Classification of C*-algebras

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Classifying C*-algebras

The C*-algebra classification theorem

Let A, B be simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras which satisfy the UCT. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Regarded as the C*-analogue of the Connes–Haagerup classification of injective factors.

Plan:

- The hypotheses
- Examples of classifiable C*-algebras
- The invariant
- Glimpse of our proof

$\mathcal{Z}$-stability

Recall:

McDuff's theorem

A II_1 factor M is $\mathcal{R}$-stable (it satisfies $M \cong M \bar{\otimes} \mathcal{R}$) iff $M_n(\mathbb{C})$ embeds into $M^\omega \cap M'$ (for some/any $n > 1$).

For a II_1 factor:

- A tensorial copy of $\mathcal{R}$ provides useful space.
- Is characterized by a richness of the central sequence algebra.

In C*-algebras, a rich central sequence algebra and tensorial space are equally useful. However, an appropriate object analogous to $\mathcal{R}$ is more elusive.

$\mathcal{Z}$-stability

The most direct analogue to $\mathcal{R}$ is a UHF algebra M_{n^∞} (where n is a natural – or even supernatural – number).

However, M_{n^∞}-stability is a rather unnatural condition, as it imposes severe K-theoretic restrictions. (If $A \cong A \otimes M_{n^\infty}$ then every projection in A can be divided into n pairwise equivalent subprojections. E.g. M_{2^∞} is not M_{3^∞}-stable.)

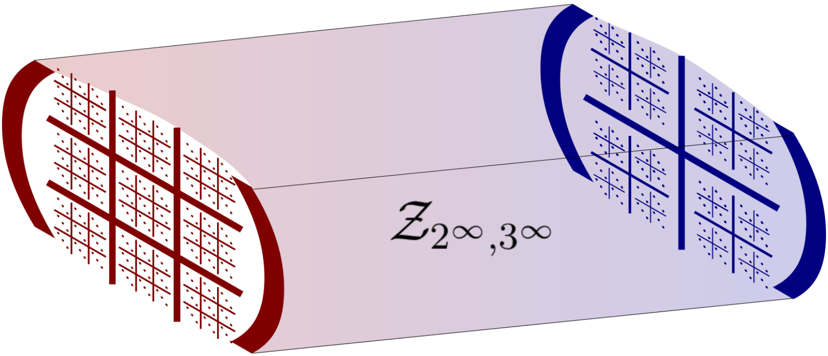
The *Jiang–Su algebra* $\mathcal{Z}$ can be thought of as a UHF algebra, with no non-trivial projections.

It is the universal *strongly self-absorbing* C*-algebra. (Some others are M_{n^∞}, $\mathcal{O}_2$, $\mathcal{O}_\infty$.)

$\mathcal{Z}$-stability

The *Jiang–Su algebra* $\mathcal{Z}$ is an inductive limit of C*-algebras of the form

$$\mathcal{Z}_{n^\infty,m^\infty} := \left\{ f \in C([0,1], M_{n^\infty} \otimes M_{m^\infty}) : \begin{array}{l} f(0) \in M_{n^\infty} \otimes 1 \\ f(1) \in 1 \otimes M_{m^\infty} \end{array} \right\}.$$



McDuff characterization of $\mathcal{Z}$-stability (Dadarlat–Toms ’09)

A unital C*-algebra A satisfies $A \cong A \otimes \mathcal{Z}$ if and only if some subhomogeneous C*-algebra without characters embeds into $A_\omega \cap A'$.

Here,

$$A_\omega := \ell_\infty(\mathbb{N}, A)/\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\| = 0\}.$$

(Cf. other McDuff-type characterizations by Kirchberg ’04, Toms–Winter ’07.)

KK-theory and the Universal Coefficient Theorem (UCT)

Kasparov's *KK*-theory is a bivariant functor unifying (and generalizing) *K*-theory and *K*-homology.

It is important in C*-algebra classification and index theory.

