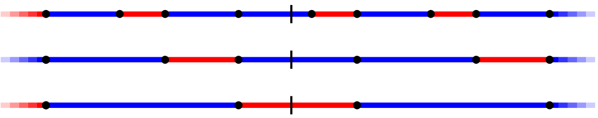


To compute the K-theory of $C^*(\sigma)$, Anderson and Putnam introduced a cell complex, called the Anderson-Putnam complex, or $AP(\sigma)$.

- There is a map from the tiling space to the inverse limit $\varprojlim AP(\sigma)$:
- The origin in T gives a point $x_0 \in AP(\sigma)$.
- The origin in $\sigma^{-1}(T)$ gives a point $x_1 \in AP(\sigma)$ that maps to x_0 under the self map.
- The sequence $(x_i)_{i\geq 1}$ is an element of $\varprojlim AP(\sigma)$.



$AP(\sigma_{fib})$



$x_0 \in AP(\sigma_{fib})$ — T

$x_1 \in AP(\sigma_{fib})$ — $\lambda\sigma_{fib}^{-1}(T)$

$x_2 \in AP(\sigma_{fib})$ — $\lambda^2\sigma_{fib}^{-2}(T)$

$\vdots$

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition ...

Definition (Cech cohomology, summary from a talk by Putnam)

To find the Cech cohomology of a space X:

1. Take a finite open cover $\mathcal{U}$ of X.
2. Associated to $\mathcal{U}$ is a simplicial complex: vertices are the elements of $\mathcal{U}$, edges are non-empty intersections of two elements of $\mathcal{U}$, etc.
3. Take the cohomology of the simplicial complex.
4. Refine the open cover, get an inductive system of cohomologies and take the limit.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition ... but it satisfies two important conditions:

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition . . . but it satisfies two important conditions:
    1. For *CW*-complexes, it agrees with singular cohomology, and hence cellular cohomology.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition … but it satisfies two important conditions:
  1. For *CW*-complexes, it agrees with singular cohomology, and hence cellular cohomology.
  2. It behaves well under taking limits.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition ... but it satisfies two important conditions:
    1. For *CW*-complexes, it agrees with singular cohomology, and hence cellular cohomology.
    2. It behaves well under taking limits.
- Then $H^*(\Omega_\sigma) \cong \varinjlim H^*(AP(\sigma))$, where H^* on the right hand side can be computed as cellular cohomology.

Computing Cech cohomology

- In some cases, the map $\Omega_\sigma \to AP(\sigma)$ is a homeomorphism.
- Then we can use $AP(\sigma)$ to compute the Cech cohomology of Ω_σ.
- Cech cohomology has a difficult definition ... but it satisfies two important conditions:
  1. For *CW*-complexes, it agrees with singular cohomology, and hence cellular cohomology.
  2. It behaves well under taking limits.
- Then $H^*(\Omega_\sigma) \cong \varinjlim H^*(AP(\sigma))$, where H^* on the right hand side can be computed as cellular cohomology.
- If σ *forces its border*, then $\Omega_\sigma \to \varprojlim AP(\sigma)$ is a homeomorphism.

Definition (Forcing the border)

σ *forces its border* if there exists some n such that, for any tile t and any two tilings T, T' containing t, $\sigma^n(T)$ and $\sigma^n(T')$ coincide, not just on $\sigma^n(t)$, but also on all tiles that meet $\sigma^n(t)$.

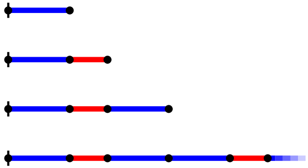
The collared Anderson-Putnam complex

But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

The collared Anderson-Putnam complex

But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

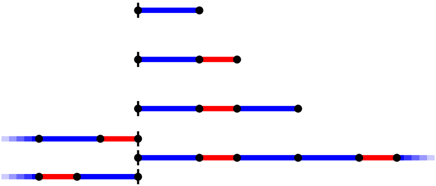
- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.



