Semigroup actions on operator algebras

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I. Preliminaries

Operator algebras are subalgebras of $\mathscr{B}(H)$

1. Selfadjoint norm-closed subalgebras, i.e. *C*-algebras*.
2. Non-involutive, i.e. *nonselfadjoint operator algebras (nsa)*.

By definition every nsa $\mathscr{A} \subseteq \mathscr{B}(H)$ generates a C*-algebra $\mathrm{C}^*(\mathscr{A})$

It may happen that $\iota_1 \colon \mathscr{A} \to \mathscr{B}(H_1)$ and $\iota_2 \colon \mathscr{A} \to \mathscr{B}(H_2)$ but

$$\mathrm{C}^*(\iota_1(\mathscr{A})) \not\simeq \mathrm{C}^*(\iota_2(\mathscr{A})).$$

Example

The disc algebra $\mathbb{A}(\mathbb{D})$ generates the Toeplitz algebra, $C(\overline{\mathbb{D}})$, and $C(\mathbb{T})$. However $C(\mathbb{T})$ is the *minimal C*-algebra generated by* $\mathbb{A}(\mathbb{D})$, and we call $C(\mathbb{T})$ *the C*-envelope of* $\mathbb{A}(\mathbb{D})$.

I. Preliminaries

Question, Arveson (1969)

Does every nsa have a C*-envelope?

Answer: Yes

$\exists\ \iota\colon \mathscr{A} \to \mathscr{B}(H)$ s.t. for any other $\iota'\colon \mathscr{A} \to \mathscr{B}(K)$, $\exists$ a $*$-epimorphism $\Phi\colon \mathrm{C}^*(\iota'(\mathscr{A})) \to \mathrm{C}^*(\iota(\mathscr{A}))$ with $\Phi\iota'(a) = \iota(a)$, $\forall a \in \mathscr{A}$.

The $\mathrm{C}^*(\iota(\mathscr{A}))$ is *the C*-envelope of* $\mathscr{A}$. We write $\mathrm{C}^*_{\text{env}}(\mathscr{A}) = \mathrm{C}^*(\iota(\mathscr{A}))$.

Proofs by:

1. Hamana (1979): $\mathrm{C}^*_{\text{env}}(\mathscr{A})$ is generated in *the injective envelope*.
2. Dritschel-McCullough (2001): $\mathrm{C}^*_{\text{env}}(\mathscr{A})$ is generated by a *maximal dilation*.

Arveson's Program on the C*-envelope

Determine[1] and examine[2] the C*-envelope of a given nsa.

I. Preliminaries

Dilations

Let $T \in \mathscr{B}(H)$. *A power dilation* $U \in \mathscr{B}(K)$ *of* T is of the form

$$U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}.$$

A dilation is *maximal* if it has only trivial dilations.

Example

If T is a contraction ($\|T\| \leq 1$), then the maximal dilation is achieved by a unitary U ($U^*U = UU^* = I$).

Dilations

The idea is that by dilating we obtain "better-behaved" objects.

I. Preliminaries

In this talk we focus on encoding:

$$\{\ \text{C*-dynamical systems}\ \} \quad \leftrightsquigarrow \quad \{\ \text{Operator algebras}\ \}$$

• Origins: Murray, von Neumann (1936, 1940) – Type I, II, and III factors.

• *C*-crossed products*: are constructed based on a given group action $\alpha\colon G \to \mathrm{Aut}(A)$ on a C*-algebra A by *-automorphisms.

• We turn our focus to semigroup actions $\alpha\colon P \to \mathrm{End}(A)$ on a C*-algebra A by **-endomorphisms*.

• Case example: $P = \mathbb{Z}_+$.

II. Philosophy

Definition

A *C*-dynamical system* $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$ consists of a *-endomorphism $\alpha\colon A \to A$ of a C*-algebra A.

