Actions of compact quantum groups and inclusions of C^*-algebras

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Aberdeen – 28 November 2014

Overview – results and perspectives

Cpct quantum gp $\mathbb{G} \overset{\alpha}{\curvearrowright} \mathscr{H}$, dim $d \rightsquigarrow \mathbb{G} \curvearrowright \mathcal{O}_d$, Cuntz algebra.

? Fixed point algebra $\mathcal{O}^\alpha \subseteq \mathcal{O}_d$?

Result: Under some conditions,

if $\mathbb{G}_1, \mathbb{G}_2$, "same representation theory", then $\mathcal{O}^{\alpha_1} \simeq \mathcal{O}^{\alpha_2}$

Key ingredient: Abstract C^*-isom. (classification theory).

Example: $\mathbb{G} = SU_q(2)$, $q \in (0,1)$ and α_q nat. rep. on $\mathbb{C}^2$.

$\mathcal{O}^{\alpha_q} = \mathcal{O}_\infty$ indep. of q, inside $\mathcal{O}_2$.

Algebraic case (CPZ '00): recover q from $\mathcal{O}^{\alpha_q} \subseteq \mathcal{O}_d$

Change pt of view: fixed alg. $\mathcal{O}^\alpha$, family inclusions $\mathcal{O}^\alpha \hookrightarrow \mathcal{O}_d$.

? Recover $\mathbb{G}$ from $\mathcal{O}^\alpha \hookrightarrow \mathcal{O}_d$?

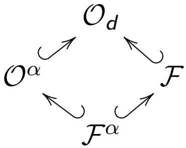
Overview – objects involved

Two actions on $\mathcal{O}_d$:

- $\underline{\alpha}\colon \mathbb{G} \curvearrowright \mathcal{O}_d$ induced from $\alpha\colon \mathbb{G} \curvearrowright \mathscr{H}$.
- "Gauge action" $\gamma\colon U(1) \curvearrowright \mathcal{O}_d$ induced from

$$\gamma_z(S_j) = zS_j.$$

Fixed point algebras $\mathcal{O}^\alpha$ and $\mathcal{F}$, respectively.



"Meaning" of those algebras?

- $\mathcal{O}_d$ contains $\mathscr{H}$, its tensor products $\mathscr{H}^{\otimes k}$ and duals...
  ... and thus all endomorphisms $\mathscr{H}^{\otimes k} \to \mathscr{H}^{\otimes \ell}$!
- $\mathcal{O}^\alpha$: keeps only intertwiners $\mathscr{H}^{\otimes k} \to \mathscr{H}^{\otimes \ell}$.
- $\mathcal{F}^\alpha = \lim_\to \mathrm{Mor}_{\mathbb{G}}(\mathscr{H}^{\otimes \ell}, \mathscr{H}^{\otimes \ell})$, AF algebra.

Outline

Compact Quantum Groups (CQG): definition

Informally, a CQG is a compact NC space with a group law.

Formally, unital C^*-algebra A with coproduct $\Delta\colon A \to A \otimes A$, and certain properties. Denote $\mathbb{G} = (A, \Delta)$, $A = C(\mathbb{G})$. Def

- Example: let q, real number with $-1 \leqslant q \leqslant 1$, $q \neq 0$,

Definition (Woronowicz – 1987)

$SU_q(2)$: universal C^*-algebra generated by the entries of U

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

with conditions $U^*U = 1 = UU^*$ and $\Delta(U) = U \otimes U$.

- For $q = 1$: A comm., recover $SU(2)$, a and c functions.
- Deformations $SU_q(N)$ of $SU(N)$ exist for $0 < q \leqslant 1$.

Compact quantum groups: representations

Notion of *representation*: $\alpha\colon \mathscr{H} \to \mathscr{H} \otimes A$... ▸ Def. rep.

... under some conditions.

For $SU_q(2)$, the *natural representation* $\alpha = (1)$ acts on $\mathbb{C}^2$:

$$\alpha_g(S_1) = S_1 \otimes a(g) + S_2 \otimes c(g)$$
$$\alpha_g(S_2) = -S_1 \otimes qc^*(g) + S_2 \otimes a^*(g).$$

For $q = 1$, a and c are $\mathbb{C}$-valued functions on $SU(2)$. ▸ Action $\mathcal{O}_d$

For $\mathbb{G} = (A, \Delta)$, general CQG:

- Notions of *unitary* and *irreducible* rep. for $\mathbb{G}$.
- *Direct sums* and *tensor products* of unit. rep. of $\mathbb{G}$.
- Schur lemma applies.
- Unit. rep. α decomposes in sum of finite dim. unit. irrep.

Example: the irreps of $SU_q(2)$ are (n) with $n \in \mathbb{N}$ and

$$(k) \otimes (k') = (|k - k'|) \oplus (|k - k'| + 2) \oplus \cdots \oplus (k + k').$$

Clebsch-Gordan relations: no difference with $SU(2)$!

Fusion semiring and $\mathcal{R}^+$-isomorphism

Definition ($\mathcal{R}^+$-isomorphism, Banica – 1999)

Given a CQG $\mathbb{G}$, denote $\mathcal{R}^+(\mathbb{G})$ the *fusion semiring* of finite dim. rep. of $\mathbb{G}$, endowed with

- the direct sum $\bigoplus$,
- the tensor product $\bigotimes$.

If $\mathcal{R}^+(\mathbb{G}_1)$ and $\mathcal{R}^+(\mathbb{G}_2)$ are isomorphic as semirings,

then $\mathbb{G}_1$ and $\mathbb{G}_2$ are $\mathcal{R}^+$-isomorphic (same *fusion rules*).

For representations, no differences between $\mathbb{G}_1$ and $\mathbb{G}_2$!

Examples:

- $SU(2)$ and $SU_q(2)$ are $\mathcal{R}^+$-isomorphic.
- More generally, $SU(N)$ and $SU_q(N)$ are $\mathcal{R}^+$-isomorphic.

▸ Stability results

Fixed point algebras from CQG

Theorem (Konishi, Nagisa & Watatani – 1992)

Given $\mathbb{G}$, CQG and a unit. rep. $\alpha = (\alpha_{ji}) \in M_d(A)$ on $\mathbb{C}^d$,

$$\underline{\alpha}(S_i) = \sum_{j=1}^{d} S_j \otimes \alpha_{ji}$$

induces $\underline{\alpha}\colon \mathcal{O}_d \to \mathcal{O}_d \otimes C(\mathbb{G})$, action on $\mathcal{O}_d$.

Reminder

Definition (fixed point algebra)

The *fixed point algebra* $\mathcal{O}^\alpha$ of $\underline{\alpha}$ is

$$\mathcal{O}^\alpha = \left\{ T \in \mathcal{O}_d \middle| \underline{\alpha}(T) = T \otimes 1 \right\}.$$

- The gauge action γ on $\mathcal{O}_d$ restricts to $\mathcal{O}^\alpha$.
- Consider the *spectral subspaces* (for γ):
  $$(\mathcal{O}^\alpha)^{(k)} = \{ T \in \mathcal{O}^\alpha : \gamma_z(T) = z^k T \} \text{ where } k \in \mathbb{Z}.$$
- For $k = 0$, set $\mathcal{F}^\alpha := (\mathcal{O}^\alpha)^{(0)}$ – gauge inv. subalgebra.

