Cuntz-Pimsner algebras arising from vector bundles and minimal homepmorphism

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Right Hillbert A-modules

Let A be a C*-algebra. We say that $\mathcal{E}$ is a right Hilbert A-module if

i) $\mathcal{E}$ is is a right A-module

ii) $\mathcal{E}$ is equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to A$, such that, for every $\xi, \eta \in \mathcal{E}$ and every $a \in A$ we have

1. $\langle \xi, \eta_1 + \eta_2 a \rangle = \langle \xi, \eta_1 \rangle + \langle \xi, \eta_2 \rangle a$
2. $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$
3. $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$

iii) $\mathcal{E}$ is complete with respect to the norm $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

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Bimodule Assume $\mathcal{E}$ is a right Hilbert A-module and a left Hilbert A-module. $\mathcal{E}$ is an *A-Hilbert bimodule* if

$$\xi \langle \eta, \zeta \rangle_{\mathcal{E}} = {}_{\mathcal{E}}\langle \xi, \eta \rangle \zeta \qquad \xi, \eta, \zeta \in \mathcal{E}.$$

where $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ denotes the right inner product, and ${}_{\mathcal{E}}\langle \cdot, \cdot \rangle$ the left inner product.

We denote by $\mathcal{L}(\mathcal{E})$ the C*-algebra of adjointable operators.

$\mathcal{K}(\mathcal{E})$ is the closed two-sided ideal of compact operators given by

$$\mathcal{K}(\mathcal{E}) = \overline{span}\{\theta_{\xi,\eta} \mid \xi, \eta \in \mathcal{E}\}$$

where $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle_{\mathcal{E}}$.

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Definition

Let A and B be C^*-algebras. An A–B C^**-correspondence* is a right Hilbert B-module $\mathcal{E}$ together with a *-homomorphism. $\varphi\colon A \to \mathcal{L}(\mathcal{E})$. If $A = B$ then we call $\mathcal{E}$ a C^*-correspondence over A.

C*-correspondences

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The map φ gives $\mathcal{E}$ a left A-module structure

$$a \cdot \xi = \varphi(a)(\xi) \ \text{ for } \xi \in \mathcal{E}.$$

Covariant representations

Definition (Katsura 2004)

Let $(\mathcal{E}, \varphi)$ be an C^*-correspondence over a C*-algebra A. A *representation* (π, τ) on a C*-algebra B consists of a *-homomorphism $\pi\colon A \to B$ and a linear map $\tau\colon \mathcal{E} \to B$ satisfying

i) $\pi(\langle \xi, \eta \rangle) = \tau(\xi)^* \tau(\eta)$, for every $\xi, \eta \in \mathcal{E}$;

ii) $\tau(\varphi(a)\xi) = \pi(a)\tau(\xi)$, for every $\xi \in \mathcal{E}$, $a \in A$.

Let $\psi_\tau : K(\mathcal{E}) \to B$ be the *-homomorphism defined on rank one operators by

$$\psi_\tau(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^* \quad \text{for } \xi, \eta \in \mathcal{E}.$$

The representation (π, τ) is *covariant* if

iii) $\pi(a) = \psi_\tau(\varphi(a))$ for every $a \in \varphi^{-1}(\mathcal{K}(\mathcal{E}))$.

Cuntz-Pimsner Algebras

Definition (Pimsner 1997, Katsura 2004)

Let A be a C^*-algebra and $(\mathcal{E}, \varphi)$ a C^*-correspondence over A. The *Cuntz–Pimsner algebra of $\mathcal{E}$ over A*, denoted by $\mathcal{O}_A(\mathcal{E})$ is the C^*-algebra generated by the universal covariant representation of $(\mathcal{E}, \varphi)$.

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Examples

1. If A is commutative and finite dimensional and $\mathcal{E}$ is finitely generated projective, then $\mathcal{O}_A(\mathcal{E})$ is a Cuntz-Krieger algebra.
2. Let A be an arbitrary C*-algebra. Take $\mathcal{E} = A$, with right/left action given by right/left multiplication, and inner product given by $\langle a, b \rangle_A = a^* b$. If $\pi \colon A \to A$ is an automorphism, then π defines a Hilbert bimodule structure on $\mathcal{E}$, and one can shows that $\mathcal{O}_A(\mathcal{E})$ is simply the crossed product $A \rtimes_\pi \mathbb{Z}$.