- Use operators to encode the evolution of the system (in discrete time):

$$\begin{array}{cccc} a & \alpha(a) & \alpha^2(a) & \cdots \\ | & | & | & \\ t{=}0 & t{=}1 & t{=}2 & \end{array}$$

- The key is to introduce an "external" operator V that satisfies the *covariance relation*

$$a \cdot V = V \cdot \alpha(a) \text{ for all } a \in A.$$

This defines a convolution on monomials $V^n a$ for $n \in \mathbb{Z}_+$ and $a \in A$.

II. Operator algebras over $\alpha \colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Semicrossed product $\mathscr{T}^+_{(A,\alpha)}$ (no involution)

Universal nonselafdjoint operator algebra generated by

$$V^n a, \text{ with } a \in A,\ n \in \mathbb{Z}_+,$$

such that $a \cdot V = V \cdot \alpha(a)$ and V is a contraction ($\|V\| \leq 1$).

Remark

Inititated by Arveson (1967), formally defined by Peters (1984).

Theorem (Muhly-Solel 2006)

The scp $\mathscr{T}^+_{(A,\alpha)}$ *coincides with the nsa generated by*

$$V^n a, \text{ with } a \in A,\ n \in \mathbb{Z}_+,$$

such that $a \cdot V = V \cdot \alpha(a)$ *and* V **is an isometry** ($V^* V = I$).

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (with involution)

Universal C*-algebra generated by

$$V^n a, \text{ with } a \in A,\ n \in \mathbb{Z}_+,$$

such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^* V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker\alpha^\perp := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Remarks

1. Example of a C*-correspondence.
2. Notice that $a = V\alpha(a)V^*$ for all $a \in \ker\alpha^\perp$.
3. $A \hookrightarrow \mathscr{O}(A,\alpha)$ (Katsura 2004).
4. When $\alpha \in \mathrm{Aut}(A)$ then $\ker\alpha^\perp = A$.Thus V is a unitary and $\mathscr{O}_{(A,\alpha)}$ is *the C*-crossed product* $A \rtimes_\alpha \mathbb{Z}$.

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Question

Why such complexity?

Remark

1. Let a faithful $\rho\colon A \to \mathscr{B}(H)$ and an isometry V such that
$$\rho(a)V = V\rho\alpha(a).$$
2. If $\rho(a_0) + \sum_{n>0} V_n \rho(a_s) V_n^* = 0$ then $a_0 \in \ker\alpha^\perp$ (Katsura 2004).
3. This happens because such equations magically transform into
$$\rho(a_0)(I - VV^*) = 0.$$

II. Two interpretations of dilation

(1) Identification of the C*-envelope (Katsoulis-Kribs 2005)

The C*-envelope of $\mathscr{T}^{+}_{(A,\alpha)}$ is $\mathscr{O}_{(A,\alpha)}$.

(2) Connecting it to a natural C*-object (K. 2011)

$$
\begin{array}{ccc}
\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A) & \longrightarrow & \mathscr{O}_{(A,\alpha)} \\
\text{dilation} \downarrow & & \wr \ \text{strong Morita equivalent} \\
\beta\colon \mathbb{Z} \to \mathrm{Aut}(B) & \longrightarrow & \mathscr{O}_{(B,\beta)} \simeq B \rtimes_\beta \mathbb{Z}
\end{array}
$$

II. Application: Ideal Structure

Theorem (K.-Katsoulis 2011)

$$
\begin{array}{ccc}
\alpha : \mathbb{Z}_+ \to \mathrm{End}(A) & \longrightarrow & \mathscr{O}_{(A,\alpha)} \\
\Big\downarrow \text{dilation} & & \Big\updownarrow \text{strong Morita equivalent} \\
\beta : \mathbb{Z} \to \mathrm{Aut}(B) & \longrightarrow & \mathscr{O}_{(B,\beta)} \simeq B \rtimes_\beta \mathbb{Z}
\end{array}
$$

Corollary (K. 2011)

Let $A = C(X)$. TFAE:

