Ultrapowers and relative commutants of operator algebras

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Nonprincipal ultrafilters on $\mathbb{N}$

A subset $\mathcal{U}$ of the power-set of $\mathbb{N}$ is an *nonprincipal (or free, or uniform) ultrafilter on* $\mathbb{N}$ if

1. $x \in \mathcal{U}$ and $y \in \mathcal{U}$ implies $x \cap y \in \mathcal{U}$.
2. $x \in \mathcal{U}$ and $x \subseteq y$ implies $y \in \mathcal{U}$.
3. for every x, either $x \in \mathcal{U}$ or $\mathbb{N} \setminus x \in \mathcal{U}$.
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In short, $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$.
We fix such $\mathcal{U}$ throughout.

$\mathcal{U}$-limits

Assume x_n, for $n \in \mathbb{N}$, is a sequence in a compact Hausdorff space X. Then function $n \mapsto x_n$ extends to a unique continuous

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We define

$$\lim_{n \to \mathcal{U}} x_n := f(\mathcal{U}).$$

Ultrapower of a Banach space

Let Z_n be Banach spaces. Then

$$c_{\mathcal{U}}((Z_n)) := \{\bar{z} \in \prod_n Z_n : \lim_{n \to \mathcal{U}} \|z_n\| = 0\}$$

is a closed subspace of $\prod_n Z_n$.

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I will concentrate on the *ultrapowers*,

$$Z^{\mathcal{U}} := \prod_{\mathcal{U}} Z.$$

Example

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Every ultrapower of an infinte-dimensional Banach space contains an isometric copy of $\ell^2(2^{\aleph_0})$.

Proposition

The following are equivalent for all Z and p.

1. ℓ^p *is finitely represented in* Z.
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Proof that (1) $\Rightarrow$ (2).

Fix $f_n\colon \ell^p(n) \to Z$ such that

$$(1 - \frac{1}{n})\|z\| \leq \|f(z)\| \leq (1 + \frac{1}{n})\|z\|.$$

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$$f(z) = (f_n(z))/\mathcal{U}.$$

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Exercise

(2) implies $\ell^p(2^{\aleph_0})$ *embeds into* $Z^{\mathcal{U}}$ *isometrically.*

Ultrapowers of C*-algebras

Let A be a C*-algebra. Let

$$c_{\mathcal{U}}(A) = \{\bar{a} \in \ell^{\infty}(A) : \lim_{n \to \mathcal{U}} \|a_n\| = 0\}$$

and

$$A^{\mathcal{U}} := \ell^{\infty}(A)/c_{\mathcal{U}}(A).$$

Proposition (Choi–F.–Ozawa)

Let Γ *be a countable amenable group and let* A *be a unital* C^**-algebra. Then every bounded homomorphism* $\Phi\colon \Gamma \to \mathrm{GL}(A^{\mathcal{U}})$ *is unitarizable.*

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$$\|\Phi\|^{-2} \le x \le \|\Phi\|^2 \tag{1}$$

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then

$$g \mapsto x^{1/2}\Phi(g)x^{-1/2}$$

is a homomorphism from Γ into $U(A^{\mathcal{U}})$.

Unitarizing $\Phi\colon \Gamma \to A^{\mathcal{U}}$, continued

For a finite $F \subseteq \Gamma$ let

$$a_F := \frac{1}{|F|} \sum_{f \in F} \Phi(f)\Phi(f)^*.$$

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If $F(n)$, for $n \in \mathbb{N}$, is a Følner sequence then

$$\|\Phi\|^{-2} \leq a_{F(n)} \leq \|\Phi\|^2, \tag{3}$$

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hence every finite subset of the system (1), (2) is approximately satisfied by $a_{F(n)}$ for some n.
Since $A^{\mathcal{U}}$ is an ultrapower, we can find an exact solution to this system and therefore unitarize Φ.

Tracial ultrapower

Let (M, τ) be a tracial von Neumann algebra with normalized trace tr and

$$\|a\|_2 := \mathrm{tr}(a^* a)^{1/2}.$$

Then

$$c_{\mathcal{U}}(M) = \{\bar{a} \in \ell^\infty(M) : \lim_{n \to \mathcal{U}} \|a_n\|_2 = 0\}$$

is a closed ideal and

$$M^{\mathcal{U}} := \ell^\infty(M)/c_{\mathcal{U}}(M)$$

is a tracial von Neumann algebra.