Generalized Crossed Products

Covariant representations for Hilbert A-bimodules were defined by Abadie, Eilers, and Exel. in the late '90s.

Let $\mathcal{E}$ be a Hilbert A-bimodule with left and right inner products given by ${}_{\mathcal{E}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, respectively. A covariant representation (π,τ) of $\mathcal{E}$ on a C^*-algebra B consists of a *-homomorphism $\pi : A \to B$ and linear map $\tau : \mathcal{E} \to B$ satisfying

1. $\pi(\langle\xi,\eta\rangle_{\mathcal{E}}) = \tau(\xi)^*\tau(\eta)$, for every $\xi,\eta \in \mathcal{E}$;
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Definition

Let $\mathcal{E}$ be a Hilbert A-bimodule . We denote by $A \rtimes_{\mathcal{E}} \mathbb{Z}$ the C^*-algebra generated by the universal covariant representation of $\mathcal{E}$, and refer to it as *the crossed product of A by the Hilbert bimodule $\mathcal{E}$*.

Example

Let X be a compact metric space and let $\mathcal{V} = [V, p, X]$ be a vector bundle of finite rank (that means that $p : V \to X$ is a continuous surjective map and for every $x \in X$, $p^{-1}(x) \cong \mathbb{C}^{n_x}$ for some n_x). Set

$$\Gamma(\mathcal{V}) = \{\xi \colon X \to V \mid \xi \text{ is continuous and } p \circ \xi = id_X\}.$$

Define a right $C(X)$-action on $\Gamma(\mathcal{V})$ by

$$(\xi \cdot f)(x) = \xi(x) f(x) \text{ for } \xi \in \mathcal{E},\ f \in C(X).$$

We can find open sets $U_1, \ldots, U_n$ that cover X, and homeomorphisms $t_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}$. Let $\gamma_1, \ldots, \gamma_n$ be a partition of unity subordinate to $U_1, \ldots, U_n$. Define for $\xi, \eta \in \Gamma(\mathcal{V})$

$$\langle \xi, \eta \rangle(x) := \sum_{i=1}^{n} \gamma_i(x) \langle t_i(\xi(x)), t_i(\eta(x)) \rangle_{\mathbb{C}^{n_i}}.$$

Then $\Gamma(\mathcal{V})$ is a right Hilbert C(X)-module.

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$$\langle \xi, \eta \rangle(x) := \sum_{i=1}^{n} \gamma_i(x) \langle t_i(\xi(x)), t_i(\eta(x)) \rangle_{\mathbb{C}^{n_i}}.$$

Then $\Gamma(\mathcal{V})$ is a right Hilbert C(X)-module. Given a homeomorphism $\alpha\colon X \to X$ define $\varphi : C(X) \to K(\Gamma(\mathcal{V}))$ by $\varphi(f)(\xi) = \xi \cdot (f \circ \alpha)$. Denote the associate C*-correspondence by $\Gamma(\mathcal{V}, \alpha)$.

In general the C*-correspondence $\Gamma(\mathcal{V}, \alpha)$ will not admit a Hilbert $C(X)$–bimodule structure, only a $\mathcal{K}(\Gamma(\mathcal{V}, \alpha))$-$C(X)$–bimodule structure, In general the C*-correspondence $\Gamma(\mathcal{V},\alpha)$ will not admit a Hilbert $C(X)$–bimodule structure, only a $\mathcal{K}(\Gamma(\mathcal{V},\alpha))$-$C(X)$–bimodule structure,

Proposition (AAFGJSV)

Let X be a compact metric space. Suppose that $\mathcal{E} = \Gamma(\mathcal{V})$ is the right Hilbert $C(X)$-module of continuous sections of a complex line bundle over X. Then for any homeomorphism $\alpha\colon X \to X$ the left multiplication

$$f \cdot \xi := \xi(f \circ \alpha), \qquad f \in C(X), \xi \in \mathcal{E},$$

and left $C(X)$-inner product

$${}_{\mathcal{E}}\langle \xi, \eta\rangle := \langle \eta, \xi\rangle_{\mathcal{E}} \circ \alpha^{-1}, \qquad \xi, \eta \in \mathcal{E},$$

make $\mathcal{E}$ into a Hilbert $C(X)$-bimodule. Moreover, if $\varphi : C(X) \to \mathcal{K}(\mathcal{E})$ is the *-homomorphism

$$\varphi(f)(\xi) = \xi(f \circ \alpha), \qquad \xi \in \mathcal{E}, f \in C(X),$$

then

$${}_{\mathcal{E}}\langle \xi, \eta\rangle = \varphi^{-1}(\theta_{\xi,\eta}), \qquad \xi, \eta \in \mathcal{E}.$$