1. (A, α) *is minimal and* $\alpha^n \neq \alpha^m$ *for all* $n, m \in \mathbb{Z}_+$*;*
2. (B, β) *is minimal and* $\beta^n \neq \mathrm{id}$ *for all* $n \in \mathbb{Z}$ *(topol. free);*
3. $B \rtimes_\beta \mathbb{Z}$ *is simple;*
4. $\mathscr{O}_{(A,\alpha)}$ *is simple (has no non-trivial two-sided closed ideals).*

III. Program on semigroup actions

Question 1

$$\begin{array}{ccc} \alpha\colon P \to \mathrm{End}(A) & \longrightarrow & \text{C*-envelope of } a \text{ scp} \\ \Big\downarrow \text{dilation} & & \Big\} \text{strong Morita equivalent ?} \\ \beta\colon G \to \mathrm{Aut}(B) & \longrightarrow & \text{C*-crossed product} \end{array}$$

Question 2

Is the C*-envelope a Cuntz-type C*-algebra? Can we describe it by $*$-algebraic relations?

Applications 3

Relate the intrinsic properties of $\alpha\colon P \to \mathrm{End}(A)$ to C*-properties of the obtained object.

III. Program on semigroup actions

Davidson-Fuller-K. (2014)

$$\begin{array}{ccc} \alpha\colon P \to \mathrm{End}(A) & \longrightarrow & \text{C*-envelope of a sem. prod.} \\ \text{dilation} \downarrow & & \text{strong} \wr \text{ Morita equivalent} \\ \beta\colon G \to \mathrm{Aut}(B) & \longrightarrow & \text{C*-crossed product} \end{array}$$

1. We confirm this when P is $\mathbb{Z}_+^n$, $\mathbb{F}_n^+$, a spanning cone, an Ore sgrp.
2. For $P = \mathbb{Z}_+^n$ we coin the Cuntz-Nica-Pimsner algebra.
3. We study the Cuntz-Nica-Pimsner algebras in terms of ideal structure.

K. (2014)

4. We study the Nica-Pimsner algebras in terms of nuclearity, exactness, KMS states.

III. Operator algebras over $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

Notation

We write $\mathbf{i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ for all $i = 1, \ldots, n$.

Thus $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ is defined by n commuting $\alpha_{\mathbf{i}} \in End(A)$.

Requirements

1. n contractions $V_{\mathbf{i}}$ such that $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.
2. The $V_{\mathbf{i}}$ commute.

Is this enough?

The aim is to reach a crossed product. For $A = \mathbb{C}$ we would like to dilate the $V_{\mathbf{i}}$ to unitaries. Parrott's counterexample shows that this cannot be done for general n.

3. We focus on doubly commuting $V_{\mathbf{i}}$, i.e. $V_{\mathbf{i}} V_{\mathbf{j}}^* = V_{\mathbf{j}}^* V_{\mathbf{i}}$ for $i \neq j$.

III. Operator algebras over $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

The Nica-covariant semicrossed product $A \times_\alpha^{nc} \mathbb{Z}_+^n$ (no involution)

Universal nonselafdjoint operator algebra generated by

$$V_s a, \text{ with } a \in A,\ s \in \mathbb{Z}_+^n,$$

for n doubly commuting contractions $V_{\mathbf{i}}$ with $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.

Remark

A embeds in $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$.

Example

For $A \subseteq H$ let $K = H \otimes \ell^2(\mathbb{Z}_+^n)$ and define

$$S_{\mathbf{i}}(\xi \otimes e_s) = \xi \otimes e_{\mathbf{i}+s} \text{ and } \pi(a)(\xi \otimes e_s) = \alpha_s(a)\xi \otimes e_s$$

for all $s \in \mathbb{Z}_+^n$ and $\xi \in H$. Then π is a faithful representation of A.

III. Operator algebras over $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

Question

Why do we call it *Nica* covariant?

Theorem (Davidson-Fuller-K. 2014)

The Nc-scp $A \times_\alpha^{nc} \mathbb{Z}_+^n$ coincides with the nsa generated by doubly commuting isometries $V_{\mathbf{i}}$ and A such that $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.

Remark

Doubly commuting isometries form a representation of $\mathbb{Z}_+^n$ in the sense of Nica.