The collared Anderson-Putnam complex

But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

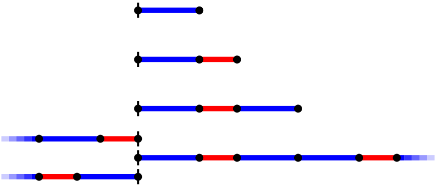
- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.
- There are two ways to extend this to the left half-line.

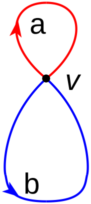


The collared Anderson-Putnam complex

But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.
- There are two ways to extend this to the left half-line.
- So two different tilings are sent to $(v, v, v, \ldots) \in \varprojlim AP(\sigma_{fib})$.



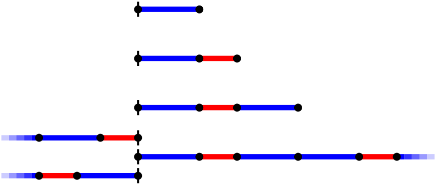


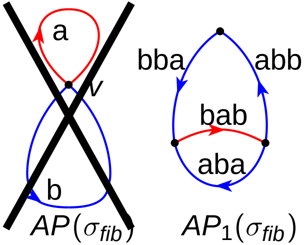
$AP(\sigma_{fib})$

The collared Anderson-Putnam complex

But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.
- There are two ways to extend this to the left half-line.
- So two different tilings are sent to $(v, v, v, \dots) \in \varprojlim AP(\sigma_{fib})$.
- To get injectivity, use the *collared AP complex* $AP_1(\sigma)$ instead.

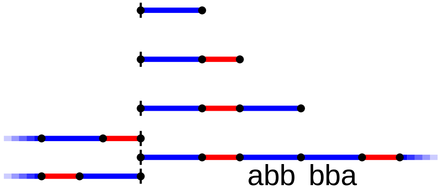


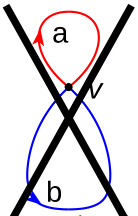


The collared Anderson-Putnam complex

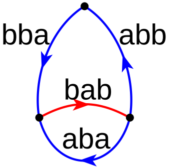
But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.
- There are two ways to extend this to the left half-line.
- So two different tilings are sent to $(v, v, v, \ldots) \in \varprojlim AP(\sigma_{fib})$.
- To get injectivity, use the *collared AP complex* $AP_1(\sigma)$ instead.
- This is like $AP(\sigma)$, but it distinguishes between two translates of the same tile if they have different neighbours.





$AP(\sigma_{fib})$

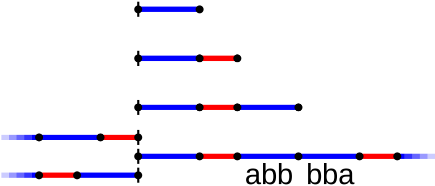


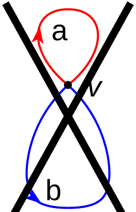
$AP_1(\sigma_{fib})$

The collared Anderson-Putnam complex

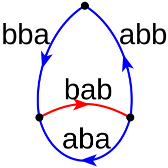
But $\Omega_\sigma \to \varprojlim AP(\sigma)$ is not necessarily injective.

- Apply σ_{fib} repeatedly to a b-tile that has its left end on the origin; get a tiling of the right half-line.
- There are two ways to extend this to the left half-line.
- So two different tilings are sent to $(v, v, v, \ldots) \in \varprojlim AP(\sigma_{fib})$.
- To get injectivity, use the *collared AP complex* $AP_1(\sigma)$ instead.
- This is like $AP(\sigma)$, but it distinguishes between two translates of the same tile if they have different neighbours.
- Then $\Omega_\sigma \cong \varprojlim AP_1(\sigma)$.