Classification theory

Definition : Kirchberg algebra

A *Kirchberg algebra* is a C^*-algebra A which is

1. Purely Infinite (PI) 2. simple 3. nuclear 4. separable

First two properties: $\forall a \neq 0 \in A$, $\forall \varepsilon > 0$, $\exists u, v \in A$ s.t.

$$\|uav - 1\| \leqslant \varepsilon.$$

Theorem (Kirchberg & Phillips – 1994, 2000)

Let A and B, be unital Kirchberg algebras in $\mathscr{N}$. Def. UCT

$A \simeq B$ as C^*-algebras iff Abelian groups isomorphisms

$$\alpha_0 \colon K_0(A) \to K_0(B) \qquad \alpha_1 \colon K_1(A) \to K_1(B)$$

with $\alpha_0([1_A]) = [1_B]$ in K_0.

Proposition (Kirchberg, Phillips – 1994, 2000)

Given Kirchberg algebras A, B, every element of $KK(A, B)$ lifts to a $*$-hom. from A to $B \otimes \mathbb{K}$.

Conditions and stability result

Given $\mathbb{G}$, CQG, and α, unitary representation of $\mathbb{G}$,
$\mathcal{T}_\alpha$, union of (classes of) irrep. contained in $\alpha^{\otimes \ell}$ for $\ell \geqslant 0$.

(C1) If $\beta \in \mathcal{T}_\alpha$, $\exists \beta' \in \mathcal{T}_\alpha$ s.t. $\beta \otimes \beta'$ contains trivial rep. ε.

(C2) There are integers N, k_0 s.t.

- $\alpha^{\otimes N}$ is contained in $\alpha^{\otimes (N+k_0)}$ and
- $\forall k, \ell \in \mathbb{N}$ with $0 < k < k_0$, $\mathrm{Mor}_{\mathbb{G}}(\alpha^{\otimes \ell}, \alpha^{\otimes (\ell+k)}) = \{0\}$.

Theorem (G. – 2014)

If α satisfies (C1) and (C2), then $\mathcal{O}^\alpha$ is Kirchberg, in $\mathcal{N}$ and

$$\mathcal{O}^\alpha \text{ only depends on } \mathbb{G} \textit{ via } \mathcal{R}^+(\mathbb{G}).$$

Last property: for isom. $\Phi\colon \mathcal{R}^+(\mathbb{G}_1) \to \mathcal{R}^+(\mathbb{G}_2)$ Reminder,

$$\mathcal{O}^\alpha \text{ and } \mathcal{O}^{\Phi(\alpha)} \text{ are (abstractly) isomorphic as } C^*\text{-algebras.}$$

Free actions

Definition (Ellwood – 2000)

An action $\underline{\alpha}\colon A \to A \otimes C(\mathbb{G})$ on a C^*-alg. A is *free* if

$$\overline{\underline{\alpha}(A)(A \otimes 1)} = A \otimes C(\mathbb{G}).$$

Theorem (G. – 2014)

Assume that

- $\mathbb{G}$ is a semisimple cpct Lie group (or a $\mathcal{R}^+$-def. thereof)
- α is (a $\mathcal{R}^+$-def. of) a faithful rep. of $\mathbb{G}$

then the induced action $\underline{\alpha}\colon \mathcal{O}_d \to \mathcal{O}_d \otimes C(\mathbb{G})$ is free.

- Actually, suffices that $\mathcal{T}_\alpha$ contains all irreps. of $\mathbb{G}$.
- Equivalently (De Commer & Yamashita – 2013), there is a Morita equivalence between $\mathcal{O}^\alpha$ and $\mathcal{O}_d \rtimes \mathbb{G}$.
- Combining both theorems: source of noncommutative principal bundles!

Condition (C1)

(C1) If $\beta \in \mathcal{T}_\alpha$, $\exists \beta' \in \mathcal{T}_\alpha$ s.t. $\beta \otimes \beta'$ contains trivial rep. ε.

Proposition

If $\mathcal{R}^+(\mathbb{G}) \simeq \mathcal{R}^+(G)$ for

- some semisimple Lie group G or
- some finite group G,

then (C1) is satisfied for any irreducible rep. α.

Proof: in both cases, we prove that the trivial rep. ε appears in β^L for some L.

- Assume that $\mathbb{G}$ is a *group* in one of the above classes.
- if β is a representation of dim. d, consider $\delta := \bigwedge^d \beta$.
  - for semisimple $\mathbb{G}$, unique dim. 1 rep. thus δ trivial;
  - if $\mathbb{G}$ is finite, for K large enough, δ^K is trivial.

Condition (C2)

(C2) There are integers N, k_0 s.t.

- $\alpha^{\otimes N}$ is contained in $\alpha^{\otimes(N+k_0)}$ and
- $\forall k, \ell \in \mathbb{N}$ with $0 < k < k_0$, $\mathrm{Mor}_{\mathbb{G}}(\alpha^{\otimes \ell}, \alpha^{\otimes(\ell+k)}) = \{0\}$.

Alternative statement:

(C2') $\forall k \in \mathbb{N} \setminus \{0\}$, if $(\mathcal{O}^{\alpha})^{(k)} \neq \{0\}$, then it contains an isometry. Moreover, not all $(\mathcal{O}^{\alpha})^{(k)}$ are trivial.

- Condition (C2) gives a "generating isometry" $\mathbb{V}$.
- (C2) difficult to check, in general.
- Examples:
  - $\alpha = \varepsilon \oplus t$, ε trivial representation $\rightsquigarrow k_0 = 1$.
  - For $\mathbb{G} = SU_q(d)$ and $\alpha = \nu$ $\rightsquigarrow k_0 = d$.

To prove that (C2) $\Leftrightarrow$ (C2'):

- Inclusion of $\mathscr{H}$, $\mathscr{H}^{\otimes k}$ inside $\mathcal{O}^{\alpha}$.
- *Fourier coefficients maps* $m_k \colon \mathcal{O}^{\alpha} \to (\mathcal{O}^{\alpha})^{(k)}$ for $k \in \mathbb{Z}$:

$$m_k(T) := \int_{S^1} z^{-k} \gamma_z(T) dz.$$

Natural representation ν of $SU(d)$: chain group

Consider $\mathbb{G} = SU_q(d)$ and its natural rep. $\alpha = \nu$ on $\mathbb{C}^d$.

- $\mathcal{R}^+(\mathbb{G}) = \mathcal{R}^+(SU(d))$ and $G = SU(d)$ semisimple, so (C1) is satisfied.
- By def. of $SU(d)$, $\nu^d = \varepsilon \oplus t$, hence α^d satisfies (C2).
- For ν, the property is less obvious...