Early timeline (incomplete)

1954	F.B. Wright	ultrapowers of AW^* II_1 factors.
1962	S. Sakai	ultrapowers of II_1 factors
1970	McDuff	relative commutants of II_1 factors
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1976–present	...	more applications

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Proposition

In each category equipped with an ultrapower, it is a functor which preserves exact sequences.

Early timeline (slightly more complete)

1954	F.B. Wright	ultrapowers of AW* II_1 factors.
1955	J. Łos	fundamental theorem
1960	A. Robinson	nonstandard analysis
1962	S. Sakai	ultrapowers of II_1 factors
1966	H.J. Keisler	countable saturation
1969	W.A.J. Luxembourg	nonstandard hulls of Banach spaces
1970	McDuff	relative commutants
1972	Dacunha-Costelle–Krivine	ultrapowers of Banach spaces
1976	W.H. Woodin	discrete ultraproducts in automati continuity of Banach algebras
1976	A. Connes	applications
1976–present	. . . and . . .	more applications

Logic of metric structures

Ben Yaacov–Berenstein–Henson–Usvyatsov (2008),
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classical logic	Banach spaces	C*-algebras	tracial vNa
terms	linear combinations	noncommutative *-polynomials	
$a = b$	$\|a-b\|$	$\|a-b\|$	$\|a-b\|_2$
$\top, \bot$	$[0,\infty)$		
$\wedge, \vee, \leftrightarrow$	continuous $f : \mathbb{R}^n \to [0,\infty)$		
$\forall, \exists$	$\sup_{\|x\|\leq 1}$, $\inf_{\|x\|\leq 1}$		

Examples of sentences in logic of metric structures

For a sentence φ and a C*-algebra A one recursively defines interpretation of φ in A, φ^A.

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   $\sup_{x,y} \left| \|x + y\|^2 + \|x - y\|^2 - 2(\|x\|^2 + \|y\|^2) \right|$.

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$$\sup_x \max\{0, (\|x - \mathrm{tr}(x) \cdot 1\|_2 - \sup_y \|[x,y]\|_2)\}.$$

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4. $\inf_{x_1} \sup_{x_2} \inf_{x_3} \sup_{x_4} \inf_{x_5, x_6} \max(\|x_2 x_2^* - x_1 x_1^*\|, \frac{3}{4}\|x_3^* x_3 - x_4\| - \frac{2}{3}\|x_1^* x_4 x_2 - x_2^* x_5^* x_1\|)$

Elementary embeddings

A map $\Phi\colon A \to B$ is an *elementary embedding* if for every $\psi(\bar{x})$ and $\bar{a}$ in A we have

$$\psi(\bar{a})^A = \psi(\Phi(\bar{a}))^B.$$

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Theorem (Fundamental Theorem of Ultraproducts. Łos, 1955)
The diagonal embedding of A into $A^{\mathcal{U}}$ is elementary.

Types

A *condition on* $\bar{x} = (x_1, \ldots, x_n)$ is an expression of the form $\varphi(\bar{x}) \leq r$, $\varphi(\bar{x}) \geq r$, or $\varphi(\bar{x}) = 0$.

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Example

A type in x, with parameters in algebra C.

$$M^{-2} \leq \|x^*x\| \leq M^2$$
$$\|a_n(x^*x){a_n}^* - x^*x\| = 0, \text{ for all } n \in \mathbb{N}.$$

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Type is *satisfied* in C if some $\bar{c}$ satisfies all of its conditions. Type is *consistent* if each of its finite subsets is approximately satisfiable.

All you need to know about ultrapowers

Theorem (Countable saturation. Keisler, 1966)

Every consistent countable type with parameters in $A^{\mathcal{U}}$ *is satisfied in* $A^{\mathcal{U}}$.

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Corollary (to Łos and Keisler)

C is an ultrapower of $A \subseteq C$ iff

(i) $\mathrm{id}\colon A \to C$ *is elementary and*

(ii) *C is countably saturated.*

(Assuming A is separable, C has cardinality $2^{\aleph_0}$, and the Continuum Hypothesis holds.)

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Theorem (Keisler–Shelah)

For all A and B, $\mathrm{Th}(A) = \mathrm{Th}(B)$ if and only if A and B have isomorphic ultrapowers.