$AP(\sigma_{fib})$



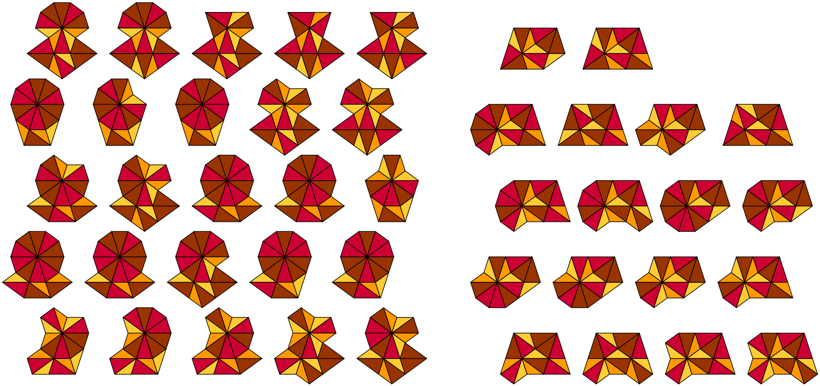
$AP_1(\sigma_{fib})$

Combinatorial explosion

Tiles: $(1 + 1) \times 20 = 40$ total

Combinatorial explosion

Tiles: $(1 + 1) \times 20 = 40$ total



Collared tiles: $(25 + 18) \times 20 = 860$ total

We need to use a computer

Calculating K-theory involves several steps.

1. Find all the collared tiles and lower-dimensional cells.
2. Compute matrices for the coboundary maps.
3. Compute matrices induced on cohomology by the self-map of the complex.
4. Compute the inductive limit groups.

We need to use a computer

Calculating K-theory involves several steps.

1. Find all the collared tiles and lower-dimensional cells.
2. Compute matrices for the coboundary maps.
3. Compute matrices induced on cohomology by the self-map of the complex.
4. Compute the inductive limit groups.

If there are 860 collared tiles, then we need to use a computer.

We need to use a computer

Calculating K-theory involves several steps.

1. Find all the collared tiles and lower-dimensional cells.
2. Compute matrices for the coboundary maps.
3. Compute matrices induced on cohomology by the self-map of the complex.
4. Compute the inductive limit groups.

If there are 860 collared tiles, then we need to use a computer.
Is it worth it?

We need to use a computer

Calculating K-theory involves several steps.

1. Find all the collared tiles and lower-dimensional cells.
2. Compute matrices for the coboundary maps.
3. Compute matrices induced on cohomology by the self-map of the complex.
4. Compute the inductive limit groups.

If there are 860 collared tiles, then we need to use a computer.
Is it worth it? Yes. Gähler computed the Cech cohomology of the Tübingen triangle substitution using the Anderson-Putnam method, and found the following.

$$H^0 : \mathbb{Z}, \quad H^1 : \mathbb{Z}^5, \quad H^2 : \mathbb{Z}^{24} \oplus \mathbb{Z}_5^2.$$

This was the first known example of torsion in tiling cohomology.

A Problem

- 860 is a lot of cells.
- The Tübingen triangle substitution is still a very basic one.
- We want to compute K-theory for bigger 2-d substitutions.
- We want to compute K-theory for 3-d substitutions.

Passing from $AP(\sigma)$ to $AP_1(\sigma)$ introduces too many new cells. We would like to replace $AP_1(\sigma)$ with a different, smaller complex.

A Problem

- 860 is a lot of cells.
- The Tübingen triangle substitution is still a very basic one.
- We want to compute K-theory for bigger 2-d substitutions.
- We want to compute K-theory for 3-d substitutions.

Passing from $AP(\sigma)$ to $AP_1(\sigma)$ introduces too many new cells. We would like to replace $AP_1(\sigma)$ with a different, smaller complex.
Ideas:

1. (Barge-Diamond, Barge-Diamond-Hunton-Sadun) Blow up all the subcells in $AP(\sigma)$ to cells of full dimension.
2. (Gähler-M) In one dimension, collar on the left only.