Corollary

Then $\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{nc} \mathbb{Z}_+^n) \simeq \overline{\mathrm{span}}\{V_s a V_t^* : a \in \mathscr{A}$ and $s, t \in \mathbb{Z}_+^n\}$.

III. Reductions

The plan

Dilate a system $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ to a group action $\beta\colon \mathbb{Z}^n \to \mathrm{Aut}(B)$.

Injective case: $\ker \alpha_{\mathbf{i}} = (0)$ for all $i = 1, \ldots, n$.

We can then construct the direct limit $\beta_{\mathbf{i}} \in \mathrm{Aut}(B)$ s.t.

$$\begin{array}{ccccc} A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B \\ \downarrow{\scriptstyle \alpha_{\mathbf{i}}} & & \downarrow{\scriptstyle \alpha_{\mathbf{i}}} & & \downarrow{\scriptstyle \beta_{\mathbf{i}}} \\ A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B \end{array}$$

where $A_s = A$ for all $s \in \mathbb{Z}_+^2$.

Then $\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n) \simeq B \rtimes_\beta \mathbb{Z}^n$ (Corollary Davidson-Fuller-K. 2014).

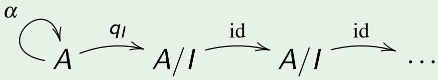
III. Reductions

The (revised) plan

Dilate a system $\alpha : \mathbb{Z}_+^n \to \mathrm{End}(A)$ where $\ker \alpha_{\mathbf{i}} \neq (0)$ to a system $\beta : \mathbb{Z}_+^n \to \mathrm{End}(B)$ such that $\ker \beta_{\mathbf{i}} = (0)$.

The $n = 1$ case (K. 2011)

For $I = \ker \alpha^{\perp}$ let $B = A \oplus c_0(A/I)$ and $\beta(a, (x_n)) = (\alpha(a), a + I, (x_n))$.



The $n = 2$ case

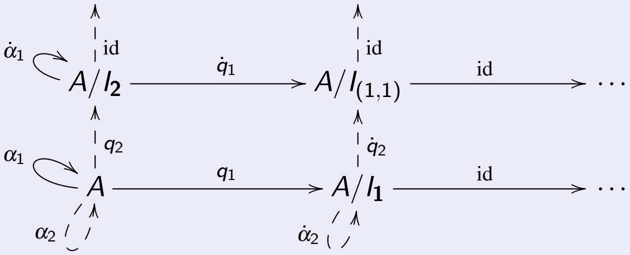
Let $\alpha_1, \alpha_2 \in \mathrm{End}(A)$ such that $\alpha_1 \alpha_2 = \alpha_2 \alpha_1$. We want two injective commuting β_1, β_2 on some $B \supseteq A$ that dilate α_1, α_2.

III. Non-injective case

A first attempt

Let $I_{(1,1)} := (\ker \alpha_1 \cdot \ker \alpha_2)^{\perp}$, $I_1 := \cap_n \alpha_2^{-n}(I_{(1,1)})$, $I_2 := \cap_n \alpha_1^{-n}(I_{(1,1)})$.

Let β_1 be the solid arrows and β_2 the broken arrows:



with $\dot{\alpha}_1 q_2 = q_1 \alpha_2$ and $\dot{q}_1 q_1 = q_{(1,1)}$ (plus the symmetrical ones).

Then β is injective and generalises the $n = 1$ case.

However this construction is bound to fail!

III. Non-injective case

How did we end up with $I_{(1,1)} = (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$?

1. Let a faithful $\rho\colon A \to \mathscr{B}(H)$ and doubly commuting isometries $V_{\mathbf{i}}$ such that

$$\rho(a) V_{\mathbf{i}} = V_{\mathbf{i}} \rho \alpha_{\mathbf{i}}(a).$$

2. Because of a gauge action, we will have to deal with equations

$$\rho(a_0) + \sum_{s > \underline{0}} V_s \rho(a_s) V_s^* = 0.$$

3. This magically transforms into

$$\rho(a_0)(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0.$$

4. From this we get that $a_0 \perp \ker \alpha_{\mathbf{1}}, \ker \alpha_{\mathbf{2}}$.

III. Non-injective case

Why isn't $I_{(1,1)} = (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$ enough?