Examples

Example

1) Let $\mathcal{V} = [V, p, X]$ be a line bundle and take $\alpha = id_X$. Then $C(X) \cong \mathcal{K}(\Gamma(\mathcal{V}))$, and $\Gamma(\mathcal{V}, id_X)$ has the structure of a Hilbert $C(X)$-bimodule. Moreover, $\mathcal{O}(\Gamma(\mathcal{V}, id)_X) \cong C(X) \rtimes_{\Gamma(\mathcal{V}, id_X)} \mathbb{Z}$.

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1) Let $\mathcal{V} = [V, p, X]$ be a line bundle and take $\alpha = id_X$. Then $C(X) \cong \mathcal{K}(\Gamma(\mathcal{V}))$, and $\Gamma(\mathcal{V}, id_X)$ has the structure of a Hilbert $C(X)$-bimodule. Moreover, $\mathcal{O}(\Gamma(\mathcal{V}, id)_X) \cong C(X) \rtimes_{\Gamma(\mathcal{V}, id_X)} \mathbb{Z}$.

2) Let $\mathcal{V} = [V, p, X]$ be a trivial line bundle and $\alpha\colon X \to X$ a homeomorphism. Then $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ is generated by a single element, which is easily seen to be a unitary element $u \in \mathcal{O}(\mathcal{E})$ and satisfies $ufu^* = f \circ \alpha^{-1}$. Therefore, $\mathcal{O}(\mathcal{E}) = C(X) \rtimes_\alpha \mathbb{Z}$.

Proposition (AAFGJSV)

Let $\mathcal{E}$ be a non-zero Hilbert $C(X)$-bimodule which is finitely generated projective as a right Hilbert $C(X)$-module, and full as a left Hilbert $C(X)$-module. Then there exist a compact metric space X, a line bundle $\mathcal{V} = [V, p, X]$ and a homeomorphisms $\alpha : X \to X$ such that

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Proposition

Let X be an infinite compact metric space, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha\colon X \to X$ an homeomorphism. Then, the following are equivalent.

1. *$\mathcal{O}(\Gamma(\mathcal{V}, \alpha)$ is simple.*
2. *α is minimal.*

Proposition (AAFGJSV)

Let $\mathcal{V} = [V, p, X]$ be a vector bundle and $\alpha \colon X \to X$ a homeomorphism.

1. $T(\mathcal{O}(\Gamma(\mathcal{V}, \alpha))) \neq \emptyset$ *if and only if* $\mathcal{V}$ *is a line bundle.*

Tracial state space

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1. $T(\mathcal{O}(\Gamma(\mathcal{V}, \alpha))) \neq \emptyset$ *if and only if* $\mathcal{V}$ *is a line bundle.*
2. *If* $\mathcal{V} = [V, p, X]$ *is a line bundle and* $\alpha\colon X \to X$ *an aperiodic homeomorphism then there are affine homeomorphisms*

$$T(\mathcal{O}(\Gamma(\mathcal{V}, \alpha))) \cong M^1(X, \alpha),$$

where $M^1(X, \alpha)$ *denotes the space of* α*-invariant Borel probability measures.*

Regularities properties of $\mathcal{O}(\Gamma(V, \alpha))$

Corollary (AAFGJSV)

Let $\mathcal{V} = [V, p, X]$ be a line bundle and $\alpha\colon X \to X$ a minimal homeomorphism. Then $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ is stably finite.

Regularities properties of $\mathcal{O}(\Gamma(V,\alpha))$

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Let $\mathcal{V} = [V, p, X]$ be a line bundle and $\alpha\colon X \to X$ a minimal homeomorphism. Then $\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ is stably finite.