A Problem

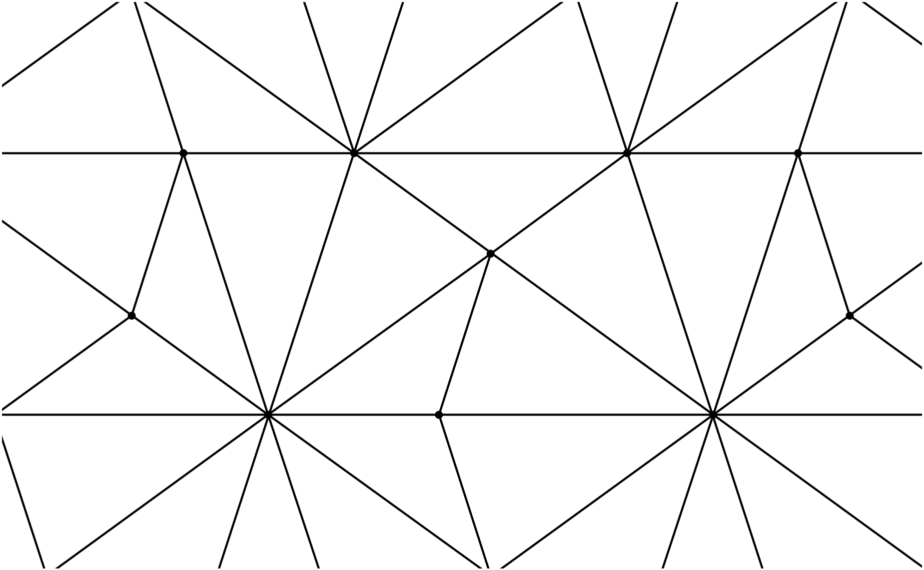
- 860 is a lot of cells.
- The Tübingen triangle substitution is still a very basic one.
- We want to compute K-theory for bigger 2-d substitutions.
- We want to compute K-theory for 3-d substitutions.

Passing from $AP(\sigma)$ to $AP_1(\sigma)$ introduces too many new cells. We would like to replace $AP_1(\sigma)$ with a different, smaller complex.

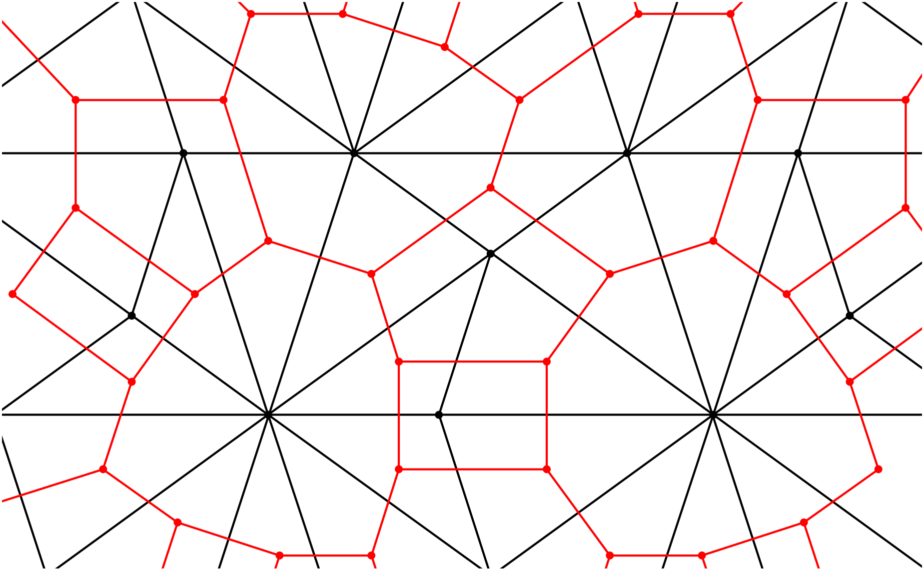
Ideas:

1. (Barge-Diamond, Barge-Diamond-Hunton-Sadun) Blow up all the subcells in $AP(\sigma)$ to cells of full dimension.
2. (Gähler-M) In one dimension, collar on the left only.
3. New: use dual tilings.