Can we compare $\mathcal{O}^\alpha$ and $\mathcal{O}^{\alpha^N}$?

Definition (Chain group, Baumgärtel & Lledó – 2004)

Chain group $\mathfrak{C}(\mathbb{G})$: equivalence classes $[t]$ of irreps

- under $t \sim t'$ if there is a chain of irreps. $\tau_1, \dots, \tau_n$ s.t. both t and t' appear in $\tau_1 \otimes \cdots \otimes \tau_n$,
- product structure $[t][t'] = [t \otimes t']$.

This actually defines a *group* structure on $\mathfrak{C}(\mathbb{G})$. Proof

Identity of $\mathfrak{C}(\mathbb{G})$: given by trivial rep. ε: $[\varepsilon] = 1_{\mathfrak{C}(\mathbb{G})}$.

Inclusions of fixed points: statement

If $\mathcal{R}^+(\mathbb{G}_1) \simeq \mathcal{R}^+(\mathbb{G}_2)$, then $\mathfrak{C}(\mathbb{G}_1) \simeq \mathfrak{C}(\mathbb{G}_2)$.

Theorem (Baumgärtel & Lledó – 2004)

For compact (ordinary) groups G, $\mathfrak{C}(G)$ is the character group of the center $Z(G)$. Explicit isomorphism:

$$[t] \mapsto t \restriction Z(G).$$

E.g. $G = SU(d)$, get $\mathfrak{C}(G) = \mathbb{Z}/d\,\mathbb{Z}$ ("Grading of irreps").

Proposition (G. – 2014)

Given a rep. α of a CQG $\mathbb{G}$,

1. For any $M \geqslant 1$, there is an injective map $\mathcal{O}^{\alpha^M} \to \mathcal{O}^\alpha$.
2. If all irrep. in α have the same class $[\alpha] \neq e$ in $\mathfrak{C}(\mathbb{G})$, for the order M of $[\alpha]$ in $\mathfrak{C}(\mathbb{G})$, $\mathcal{O}^{\alpha^M} \simeq \mathcal{O}^\alpha$.

Point 2 clearly applies to $\mathbb{G} = SU_q(d)$ and α nat. rep. ▸ More

$SU_q(2)$ and $\alpha = (1)$: explicit identification

Canonical endomorphism of $\mathcal{O}_2$: $\rho(T) = S_1 T S_1^* + S_2 T S_2^*$.

Theorem (Marciniak – 1998)

$\mathcal{O}^\alpha$ is the smallest C^*-subalgebra of $\mathcal{O}_2$ which contains $\mathbb{V}_q := \frac{1}{\sqrt{1+q^2}}(S_1 S_2 - q S_2 S_1)$ and is stable under ρ.

$\mathbb{V}_q$ is the "generating isometry" and satisfies the relations:

$$\mathbb{V}_q^* \mathbb{V}_q = 1 \qquad \mathbb{V}_q^* \rho(\mathbb{V}_q) = -\frac{1}{q+q^{-1}} 1.$$

Thus, using ρ, the projections $p_n := \rho^n(\mathbb{V}_q \mathbb{V}_q^*)$ satisfy:

$$p_n p_m = p_m p_n \qquad p_n p_k p_n = \tau\, p_n$$

where $|n-m| > 1$, $|n-k| = 1$ and $\tau = (q+q^{-1})^{-2}$ (Temperley-Lieb relations).

Theorem (G. – 2014)

The K-theory of $\mathcal{O}^\alpha$ is $K_0(\mathcal{O}^\alpha) = \mathbb{Z}$, $K_1(\mathcal{O}^\alpha) = 0$.

Moreover, $[1_{\mathcal{O}^\alpha}]_0 = 1$ therefore $\mathcal{O}^\alpha \simeq \mathcal{O}_\infty$.

Natural representation α_q of $SU_q(d)$: results

From now on: $\mathbb{G} = SU_q(d)$ and α_q, natural rep. on $\mathbb{C}^d$.

Corollary

$\mathcal{O}^{\alpha_1}$ and $\mathcal{O}^{\alpha_q}$ are (abstractly) isomorphic as C^*-algebras.

- In other words, the construction $\mathcal{O}^{\alpha}$ doesn't "feel" the deformation parameter q.

Explicit description using ρ, canonical endomorphism of $\mathcal{O}_d$:

Theorem (Paolucci – 1997)

- Embedding θ of the braid group B_∞ in $\mathcal{O}_d$; ▸ Details
- $\mathcal{O}^{\alpha}$ contains the *q-antisymmetric tensor* $\mathbb{V}_q$;
- $\mathcal{O}^{\alpha}$ is the smallest C^*-subalgebra of $\mathcal{O}_d$ s.t.
  - $\theta(g)$, for any $g \in B_\infty$, and $\mathbb{V}_q$ are in $\mathcal{O}^{\alpha}$;
  - $\mathcal{O}^{\alpha}$ is stable under ρ (with $\rho(T) := \sum S_j T S_j^*$).

Comparison algebraic – C^*-algebraic

Case $\mathbb{G} = SU_q(d)$ and α_q, nat. rep.

In the algebraic setting:

Def. $*$-Hopf alg.

Theorem (Carey, Paolucci & Zhang – 2000)

If $\mathscr{A}$, $*$-Hopf algebra with generators u_{ij} and $\overline{u_{ij}} = (u_{ij})^*$ and

- $\mathcal{O}_d^{alg} \subseteq \mathcal{O}_d$ possesses an action $\underline{\alpha} \colon \mathcal{O}_d^{alg} \to \mathcal{O}_d^{alg} \otimes \mathscr{A}$:

$$\underline{\alpha}(S_i) = \sum_{j=1}^{d} S_j \otimes u_{ji} \qquad \underline{\alpha}(S_i^*) = \sum_{j=1}^{d} S_j^* \otimes \overline{u_{ji}},$$

- the algebra $\mathcal{O}^{\alpha,alg}$ is generated by $\mathbb{V}_q$ and $\theta(g)$ for $g \in B_\infty$

then $\mathscr{A}$ is $SU_q(d)$.

- In other words, we can recover q in the algebraic setting...
- ... but for C^*-algebras, $\mathcal{O}^\alpha$ "doesn't feel" the q.

Inclusions and recovery of CQG

Case $\mathbb{G} = SU_q(d)$ and α_q, nat. rep. Why such difference?

- Carey, Paolucci & Zhang consider the full inclusion...
- ... we consider only the fixed point algebra $\mathcal{O}^\alpha$.

New problem: classify inclusions $\mathcal{O}_\infty \hookrightarrow \mathcal{O}_2$!

- KK-theory not helping: $KK(\mathcal{O}_\infty, \mathcal{O}_2) = 0$ (UCT).
- For irreducible representations,
  $\mathbb{V}^* \rho(\mathbb{V}) \in (\mathscr{H}^*)^N \mathscr{H}^{N+1} \mathscr{H}^*$ thus it is a scalar.