The *Universal Coefficient Theorem* is an exact sequence that Rosenberg and Schochet found to hold among a large class of separable nuclear C*-algebras, with good permanence properties. It expresses *KK*-theory in terms of *K*-theory.

C*-algebras satisfying this exact sequence are said to *satisfy the UCT*.

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C*-algebras satisfying this exact sequence are said to *satisfy the UCT*.

Proposition

A separable nuclear C*-algebra A satisfies the UCT iff it is *KK*-equivalent to an abelian C*-algebra.

(*KK*-equivalence is defined in terms of *KK*-theory; it can be thought of as a very weak form of homotopy equivalence.)

Examples

Definition

The *classifiable class* consists of simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras which satisfy the UCT.

The C*-algebra classification theorem (restated)

Let A, B be in the classifiable class. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

What C*-algebras are in the classifiable class?

Examples: approximately subhomogeneous C*-algebras

Approximately subhomogeneous C*-algebras

A C*-algebra is *subhomogeneous* if there is a bound on the dimension of irreducible representations.

An *approximately subhomogeneous* C*-algebra is an inductive limit of subhomogeneous C*-algebras.

It has *slow dimension growth* if (topological dimension)/(matricial dimension) $\to 0$.

All simple approximately subhomogeneous C*-algebras with slow dimension growth are in the classifiable class ($\mathcal{Z}$-stability: Toms '11, Winter '12).

In fact, every C*-algebra in the classifiable class is of this form (Elliott '96 + classification).

Examples: group representations

If G is a nilpotent group and $\pi : G \to \mathcal{B}(\mathcal{H})$ is an irreducible representation then $C^*(\pi(G))$ is in the classifiable class.

Eckhardt–Gillaspy '16: UCT.
Eckhardt–Gillaspy–McKenney '19: $\mathcal{Z}$-stability.

Question

If G is virtually nilpotent and $\pi : G \to \mathcal{B}(\mathcal{H})$ is an irreducible representation, does $C^*(\pi(G))$ satisfy the UCT?

If so, then $C^*(\pi(G))$ is in the classifiable class.

Examples: dynamical systems

Let G be a countable amenable group, X a compact metrizable space, and $\alpha : G \curvearrowright X$ a free minimal action.

$C(X) \rtimes_\alpha G$ always satisfies the UCT (Tu '99) and is simple and separable. The challenge is to prove $\mathcal{Z}$-stability.

Examples: dynamical systems

Let G be a countable amenable group, X a compact metrizable space, and $\alpha : G \curvearrowright X$ a free minimal action.

$C(X) \rtimes_\alpha G$ is in the classifiable class in the following cases:

- $\dim(X) < \infty$ and G has locally subexponential growth (Kerr–Szabó '20, Downarowicz–Zhang '23).
- $\dim(X) < \infty$ and G is elementary amenable (Kerr–Naryshkin '21).
- X is the Cantor set, for generic actions α (Conley–Jackson–Kerr–Marks–Seward–Tucker-Drob '18).
- $G = \mathbb{Z}^d$ and the action has mean dimension zero (Elliott–Niu '17, Niu arXiv'19).

Examples: dynamical systems

Let G be a countable amenable group, X a compact metrizable space, and $\alpha : G \curvearrowright X$ a free minimal action.

Questions

1. Is $C(X) \rtimes_\alpha G$ always in the classifiable class for $\dim(X) < \infty$?
2. Is there a dynamical characterization of when $C(X) \rtimes_\alpha G$ is in the classifiable class? (Mean dimension zero? Small boundary property?)

Examples: crossed products (noncommutative dynamics)

Let G be a torsion-free countable amenable group, A in the classifiable class, and $\alpha : G \curvearrowright A$ an outer action.

If A has unique trace then $A \rtimes_\alpha G$ is classifiable (Sato '19, and under less restrictions on $T(A)$ by Gardella–Hirshberg arXiv'18).