Dual tilings



Dual tilings



Dual tilings

Definition (Combinatorial dual)

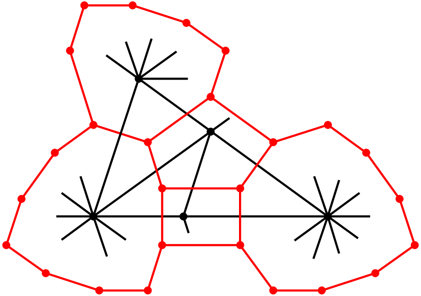
Given a tiling T containing an open cell c, the *combinatorial dual* of c is

$$c^* := \{t \in T \mid c \subset t\}.$$

- If c is a vertex, then c^* is called a *vertex star*.
- A *dual tiling* T^* is a tiling that is a geometric realisation for the set of combinatorial dual cells of T. Vertex stars play the role of tiles.
- The tiling space is homeomorphic to the space of dual tilings.

Dual substitutions

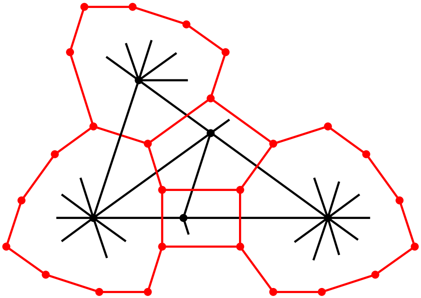
Idea: define a substitution on the dual so we can use the AP method to compute K-theory. Be lazy! Combinatorics only.



Dual substitutions

Idea: define a substitution on the dual so we can use the AP method to compute K-theory. Be lazy! Combinatorics only.

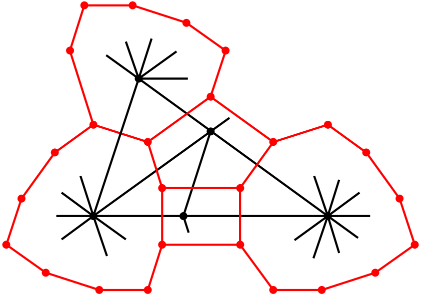
1. Make a complex from the combinatorial dual cells.
2. Use σ to define a combinatorial substitution σ^* on the vertex stars.
3. Get a self-map on the dual complex.
4. Take the inverse limit of this complex.



Dual substitutions

Idea: define a substitution on the dual so we can use the AP method to compute K-theory. Be lazy! Combinatorics only.

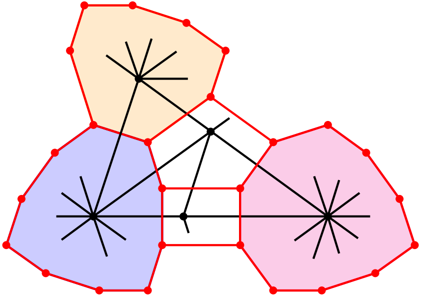
1. Make a complex from the combinatorial dual cells.
2. Use σ to define a combinatorial substitution σ^* on the vertex stars.
3. Get a self-map on the dual complex.
4. Take the inverse limit of this complex.
5. The inverse limit is homotopy equivalent, not homeomorphic, to Ω_σ.



Dual substitutions

Idea: define a substitution on the dual so we can use the AP method to compute K-theory. Be lazy! Combinatorics only.

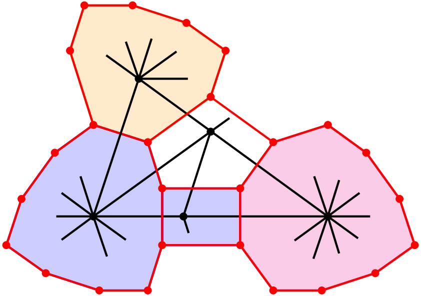
1. Make a complex from the combinatorial dual cells.
2. Use σ to define a combinatorial substitution σ^* on the vertex stars.
3. Get a self-map on the dual complex.
4. Take the inverse limit of this complex.
5. The inverse limit is homotopy equivalent, not homeomorphic, to Ω_σ.



Dual substitutions

Idea: define a substitution on the dual so we can use the AP method to compute *K*-theory. Be lazy! Combinatorics only.