For $SU_q(2)$, $\mathbb{V}^* \rho(\mathbb{V}) = -(q + q^{-1})^{-1} 1$ recovers q.

But if we consider free orthogonal quantum groups?

- Alternative approach: use von Neumann setting!

Theorem (Enock & Nest – 1996)

If $M_0 \subseteq M_1$ is a depth 2 irreducible inclusion of factors with a conditional expectation $\mathbb{E}$ from M_1 to M_0
then there is a CQG $\mathbb{G}$ and an action α of $\mathbb{G}$ s.t. $M_0 = M_1^\alpha$.

Idea: find sufficient cond. on $\mathcal{O}^\alpha \hookrightarrow \mathcal{O}_d$ to apply theorem.

From C^*-algebras to factors – gauge actions

Theorem

Fix an $\mathbb{R}$-action τ on a C^*-algebra A. Given $\beta \in \mathbb{R}$, let KMS_β be the set of τ-KMS states at value β,

$$\omega \text{ is extremal in } \mathrm{KMS}_\beta \text{ iff } \omega \text{ is a factor state.}$$

- KMS-states satisfy τ-invariance: $\omega(\tau_t(a)) = a$.
- We consider gauge actions $\rightsquigarrow$ charact. by trace on $A^{(0)}$!
- For $\mathcal{O}_d$, $\mathcal{F}$ UHF alg. d^∞ thus unique trace.
  Weak closure of $\mathcal{O}_d$ for $GNS(\phi)$: factor.

Proposition (G.)

For $\mathbb{G} = SU_q(2)$ and $\alpha = \nu$, given $\beta \neq 1$,
there is at most one KMS_β state ω.

Consequence:
from the inclusion $\mathcal{O}_\infty \hookrightarrow \mathcal{O}_2$, we get a subfactor system.

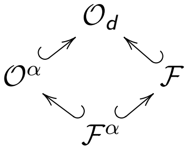
Towards the proof: fixed point algebras

Two actions on $\mathcal{O}_d$:

- $\underline{\alpha}\colon \mathbb{G} \curvearrowright \mathcal{O}_d$ induced from $\alpha\colon \mathbb{G} \curvearrowright \mathscr{H}$.
- "Gauge action" $\gamma\colon U(1) \curvearrowright \mathcal{O}_d$ induced from

$$\gamma_z(S_j) = zS_j.$$

Fixed point algebras $\mathcal{O}^\alpha$ and $\mathcal{F}$, respectively.



"Meaning" of those algebras?

- $\mathcal{O}_d$ contains $\mathscr{H}$, its tensor products $\mathscr{H}^{\otimes k}$ and duals...
  ... and thus all endomorphisms $\mathscr{H}^{\otimes k} \to \mathscr{H}^{\otimes \ell}$!
- $\mathcal{O}^\alpha$: keeps only intertwiners $\mathscr{H}^{\otimes k} \to \mathscr{H}^{\otimes \ell}$.
- $\mathcal{F}^\alpha = \lim_\to \mathrm{Mor}_\mathbb{G}(\mathscr{H}^{\otimes \ell}, \mathscr{H}^{\otimes \ell})$, AF algebra.

Stability result: parts of the proof

Theorem (G. – 2014)

If α satisfies (C1) and (C2), then $\mathcal{O}^\alpha$ is Kirchberg, in $\mathcal{N}$ and

$$\mathcal{O}^\alpha \text{ only depends on } \mathbb{G} \textit{ via } \mathcal{R}^+(\mathbb{G}).$$

Parts of the proof:

1. Prove that $\mathcal{O}^\alpha$ is PI and simple. Argument:
   Identify $\mathcal{O}^\alpha$ with a crossed product $\mathcal{F}^\alpha \rtimes \mathbb{N}$ and

Theorem (Dykema & Rørdam – 1998)

Given $A \neq \mathbb{C}$ and σ injective endomorphism of A, if

(i) $\forall a > 0$, $\exists b \in A$, $\exists L > 0$ s.t. $bab^* = \sigma^L(1)$ and

(ii) $\overline{\sigma}^m$ is outer for all $m \in \mathbb{N}$

then $A \rtimes_\sigma \mathbb{N}$ is PI and simple.

2. Compute the K-theory of $\mathcal{O}^\alpha$. Argument:
   $\mathcal{O}^\alpha \simeq \mathcal{F}^\alpha \rtimes \mathbb{N}$ and Pimsner-Voiculescu exact sequence.

Purely infinite and simple algebra

Conditions of previous theorem (Dykema & Rørdam – 1998) in our case:

(i) $\forall T > 0$, $\exists z \in \mathcal{F}^\alpha$, $\exists L > 0$ s.t. $zTz^* = \sigma^L(1)$.

(ii) $\overline{\sigma}^m$ is outer for all $m \in \mathbb{N}$.

Steps to prove that $\mathcal{O}^\alpha$ is PI and simple:

1. Consider the crossed product $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$
   for σ defined by $\sigma(T) := \mathbb{V} T \mathbb{V}^*$, $\mathbb{V}$ "generating isom."
2. Check hypothesis (i).
3. Check hypothesis (ii).
4. Prove that $\mathcal{O}^\alpha \simeq \mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$.

Checking (i): projections in $\mathcal{F}^{\alpha,\ell}$

(C1) If $\beta \in \mathcal{T}_\alpha$, $\exists \beta' \in \mathcal{T}_\alpha$ s.t. $\beta \otimes \beta'$ contains trivial rep. ε.

Proposition

Let $P \in \mathcal{F}^{\alpha,\ell}$ be a nonzero proj., $\exists L > 0$, $\exists u \in (\mathcal{O}^\alpha)^{(L)}$ s.t.

$$u^* P u = 1 \qquad u^* u = 1.$$

- P α-inv. $\rightsquigarrow$ Hilb. sp. $P\mathscr{H}^\ell =: \mathscr{K} \subseteq \mathscr{H}^\ell$ stable under α^ℓ.
- Decompose induced representation in $\mathcal{T}_\alpha$.
  WLOG, $\mathscr{K}$ equipped with $\beta \in \mathcal{T}_\alpha$.
- From (C1), $\exists q \in \mathbb{N}$ s.t. $\beta \otimes \alpha^q$ acting on $\mathscr{K} \otimes \mathscr{H}^q$ possess an invariant vector u.
- Since $u \in \mathscr{K} \otimes \mathscr{H}^q \subseteq \mathscr{H}^{\ell+q}$, satisfies $Pu = u$. It is $\mathbb{G}$-invariant thus $u \in \mathcal{O}^\alpha$.
- Up to normalisation, $u^* u = 1$ and hence

$$u^* P u = (Pu)^* P u = u^* u = 1.$$

Extension to any $a > 0$: alg. approx. and finite dim. alg. $\mathcal{F}^{\alpha,\ell}$.