However we will also have equations of the form

$$\rho(a_0) + \sum_{n > \underline{0}} V_{(n,0)} \rho(a_n) V_{(n,0)}^* = 0$$

which magically transform into

$$\rho(a_0)(I - V_{\mathbf{1}} V_{\mathbf{1}}^*) = 0.$$

From this we get that $a_0 \perp \ker \alpha_{\mathbf{1}}$.

From this we also get that $\alpha_{(0,n)}(a_0) \perp \ker \alpha_{\mathbf{1}}$ for all $n > 0$.

This happens because $\rho \alpha_{\mathbf{2}}(a) = V_{\mathbf{2}}^* \rho(a) V_{\mathbf{2}}$.

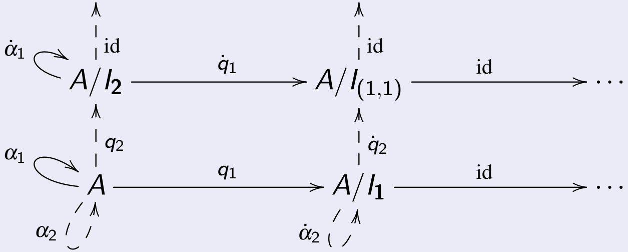
So we need the ideal $I_{\mathbf{1}} = \cap_n \alpha_{\mathbf{2}}^{-n}(\ker \alpha_{\mathbf{1}}^{\perp})$ instead of $\cap_n \alpha_{\mathbf{2}}^{-n}(I)$.

And of course its symmetrical $I_{\mathbf{2}}$.

Correct tail

$I_{(1,1)} = (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp} \quad I_{\mathbf{1}} = \cap_n \alpha_{\mathbf{2}}^{-n}(\ker \alpha_{\mathbf{1}}^{\perp}) \quad I_{\mathbf{2}} = \cap_n \alpha_{\mathbf{1}}^{-n}(\ker \alpha_{\mathbf{2}}^{\perp})$.

Then define $\beta_{\mathbf{1}}$ and $\beta_{\mathbf{2}}$ by



with $\dot{\alpha}_{\mathbf{1}} q_{\mathbf{2}} = q_{\mathbf{1}} \alpha_{\mathbf{2}}$ and $\dot{q}_{\mathbf{2}} q_{\mathbf{1}} = q_{(1,1)}$ (plus the symmetrical ones).

Then $\beta_{\mathbf{1}}$ and $\beta_{\mathbf{2}}$ generalise the $n = 1$ case.

It is not immediate but they are commuting and injective.

III. General construction

For $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_+^n$, define

$$\mathrm{supp}(\underline{x}) = \{\mathbf{i} : x_i \neq 0\} \text{ and } \underline{x}^\perp = \{\underline{y} \in \mathbb{Z}_+^n : \mathrm{supp}(\underline{y}) \cap \mathrm{supp}(\underline{x}) = \emptyset\}$$

and let the ideals

$$I_{\underline{x}} = \textstyle\bigcap_{\underline{y} \in \underline{x}^\perp} \alpha_{\underline{y}}^{-1}\Big(\big(\bigcap_{\mathbf{i} \in \mathrm{supp}(\underline{x})} \ker \alpha_{\mathbf{i}} \big)^\perp \Big).$$

Let $B_{\underline{x}} = A / I_{\underline{x}}$ and on the C*-algebra

$$B = \textstyle\sum_{\underline{x} \in \mathbb{Z}_+^n}^{\oplus} B_{\underline{x}}$$

define the $*$-endomorphisms

$$\beta_{\mathbf{i}}(q_{\underline{x}}(a) \otimes e_{\underline{x}}) = \begin{cases} q_{\underline{x}} \alpha_{\mathbf{i}}(a) \otimes e_{\underline{x}} + q_{\underline{x}+\mathbf{i}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \underline{x}^\perp, \\ q_{\underline{x}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \mathrm{supp}(\underline{x}). \end{cases}$$

Then the $\beta_{\mathbf{i}}$ commute and are injective (this is not trivial).