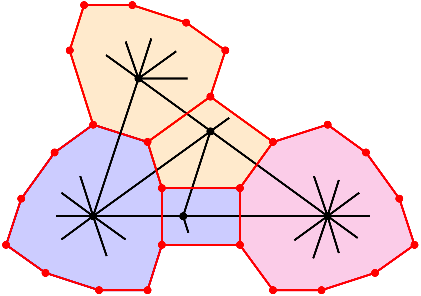
1. Make a complex from the combinatorial dual cells.
2. Use σ to define a combinatorial substitution σ^* on the vertex stars.
3. Get a self-map on the dual complex.
4. Take the inverse limit of this complex.
5. The inverse limit is homotopy equivalent, not homeomorphic, to Ω_σ.



Dual substitutions

Idea: define a substitution on the dual so we can use the AP method to compute K-theory. Be lazy! Combinatorics only.

1. Make a complex from the combinatorial dual cells.
2. Use σ to define a combinatorial substitution σ^* on the vertex stars.
3. Get a self-map on the dual complex.
4. Take the inverse limit of this complex.
5. The inverse limit is homotopy equivalent, not homeomorphic, to Ω_σ.

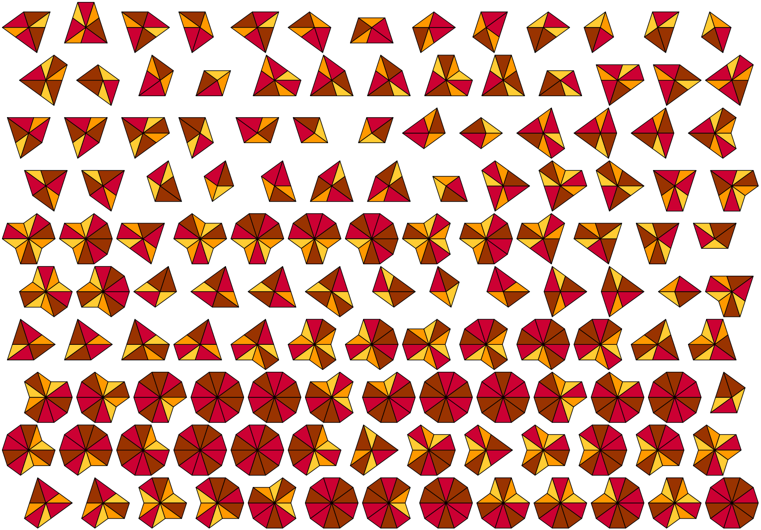


Caution

This method works, but we have to be careful how we define the dual substitution σ^*.

- Make sure σ^* is translation equivariant.
- Make sure σ^* is primitive.
- Don't introduce new adjacency.
- Don't remove existing adjacency.
- Don't let c^* and $\sigma^*(c^*)$ have different topology.

An improvement

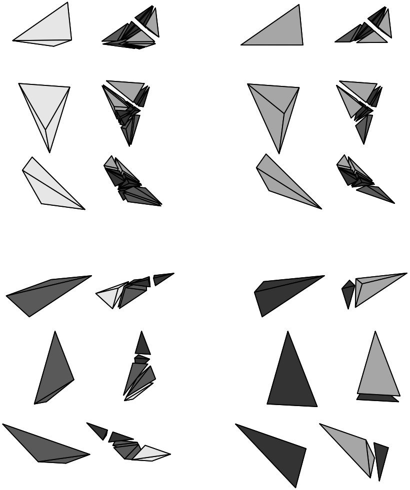


Vertex stars: 130 total

Results

- We recover the results of Gähler for the TT substitution and others.
- We have new examples with an interesting property: the substitution matrix is unimodular, but the homomorphism induced on H^2 of the complex by the substitution is not.

A 3-d substitution



L. Danzer, Discr. Math. **76** (1989) 1–7
Tetrahedra tiling with τ scaling

	0	1	2	3
$\mathrm{rk}\, C^k(\Gamma)$	480	1320	1320	480
$H^k(\Omega)$	$\mathbb{Z}$	$\mathbb{Z}^7$	$\mathbb{Z}^{16}$	$\mathbb{Z}^{20}$