Checking (ii): outer automorphism

Def. $\overline{\sigma}$ is gen. from σ on $\overline{\mathcal{F}^\alpha}$, limit of the inductive system

$$\mathcal{F}^\alpha \xrightarrow{\sigma} \mathcal{F}^\alpha \xrightarrow{\sigma} \mathcal{F}^\alpha \xrightarrow{\sigma} \cdots \to \overline{\mathcal{F}^\alpha}$$

Lemma

Under condition (C2),

for all $m \in \mathbb{N} \setminus \{0\}$, the automorphism $\overline{\sigma}^m$ is outer.

- $\sigma : \mathcal{F}^\alpha \to \mathcal{F}^\alpha$ extends to $\sigma : \mathcal{F} \to \mathcal{F}$ by $\sigma(T) := \mathbb{V} T \mathbb{V}^*$.
- There is a commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{F}^\alpha & \xrightarrow{\sigma} & \mathcal{F}^\alpha & \xrightarrow{\sigma} & \mathcal{F}^\alpha & \xrightarrow{\sigma} & \cdots & \longrightarrow & \overline{\mathcal{F}^\alpha} \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \overline{\varphi} \\
\mathcal{F} & \xrightarrow{\sigma} & \mathcal{F} & \xrightarrow{\sigma} & \mathcal{F} & \xrightarrow{\sigma} & \cdots & \longrightarrow & \overline{\mathcal{F}}.
\end{array}$$

- If $\overline{\sigma}^m$ is inner on $\overline{\mathcal{F}^\alpha}$, it must act trivially on $K_0(\overline{\mathcal{F}^\alpha})$.
- This leads to a contradiction when extended to $\overline{\mathcal{F}}$, where we can rely on the unique trace on the UHF algebra $\mathcal{F}$.

K-theory and fusion semiring

Proposition

We can identify $\mathcal{O}^\alpha$ with the crossed product $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ and using a Pimsner-Voiculescu-like sequence,

we can recover $K_*(\mathcal{O}^\alpha)$ from $K_*(\mathcal{F}^\alpha)$.

Computation of $K_*(\mathcal{O}^\alpha)$:

- $\mathcal{F}^\alpha$, AF-algebra: $\lim_\rightarrow \mathcal{F}^{\alpha,\ell} = \lim_\rightarrow \mathrm{Mor}_{\mathbb{G}}(\mathscr{H}^{\otimes \ell}, \mathscr{H}^{\otimes \ell})$.
- Describe $K_*(\mathcal{F}^{\alpha,\ell})$ *via* intertwiner interpretation.
- Continuity of K_* gives: $K_*(\mathcal{F}^\alpha) = \lim_\rightarrow K_*(\mathcal{F}^{\alpha,\ell})$.
  Description involves $\mathbb{Z}[\mathcal{T}_\alpha]\left[\frac{1}{\alpha}\right]$, constructed from $\mathcal{R}^+(\mathbb{G})$.
- From PV sequence, obtain:

$$K_0(\mathcal{O}^\alpha) = \mathrm{coker}(\mathrm{Id} - \sigma_*) \qquad K_1(\mathcal{O}^\alpha) = \ker(\mathrm{Id} - \sigma_*),$$

Conclusion

Our results:

- from a representation α of a compact quantum group $\mathbb{G}$...
- ...yield Kirchberg algebra $\mathcal{O}^\alpha$ depending only on $\mathcal{R}^+(\mathbb{G})$.
- The action of $\mathbb{G}$ on $\mathcal{O}^\alpha$ is free.

Perspectives: translate in factor setting and recover $\mathbb{G}$.

This is the end...

Thank you for your attention!

References:

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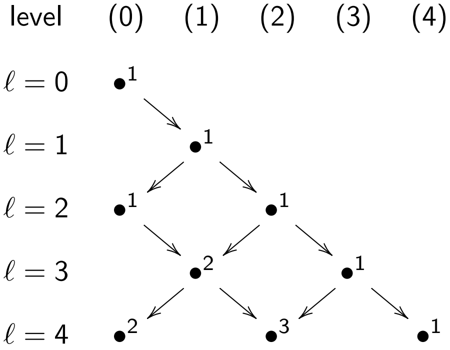
Ann. H. Poincaré **15** (2014) 5, pp 1013-1036

...

Example of Bratteli diagrams: $SU_q(2)$ and $\alpha = (1)$

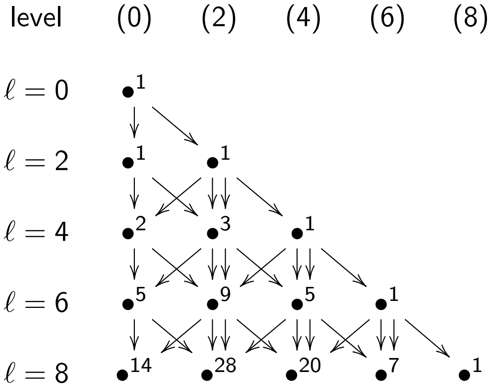
Example of identification using Bratteli diagrams:

- Tensor products: $(0) \otimes (1) = (1)$ and
  for $k > 0$, $(k) \otimes (1) = (k-1) \oplus (k+1)$.
- $\mathscr{H} \leftrightsquigarrow (1)$ and $\mathscr{H}^2 \leftrightsquigarrow (1)^2 = (0) \oplus (2)$.
- Then, $((0) \oplus (2)) \otimes (1) = (1) \oplus (1) \oplus (3) = 2.(1) \oplus (3)$.
- More generally: simple edges and addition of dimensions.



Bratteli diagram

Bratteli diagram of even lines (tensorisation by $(1)^2 = (0) \oplus (2)$): only has even representations.



- Diagram yields the direct limit explicitly.
- The action of $[E]$ on $K_0(\mathcal{F}^\alpha)$ can be easily interpreted in this diagram.

K-theory and localisation

Localisation ring $\mathcal{R}_{\mathbb{G}}^{+}$:

- formal ring on irreps with coefficients in $\mathbb{Z}$,
- product given by fusion rules.

In the inductive limit:

- at level ℓ, only irreps appearing in α^{ℓ},
- inductive limit: at each step multiply by α.

Leads to localisation by α.

Example: case of $SU_q(2)$ and $\alpha = (1)$.

- Typical elements in level ℓ:

$$\frac{a_0(0) + a_2(2) + \cdots + a_\ell(\ell)}{(1)^\ell} \quad \frac{a_1(1) + a_3(3) + \cdots + a_\ell(\ell)}{(1)^\ell}$$

  depending on the parity of ℓ.
- Pushing to level $\ell + 1$, we multiply top and bottom by (1).

Back

K-theory and localisation – continued

- To perform actual computations, we use the identification of $\mathcal{R}_{\mathbb{G}}^{+}$ and $\mathbb{Z}[t]$ under correspondences:

$$(0) \leftrightsquigarrow 1 \qquad (1) \leftrightsquigarrow t \qquad (2) \leftrightsquigarrow t^2 - 1$$

  since $(1)^2 = (2) \oplus (0)$.
- All elements of level ℓ can be written $\frac{P(t)}{t^\ell}$ where $P(t)$ is a polynomial
  - of same parity as t^ℓ,
  - the degree of P is less than ℓ.