The invariant

For a unital C*-algebra A, the Elliott invariant $\mathrm{Ell}(A)$ consists of:

- $K_0(A)$ (the Grothendieck group from homotopy classes of projections in matrix algebras over A),
- $K_1(A)$ (the Grothendieck group from homotopy classes of unitaries in matrix algebras over A),
- $T(A)$ (the set of tracial states on A),
- $\rho_A : T(A) \times K_0(A) \to \mathbb{R}$, $\rho_A(\tau, [p]) := \tau(p)$,
- $[1_A]_0 \in K_0(A)$, and
- $K_0(A)_+ := \{[p]_0 : p \in \bigcup_n M_n(A)\} \subseteq K_0(A)$ (this information is redundant for classifiable C*-algebras).

Classifying embeddings

Theorem (Carrión–Gabe–Schafhauser–T–White)

Let A be a separable exact C*-algebra which satisfies the UCT.

Let B be a separable $\mathcal{Z}$-stable C*-algebra with $T(B)$ compact and nonempty and with strict comparison with respect to traces.

Then the full nuclear *-homomorphisms from A to B (or B_∞) are classified up to approximate unitary equivalence by an augmented "total invariant" $\underline{K}T(\cdot)$ (richer than the Elliott invariant).

The hypothesis $T(B) \neq \emptyset$ can be dropped – but in this case the result is due to Phillips and Kirchberg.

Classifying embeddings

Theorem (Carrión–Gabe–Schafhauser–T–White)

Let A be separable exact UCT; B separable $\mathcal{Z}$-stable with $T(B)$ compact and with strict comparison. Then full nuclear *-homomorphisms $A \to B$ (or B_∞) are classified by $\underline{K}T(\cdot)$.

Classification means both:

- Uniqueness: given two such *-homomorphisms, if they agree on the invariant then they are approximately unitarily equivalent; and
- Existence: given a morphism of invariants, there is a *-homomorphism which realizes it.

The total invariant

Let A be a unital C*-algebra.

The total invariant $\underline{K}T(A)$ consists of K-theory and traces (as in the Elliott invariant), as well as:

- Total K-theory (a.k.a. K-theory with coefficients)
$$K_i(A; \mathbb{Z}_n) := K_i(A \otimes C_{\mathbb{Z}_n}), \quad n \in \mathbb{N}$$
where $C_{\mathbb{Z}_n}$ is a nuclear C*-algebra with $K_*(C_{\mathbb{Z}_n}) = \mathbb{Z}_n \oplus 0$,
- Hausdorffized unitary algebraic K-theory
$$\overline{K}_1^{\text{alg,u}} := \bigcup_n U(M_n(A)) / \bigcup_n \overline{\{uvu^*v^* : U \in U(M_n(A))\}},$$
- A number of maps relating these (and K-theory and traces).

The total invariant

Proposition

Let A, B be C*-algebras. Then any isomorphism $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ extends to an isomorphism $\underline{K}T(A) \to \underline{K}T(B)$.

Intertwining

The "Elliott intertwining argument" derives the C*-algebra classification theorem from the classification of embeddings.

The Intertwining Argument

Let A, B be C*-algebras. If there exist *-homomorphisms $\phi : A \to B$ and $\psi : B \to A$, such that:

- $\psi \circ \phi : A \to A$ is approximately unitarily equivalent to id_A, and
- $\phi \circ \psi : B \to B$ is approximately unitarily equivalent to id_B,

then $A \cong B$.

The trace-kernel extension

In our argument, we write B_∞ as an extension

$$0 \to J_B \to B_\infty \to B^\infty \to 0,$$

where

$$B^\infty := \ell_\infty(\mathbb{N}, B)/\{(b_n)_n : \lim_{n\to\omega} \sup_{\tau \in T(B)} \tau(b_n^* b_n) = 0\}.$$

Then

- B^∞ behaves much like a II_1 von Neumann algebra (Castillejos–Evington–T–White–Winter); in particular, we can classify nuclear maps into B^∞ via Connes' theorem.
- From there, it becomes a lifting problem, in which we employ KK-theory.