Ultrafilter not necessarily on $\mathbb{N}$ but A and B are not necessarily separable.

Does the choice of $\mathcal{U}$ matter?

Metatheorem

Assume $\mathbb{P}(B)$ is any statement that refers only to elements and separable substructures of B. Then for a separable metric structure A and all $\mathcal{U}$ and $\mathcal{V}$ we have

$$\mathbb{P}(A^{\mathcal{U}}) \Leftrightarrow \mathbb{P}(A^{\mathcal{V}})$$

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regardless of whether Continuum Hypothesis holds or not.

By results of Shelah, Dow, Ge–Hadwin, F.–Hart–Sherman, F.–Shelah, one can code many complicated total orders inside ultrapowers of A and, if Continuum Hypothesis fails, obtain $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers of the same algebra.

Relative commutant

For C*-algebras and tracial von Neumann algebras define

$$A' \cap A^{\mathcal{U}} = \{b \in A^{\mathcal{U}} : (\forall a \in A) ab = ba\}$$

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Hyperfinite II_1 factor R is the $\|\cdot\|_2$-completion of $\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$.

Theorem (McDuff, 1970)

For a II_1 factor M the following are equivalent.

1. *$M \bar{\otimes} R \cong M$, where R is the hyperfinite II_1 factor.*
2. *$M_2(\mathbb{C})$ embeds unitally into $M' \cap M^{\mathcal{U}}$.*
3. *mix-and-match (1) and (2)*

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Factors satisfynig (1)–(3) are *McDuff factors*.

Approximately inner flip

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Theorem (Effros–Rosenberg, after McDuff)

If C-algebra D has approximately inner half-flip then the following are equivalent for every (separable) A.*

1. $A \otimes D \cong A$
2. *D unitally embeds into $A' \cap A^{\mathcal{U}}$.*

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If (C-algebra) D has an approximately inner (half) flip then it is nuclear, simple, and has at most one trace.*

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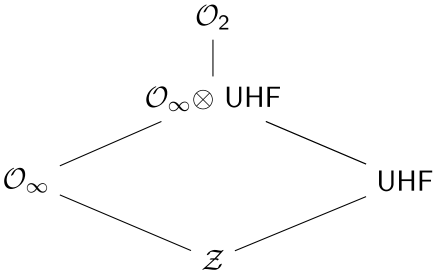

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Does every II_1 factor with separable predual embed into $R^{\mathcal{U}}$?

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Proposition (Folklore)

A II_1 factor M with separable predual embeds into $R^{\mathcal{U}}$ if and only if it embeds into $R' \cap R^{\mathcal{U}}$.

Relative commutant has no well-understood abstract analogue

On relative commutants

$A \prec B$ `stands for` '$A \subseteq B$ `and` $\mathrm{id}\colon A \to B$ `is elementary.`'

Theorem (F.–Hart–Rørdam–Tikuisis, 2015)

Assume D has approximately inner half-flip and $A \otimes D \cong A$. Then

$$D' \cap A^{\mathcal{U}} \prec C^*(D, D' \cap A^{\mathcal{U}}) \prec A^{\mathcal{U}} \prec A^{\mathcal{U}} \otimes D.$$

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For the hyperfinite II_1 factor R and a McDuff factor with separable predual M we have

$$R' \cap M^{\mathcal{U}} \prec W^*(R, R' \cap M^{\mathcal{U}}) \prec M^{\mathcal{U}} \prec M^{\mathcal{U}} \bar{\otimes} R.$$

Assume Continuum Hypothesis

Theorem (FHRT, 2015)

Assume C-algebra D has approximately inner half-flip and $A \otimes D \cong A$. Then*

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Proposition (Fang–Ge–Li, Ghasemi)

Nontrivial ultrapowers are tensorially indecomposable.

In particular, $R^{\mathcal{U}} \bar{\otimes} R \not\cong R^{\mathcal{U}}$ and $D^{\mathcal{U}} \otimes D \not\cong D^{\mathcal{U}}$.

Question

Do all free group factors $L(F_n)$, $n \geq 2$, have isomorphic ultrapowers?

Question

Can one describe automorphisms of $A^{\mathcal{U}} \otimes A^{\mathcal{U}}$ in terms of the automorphisms of $A^{\mathcal{U}}$?

(F.: Yes if A is abelian.)