  The K-theory of $\mathcal{F}^\alpha$ consists of all such fractions.

To compute $K_*(\mathcal{O}^\alpha)$ in this context, we use:

$$\begin{array}{ccccc}
K_0(\mathcal{F}^\alpha) & \xrightarrow{\cdot(1-1/t^2)} & K_0(\mathcal{F}^\alpha) & \longrightarrow & K_0(\mathcal{O}^\alpha) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{O}^\alpha) & \longleftarrow & 0 & & 0.
\end{array}$$

Back

End of the computation and results

From the previous explicit expressions, we prove:

- $\sigma = \mathrm{Id} - [E] \colon K_0(\mathcal{F}^\alpha) \to K_0(\mathcal{F}^\alpha)$ is injective,
- $\operatorname{coker} \sigma \simeq \mathbb{Z}$.

Thus, in our special case:

Proposition

If $A = SU_q(2)$ and $\alpha = (1)$, the K-theory of $\mathcal{O}^\alpha$ is

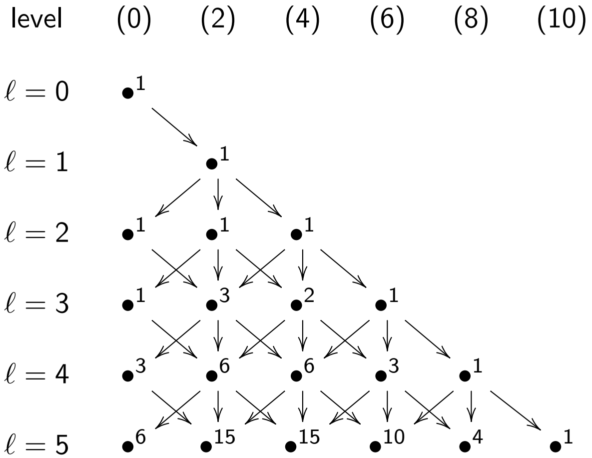
$$K_0(\mathcal{O}^\alpha) = \mathbb{Z} \qquad\qquad K_1(\mathcal{O}^\alpha) = 0,$$

and $\mathcal{O}^\alpha$ is Kirchberg, unital and in $\mathcal{N}$.
Moreover, $[1_{\mathcal{O}^\alpha}]_0 = 1 \in \mathbb{Z} \simeq K_0(\mathcal{O}^\alpha)$, hence $\mathcal{O}^\alpha \simeq \mathcal{O}_\infty$.

Back

Bratteli diagrams: example of $SU_q(2)$ and $\alpha = (2)$

Apply the procedure to $\mathbb{G} = SU_q(2)$, $d = 3$ and $\alpha = (2)$.
Only even representations occur:



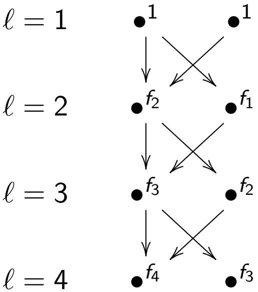
Bratteli diagrams: an example

Let $f_0, f_1, f_2, \ldots$ be the Fibonacci series given by $f_0 = f_1 = 1$ and $f_\ell = f_{\ell-1} + f_{\ell-2}$ for $\ell \geqslant 2$.

Put $A_\ell = M_{f_\ell}(\mathbb{C}) \oplus M_{f_{\ell-1}}(\mathbb{C})$ and let $\varphi_\ell \colon A_\ell \to A_{\ell+1}$ given by:

$$(x, y) \mapsto \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \right)$$

Consider the AF algebra $A = \lim_{\rightarrow} A_\ell$. Its Bratteli diagram:



Back

Bratteli diagrams

Using the equality $\mathcal{F}^\ell = M_{d^\ell}(\mathbb{C})$, the direct system $\lim_\to \mathcal{F}^\ell$ is

$$\cdots \to M_{d^\ell}(\mathbb{C}) \to M_{d^{\ell+1}}(\mathbb{C}) \to M_{d^{\ell+2}}(\mathbb{C}) \to \cdots$$

with morphisms $T \mapsto T \otimes \mathrm{Id}_{\mathbb{C}^d} \simeq \mathrm{diag}(\underbrace{T, T, \ldots, T}_{d \text{ times}})$.

Thus the Bratteli diagram is...

... which is characteristic of the type d^∞ UHF algebra.

For general AF algebras, a suitable equivalence relation on Bratteli diagrams can be defined:

Theorem (Bratteli, 1972)

Bratteli diagrams are equivalents
if and only if the AF algebras are isomorphic.

Back

Universal property of $\mathcal{O}_d$

If B is a C^*-algebra and $b_i \in B$ satisfies

$$b_i^* b_j = \delta_{ij} \qquad \sum_{j=1}^{d} b_j b_j^* = 1$$

then there is a $*$-homomorphisme $\varphi \colon \mathcal{O}_d \to B$ defined by

$$\varphi(S_j) = b_j.$$

◂ Cuntz algebras

◂ Gauge action

AF algebra: proof

- Take $T \in \mathcal{F}$. Using the universal property of $\mathcal{O}_d \supseteq \mathcal{F}$, there is an "algebraic" T_0 s.t.

$$\|T - T_0\| \leqslant \varepsilon.$$

- It suffices then to find T_0' which is
  - in $\mathcal{F}$ (gauge-invariant),
  - algebraic,
  - ε-close to T.
- If we take $T_0' = \mathbb{E}_{S^1}(T_0)$ then
  - T_0' is gauge-invariant and algebraic by construction,
  - it is ε-close to T because

$$\|T - \mathbb{E}_{S^1}(T_0)\| = \|\mathbb{E}_{S^1}(T - T_0)\| \leqslant \|T - T_0\| \leqslant \varepsilon$$

hence the result.

Back

Direct limit for $\mathcal{F}^\alpha$: proof

There is a conditional expectation $\mathbb{E}_{\mathbb{G}} \colon \mathcal{O}_d \to \mathcal{O}^\alpha$ associated to the action α defined by:

$$\mathbb{E}_{\mathbb{G}}(T) = (\mathrm{Id} \otimes h)\alpha(T),$$

where $h \colon A \to \mathbb{C}$ is the Haar measure on $A = C^*(\mathbb{G})$.
Take now $T \in \mathcal{F}^\alpha$,

- since $\mathcal{F}$ is AF, there is an algebraic T_0 in $\mathcal{F}$ s.t.

$$\|T - T_0\| \leqslant \varepsilon;$$

- consequently, $\mathbb{E}_{\mathbb{G}}(T_0) \in \mathcal{F}^\alpha$ is algebraic and ε-close to T:

$$\|T - \mathbb{E}_A(T_0)\| = \|\mathbb{E}_A(T - T_0)\| \leqslant \|T - T_0\| \leqslant \varepsilon.$$

Back

Multiplicity of map: proof

Definition (multiplicity of map)

The *multiplicity* of $\psi \colon M_k(\mathbb{C}) \to M_l(\mathbb{C})$ is defined by

$$\operatorname{Tr}(\psi(e))/\operatorname{Tr}(e) \in \mathbb{N}$$

where e is a nonzero projection in $M_k(\mathbb{C})$.