III. C*-envelope

Theorem (Davidson-Fuller-K. 2014)

Let $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ *be a semigroup action and define the Nc scp* $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$*. Apply the constructions:*

1. *dilate* α *to an injective system by adding a tail;*
2. *use the direct limit to extend it to* $\beta \colon \mathbb{Z}^n \to \mathrm{Aut}(B)$*.*

Then the C-envelope of* $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ *is strong Morita equivalent to* $B \rtimes_\beta \mathbb{Z}^n$*.*

Remarks

1. The C*-envelope is defined by a co-universal property.
2. This was one of the challenging points in the proof.

What about the structure of the C*-envelope?

Can we identify the C*-envelope by C*-algebraic relations?

III. Towards a Cuntz algebra

Recall

For $n = 2$ we arrived to the equalities

1. $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*) = 0$;
2. $a(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$;
3. $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$;

subject to a. Then we used the solutions/ideals to produce the tail. This appears to be more than an innocent coincidence!

The Cuntz-Nica-Pimsner algebra for $n = 2$ case

It is the universal C*-algebra such that: (a) $V_{\mathbf{i}}$ are doubly commuting isometries; (b) $aV_{\mathbf{i}} = V_{\mathbf{i}} \alpha_{\mathbf{i}}(a)$; and (c) we have

c.1 $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*) = 0$ for all $a \in \cap_n \alpha_{\mathbf{2}}^{-n}(\ker \alpha_{\mathbf{1}}^{\perp})$;

c.2 $a(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$ for all $a \in \cap_n \alpha_{\mathbf{1}}^{-n}(\ker \alpha_{\mathbf{2}}^{\perp})$;

c.3 $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$ for all $a \in (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$.

III. The Cuntz-Nica-Pimsner algebra

Definition (Davidson-Fuller-K. 2014)

The *Cuntz-Nica-Pimsner algebra* $\mathcal{N}\mathcal{O}(A,\alpha)$ *of* $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ is the universal C*-algebra generated by A and $V_{\mathbf{i}}$ so that:

1. $V_{\mathbf{i}}$ are commuting isometries;
2. $aV_{\mathbf{i}} = V_{\mathbf{i}}\alpha_{\mathbf{i}}(a)$; and
3. $a \cdot \prod_{\mathbf{i}\in \mathrm{supp}(\underline{x})}(I - V_{\mathbf{i}}V_{\mathbf{i}}^*) = 0$ for $a \in \bigcap_{\underline{y}\in\underline{x}^\perp} \alpha_{\underline{y}}^{-1}\Big(\big(\bigcap_{\mathbf{i}\in\mathrm{supp}(\underline{x})}\ker\alpha_{\mathbf{i}}\big)^\perp\Big)$.

Corollary (Davidson-Fuller-K. 2014)

1. *The C*-envelope of* $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ *is* $\mathcal{N}\mathcal{O}(A,\alpha)$.
2. *For* $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ *there exists a dilation* $\beta \colon \mathbb{Z}^n \to \mathrm{Aut}(B)$ *such that* $\mathcal{N}\mathcal{O}(A,\alpha) \overset{\mathrm{sMe}}{\sim} B \rtimes_\beta \mathbb{Z}^n$.