- All elementary projection $e_t \in M_{n_t}$ corresponds to a projection $P_{\mathcal{K}_t} \in \mathcal{F}^\ell$...
- ...whose range $\mathcal{K}_t \subseteq \mathscr{H}^\ell$ is
  - stable under α and
  - equipped with a type (t) irreducible representation.
- The range of $P_{\mathcal{K}_t}$ by the inclusion $\mathcal{F}^\ell \hookrightarrow \mathcal{F}^{\ell+1}$ corresponds to projection $P_{\mathcal{K}_t} \otimes \mathrm{Id} \in B(\mathscr{H}^{\ell+1})$...
- ...whose range is $\mathcal{K}_t \cdot \mathscr{H} \subseteq \mathscr{H}^{\ell+1}$ (which we decompose).

Back

Action of $[E]$: proof

The scalar product on $M \otimes E$ is

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle_B = \langle \eta, \langle \xi, \xi' \rangle_B \cdot \eta' \rangle_B$$

Proving that $(\xi_i \otimes \eta_j)$ is a frame: simple (but lengthy!) computation.

In our case,

- e can be chosen in $\mathcal{F}^\alpha$ (no matrices),
- a possible frame $\xi = e \in \mathcal{F}^\alpha$,

and then

$$\langle e \otimes \mathbb{V}^*, e \otimes \mathbb{V}^* \rangle_{\mathcal{F}^\alpha} = \mathbb{V}(e \cdot e) \cdot \mathbb{V}^* = \mathbb{V} e \mathbb{V}^*.$$

Back

Cuntz algebra $\mathcal{O}_\infty$

The Cuntz algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by infinitely many $S_1, \ldots, S_n, \ldots$ with relations

- for all i, j, $S_i^* S_j = \delta_{ij}$
- for any $r \in \mathbb{N}$,

$$\sum_{i=1}^{r} S_i S_i^* \leqslant 1.$$

Relevance of Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$:

Theorem (Kirchberg)

- $A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$ iff A simple, separable, unitary and nuclear.
- Let A be a simple, separable and nuclear C^*-algebra.

$$A \otimes \mathcal{O}_\infty \simeq A \text{ iff } A \text{ is purely infinite.}$$

Back

Co-restriction: complete argument

Prove that $\mu\colon \mathcal{O}_{d^M} \to \mathcal{O}_d$

$$S_{i_0+i_1 d+\cdots+i_{M-1}d^{M-1}} \mapsto S_{i_0}S_{i_1}\cdots S_{i_{M-1}}$$

corestricts to $\mathcal{O}^{\alpha^M} \to \mathcal{O}^{\alpha}$:

- It suffices to consider the case of algebraic $T \in \mathcal{O}^{\alpha}$ s.t. $\gamma_t(T) = e^{i2\pi nt}T$.
- Then we can then find k large enough s.t.

  $$T \in \mathscr{H}_M^{k+n}(\mathscr{H}_M^*)^k$$

  where $\mathscr{H}_M = \mathscr{H}^{\otimes M}$ is equipped with α^M.
- T is in $\mathcal{O}^{\alpha^M}$ iff it intertwins $(\mathscr{H}_M)^{k+n}$ with $(\mathscr{H}_M)^k$.

This is realised iff T seen as operator from $\mathscr{H}^{M(k+n)}$ to $\mathscr{H}^{Mk}$ is an intertwiner.

Back

Hilbert bimodule
◂ Back

We have a decomposition of algebraic elements:

$$T' = \sum_{n>0} T_n \mathbb{V}^n + T_0 + \sum_{n<0} (\mathbb{V}^*)^{|n|} T_n \qquad \text{(Decomp)}$$

Thus $\mathcal{O}^\alpha$ is generated by $\mathcal{F}^\alpha$ and $\mathbb{V}^*$, separable.

To show: $\mathcal{O}^\alpha$, Cuntz-Pimsner algebra, nuclear and in $\mathscr{N}$.

$E = \mathbb{V}^* \mathcal{F}^\alpha$ is a bimodule over $\mathcal{F}^\alpha$:

- clearly, in $\mathcal{O}^\alpha_d$, $E \cdot \mathcal{F}^\alpha = \mathbb{V}^* \mathcal{F}^\alpha$ and
- for all $T \in \mathcal{F}^\alpha$, $T\mathbb{V}^* a = \mathbb{V}^* \mathbb{V} T \mathbb{V}^* a \in E$ because $\mathbb{V}^*\mathbb{V} = 1$, $\gamma_z(\mathbb{V}) = z^2 \mathbb{V}$, $\gamma_z(T) = T$ and $\gamma_z(a) = a$.

Similarly, we have left- and right-$\mathcal{F}^\alpha$-valued scalar products:

$$_{\mathcal{F}^\alpha}\langle \mathbb{V}^* x, \mathbb{V}^* y \rangle = \mathbb{V}^* x y^* \mathbb{V} \qquad \langle \mathbb{V}^* x, \mathbb{V}^* y \rangle_{\mathcal{F}^\alpha} = x^* \mathbb{V}\mathbb{V}^* y.$$

Conclusion: E is Hilbert bimodule over $\mathcal{F}^\alpha$.

Action of $[E]$

Correspondence between proj. f.g. modules and K_0-classes:

- all proj. f.g. module M_B admits a *frame* ξ_j:

$$\sum \xi_j \langle \xi_j, \cdot \rangle_B = \mathrm{Id}_M$$

- if ξ_j is a frame for M_B then
$$e = (e_{ij}) = (\langle \xi_i, \xi_j \rangle_B) \in M_n(B), \text{ associated projector.}$$

Case of $E = \mathbb{V}^* \mathcal{F}^\alpha$: $\mathbb{V}^*$ is a frame!

Tensorisation:

- if $(\xi_j)_j$, $(\eta_k)_k$ frames of M and E, then
$$(\xi_j \otimes \eta_k)_{j,k} \text{ frame of } M \otimes_B E.$$

Therefore: $(e_{ij}) \otimes [E] \simeq (\mathbb{V} e_{ij} \mathbb{V}^*)$.

More

If e of level ℓ corresponds to (k) then
$$\mathbb{V} e \mathbb{V}^* \text{ of level } \ell + 2 \text{ corresponds to } (k) \otimes (0) = (k).$$

$[E]$ pushes down two levels without changing class!

Back

Nuclear Cuntz-Pimsner algebras in $\mathscr{N}$ ◂ Back

We know that $E = \mathbb{V}^*\mathcal{F}^\alpha$ is a Hilbert bimodule. As

$$_{\mathcal{F}^\alpha}\langle \mathbb{V}^*, \mathbb{V}^* \rangle = \mathbb{V}^*\mathbb{V} = 1,$$

the *right* module E is proj. f.g. and $\mathcal{K}(E_{\mathcal{F}^\alpha}) = \mathcal{F}^\alpha$.

Corollary

$\mathcal{O}^\alpha$ is a Cuntz-Pimsner algebra, with core $\mathcal{F}^\alpha$ and module E.

Proposition (Katsura, 2004)

A Cuntz-Pimsner algebra is nuclear as soon as its core is.

Since $\mathcal{F}^\alpha$ is AF thus nuclear, $\mathcal{O}_d^\alpha$ is nuclear.

Toeplitz extension of $\mathcal{O}_d^\alpha$ denoted $0 \to C \to \mathscr{T}_E \to \mathcal{O}^\alpha \to 0$ where C and $\mathscr{T}_E$ are KK-equivalent to $\mathcal{F}^\alpha$, AF thus in $\mathscr{N}$.

Proposition

$\mathcal{O}^\alpha$ is a unital Kirchberg algebra in $\mathscr{N}$. ▸ K-theory computations

$\mathfrak{C}(\mathbb{G})$ is a group

1. $\sim$ is an equivalence relation.
   Writing $\overline{t}$ for contragredient of t, if $t \sim u$ and $u \sim v$, with $t, u \in \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ and $u, v \in \upsilon_1 \otimes \cdots \otimes \upsilon_m$,
   then
   $$t, v \in \tau_1 \otimes \cdots \otimes \tau_n \otimes \overline{\tau_1 \otimes \cdots \otimes \tau_n} \otimes \upsilon_1 \otimes \cdots \otimes \upsilon_m$$
   since $\varepsilon \in u \otimes \overline{u}$ and $\varepsilon \in \overline{u} \otimes u$ and $u \in \tau_1 \otimes \cdots \otimes \tau_n$, $\overline{u} \in \overline{\tau_1 \otimes \cdots \otimes \tau_n}$, $u \in \upsilon_1 \otimes \cdots \otimes \upsilon_m$.

2. $[t][t'] := [t \otimes t']$ defines a group structure.
   - $\otimes$ is associative, therefore the product is too.
   - $[\varepsilon]$ is clearly the unit: $[t][\varepsilon] = [\varepsilon][t] = [t]$.
   - $[\overline{t}] = [t]^{-1}$ since $\varepsilon \in t \otimes \overline{t}$, $[t][\overline{t}] = [\varepsilon]$.

Back

Definition of the UCT class

Definition

Bootstrap class or Universal Coefficient Theorem (*UCT*) *class*: smallest class $\mathscr{N}$ of separable nuclear C^*-algebras s.t.

1. $\mathbb{C} \in \mathscr{N}$;
2. $\mathscr{N}$ is closed under inductive limit;
3. if $0 \to J \to A \to A/J \to 0$ is an exact sequence, and two C^*-algebras J, A or A/J are in $\mathscr{N}$, then so is the third;
4. $\mathscr{N}$ is closed under KK-equivalence.

Back

Braid groups and q-antisymmetric tensor $\mathbb{V}_q$

Definition (braid groups)

The finite braid groups B_n are

$$B_n := \left\langle b_i, i = 1, \ldots, n \,\middle|\, \begin{array}{c} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \\ b_i b_j = b_j b_i, |i - j| \geqslant 2 \end{array} \right\rangle$$

The *infinite braid group* B_∞ is the direct limit $\lim_{\rightarrow} B_n$

The embedding of B_∞ is obtained *via* intertwining $(t_1) \otimes (t_2)$ with $(t_2) \otimes (t_1)$.

The q-antisymmetric tensor $\mathbb{V}_q$ is

- a $\mathbb{G}$-invariant nonzero vector in $\mathbb{V}_q^{\otimes N}$,
- s.t. $\mathbb{V}_q \xrightarrow{q \to 1} \mathbb{S}$, totally antisymmetric rank N tensor.

For $\mathbb{G} = SU_q(N)$, $\alpha = \mathbb{V}_q$, $\mathbb{V}_q$ corresponds to our $\mathbb{V}$.

Back

Definition of the $*$-Hopf algebra structure

Definitions of co-multiplication Δ_0:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \qquad \Delta(\overline{u_{ij}}) = \sum_k \overline{u_{ik}} \otimes \overline{u_{kj}}.$$

Definition co-unit ε_0 and antipode γ_0 of $\mathscr{A}$:

$$\varepsilon_0(u_{ij}) = \varepsilon(\overline{u_{ij}}) = \delta_{ij} \qquad \gamma(u_{ij}) = \overline{u_{ji}}.$$

Furthermore, the involution $*$ is compatible with the coaction α: $* \circ \alpha = \alpha \circ (* \otimes *)$.

Back

Inclusions of fixed points: proof

- Easy to construct a map $\mu \colon \mathcal{O}_{d^M} \to \mathcal{O}_d$: just set
$$S_{i_0+i_1 d+\cdots+i_{M-1}d^{M-1}} \mapsto S_{i_0} S_{i_1} \cdots S_{i_{M-1}}$$
where all $i_0, i_1, \ldots, i_{M-1}$ are in $\{0, 1, \ldots, M-1\}$.
- This map is naturally injective.

$$\begin{array}{ccc} \mathcal{O}_{d^M} & \hookrightarrow & \mathcal{O}_d \\ \uparrow & & \uparrow \\ \mathcal{O}^{\alpha^M} & \dashrightarrow & \mathcal{O}^{\alpha} \end{array}$$

Aim: prove μ (co-)restricts to an isomorphism $\mathcal{O}^{\alpha^M} \to \mathcal{O}^{\alpha}$.

- Corestricting to $\mathcal{O}^{\alpha^M} \to \mathcal{O}^{\alpha}$: true in general, uses interpretation as intertwinner. Proves point 1. More
- Surjectivity: uses the chain group condition.
If $T \in \mathscr{H}^{k_0}(\mathscr{H}^*)^{k_1}$ is a nonzero intertwiner, then (same class) $[\alpha]^{k_0} = [\alpha]^{k_1}$ and thus $k_0 - k_1$ is a multiple of M.
Hence we can lift it to an element of $\mathcal{O}^{\alpha^M}$. Back

Compact quantum group: definition

Definition

A *compact quantum group* (A, Δ) is

- a unital C^*-algebra A,
- with a C^*-algebra morphism $\Delta\colon A \to A \otimes_{\min} A$

such that

- $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$ (coassociativity),
- $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

Theorem (Woronowicz – 1998)

If (A, Δ) CQG and A comm., there is a compact group G s.t.

$$A \simeq C(G) \text{ and } \Delta(f)(g, h) = f(gh).$$

- Thus, we recover all compact groups...
- ... and more, like the CQG $SU_q(2)$.

Back

Representations of CQG

Let (A, Δ) be a compact quantum group,

Definition

A matrix $w = (w_{ij}) \in M_d(\mathbb{C}) \otimes A$ is a *unitary representation* of (A, Δ) if

- w is unitary as element of $M_d(\mathbb{C}) \otimes A$,
- for all i, j, $\Delta(w_{ij}) = \sum_k w_{ik} \otimes w_{kj}$.

Back