III. Simplicity

Theorem (Davidson-Fuller-K. 2014)

$$\begin{array}{ccc}
\alpha : \mathbb{Z}_+^n \to \mathrm{End}(A) & \longrightarrow & \mathcal{N}\mathcal{O}(A,\alpha) \simeq \mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n) \\
\text{dilation} \Big\downarrow & & \text{strong} \Big\} \text{Morita equivalent} \\
\beta : \mathbb{Z}^n \to \mathrm{Aut}(B) & \longrightarrow & B \rtimes_\beta \mathbb{Z}^n
\end{array}$$

Corollary (Corollary Davidson-Fuller-K. 2014)

Let $A = C(X)$ and let $\phi_s : X \to X$ related to $\alpha_s : X \to X$. TFAE:

1. *(A,α) is minimal and $\{x \in X \mid \phi_s(x) \neq \phi_r(x)\}^o = \emptyset$ for all $s, r \in \mathbb{Z}_+^n$ (top. free);*
2. *(B,β) is minimal and topologically free;*
3. *$B \rtimes_\beta \mathbb{Z}$ is simple;*
4. *$\mathcal{N}\mathcal{O}_{(A,\alpha)}$ is simple.*

III. Exactness/Nuclearity

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (with involution)

Universal C*-algebra generated by

$$V^n a, \text{ with } a \in A,\ n \in \mathbb{Z}_+,$$

such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for } a \in \ker\alpha^\perp := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Theorem (Katsura 2004)

1. $\mathscr{O}(A,\alpha)$ *is exact if and only if* A *is exact.*

2. $\mathscr{O}(A,\alpha)$ *is nuclear if and only if: (a)* $A/\ker\alpha^\perp$ *is nuclear; and (b) the embedding* $\ker\alpha^\perp \hookrightarrow \mathrm{C}^*(V_n a V_n^* \mid a \in A, n \in \mathbb{N})$ *is nuclear.*

3. If A *is nuclear then* $\mathscr{O}(A,\alpha)$ *is nuclear. The converse is not true.*

III. Exactness/Nuclearity

Theorem (K. 2014)

$\mathcal{N}\mathcal{O}(A,\alpha)$ is exact if and only if A is exact.

Theorem (K. 2014)

Let $\beta\colon \mathbb{Z}^n \to \mathrm{Aut}(B)$ be the automorphic dilation of $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$. TFAE:

1. *the embeddings $A, A/I_s \hookrightarrow B$ are nuclear for all $s \in \mathbb{Z}_+^n$;*
2. *B is nuclear;*
3. *$B \rtimes_\beta \mathbb{Z}^n$ is nuclear;*
4. *$\mathcal{N}\mathcal{O}(A,\alpha)$ is nuclear.*

Proposition (K. 2014)

If A is nuclear or if $A \hookrightarrow \mathrm{C}^*(V_{n\mathbf{1}} a V_{n\mathbf{1}}^* \mid a \in A, n \in \mathbb{Z}_+)$ is nuclear then $\mathcal{N}\mathcal{O}(A,\alpha)$ is nuclear. The converse is not true.

IV. Remarks

Remarks on $\mathscr{N}\mathscr{T}(A,\alpha)$ (K. 2014)

1. There is a second variant, the Toeplitz-Nica-Pimsner algebra.

2. For this we get A is nuclear (resp. exact) if and only if $\mathscr{N}\mathscr{T}(A,\alpha)$ is nuclear (resp. exact).

KMS states (K. 2014)

3. The gauge action implements an action of $\mathbb{R}$ on the Nica-Pimsner algebras. We are able to identify all KMS states at finite temperature: for any $T < \infty$ there is exactly one $\text{KMS}_{1/T}$ state.

4. For $T = \infty$ the KMS states are the tracial states and there is no bijection (there might be more than one).

IV. Remarks

Remarks on simplicity

5. Recently there was a major progress in simplicity of C*-crossed product (reduced) by Kalantar-Kennedy 2014. They show that it is equivalent to topological freeness of the group action on a boundary.

6. With Ken and Adam we are working towards formulating this property for semigroups and showing its stability under the automorphic dilation.

Remarks on product systems

7. Both $\mathscr{N}\mathscr{T}(A, \alpha)$ and $\mathscr{N}\mathscr{O}(A, \alpha)$ are examples of C*-algebras associated to product systems.

8. A gauge invariance uniqueness theorem for general Toeplitz-Nica-Pimsner algebras is easy to obtain by our methods.

9. We believe that the same is true for the Cuntz-Nica-Pimsner algebras.

Thank You