Theorem (Katsura 2004)

Let A be a unital C-algebra and $(\mathcal{E},\varphi)$ a C*-correspondence over A.*

- *If A is nuclear, then $\mathcal{O}_A(\mathcal{E})$ is nuclear.*
- *If A is separable, nuclear and it satisfies the UCT, then $\mathcal{O}_A(\mathcal{E})$ satisfies the UCT*

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Since $C(X)$ is nuclear and satisfies the UCT, so does $\mathcal{O}(\Gamma(V, \alpha))$.

Regularities properties of $\mathcal{O}(\Gamma(V, \alpha))$

Question

Given a C^*-correspondence $(\mathcal{E}, \varphi)$, when does $\mathcal{O}(\mathcal{E})$ have finite nuclear dimension?

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Theorem (Brown-Tikuisis-Zelenberg 2018)

Assume the C^*-algebra A is a simple, unital, satisfies the UCT and has finite nuclear dimension. For every finitely generated projective C^*-correspondence $\mathcal{E}$ with finite Rokhlin dimension, the Cuntz-Pimsner algebra $\mathcal{O}_A(\mathcal{E})$ has finite nuclear dimension.

Definition (Brown, Tikuisis, and Zelenberg, 2018)

Let A be a separable C*-algebra and let $(\mathcal{E}, \varphi)$ be a countably generated C*-correspondence over A. We say that $(\mathcal{E}, \varphi)$ has Rokhlin dimension at most d if, for any $\varepsilon > 0$, any $p \in \mathbb{N} \setminus \{0\}$, every finite subset $F \subset A$ and every finite subset $\mathcal{G} \subset \mathcal{E}$, there exists positive contractions $\{f_k^{(l)}\}_{l=0,\dots,d;k\in\mathbb{Z}/p} \subset A$ satisfying

1. $\|f_k^{(l)} f_{k'}^{(l)}\| < \varepsilon$ when $k \neq k'$.
2. $\|\sum_{k,l} f_k^{(l)} - 1\| < \varepsilon$,
3. $\|\xi f_k^{(l)} - f_{k+1}^{(l)} \xi\| < \varepsilon$ for every $k \in \mathbb{Z}/p$, $0 \leq l \leq d$, and every $\xi \in \mathcal{G}$,
4. $\|[f_k^{(l)}, a]\| < \varepsilon$ for every $k \in \mathbb{Z}/p$, $0 \leq l \leq d$, and every $a \in F$.

Finite dimensional X

Theorem (AAFGJSV)

Let X be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha \colon X \to X$ an aperiodic homeomorphism. Then $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ has finite Rokhlin dimension.

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Let X be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha\colon X \to X$ an aperiodic homeomorphism. Then $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ has finite nuclear dimension. If α is minimal, the nuclear dimension is one.

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Corollary

Let X be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha\colon X \to X$ a minimal homeomorphism.

1. *If $\mathcal{V}$ is a line bundle, $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ has stable rank one.*
2. *If $\mathcal{V}$ has (not necessarily constant) rank greater than one, $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ is purely infinite.*

Orbit-breaking Subalgebras

Trivial line bundle case: Let $\mathcal{V} = [V, p, X]$ be the trivial line bundle and $\alpha\colon X \to X$ a minimal homeomorphism, so that

$$A = \mathcal{O}(\Gamma(\mathcal{V}, \alpha)) = C(X) \rtimes_\alpha \mathbb{Z}.$$

Let u be the unitary implementing the crossed product, and let $Y \subset X$ be a nonempty closed subset. Then

$$A_Y = C^*(C(X), C_0(X \setminus Y)u)$$

is the *orbit-breaking subalgebra* at Y of A.

Assume that Y meets every α-orbit at most once. Then

1. A_Y is a simple, separable nuclear C*-algebra,
2. the inclusion $A_Y \subset A$ induces an affine homeomorphism $T(A_Y) \cong T(A)$.
3. A_Y is a centrally large subalgebra of $C(X) \rtimes_\alpha \mathbb{Z}$ in the sense of Phillips.

Large Subalgebras (Phillips, 2014)

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital C*-subalgebra $B \subset A$ is *large* if, for every $m \in \mathbb{N} \setminus \{0\}$, $a_1, \ldots, a_m \in A$, $\epsilon > 0$, $x \in A_+$ with $\|x\| = 1$ and every $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that

1. $0 \leq g \leq 1$;
2. $\|c_j - a_j\| < \epsilon$,
3. $(1 - g)c_j \in B$,
4. $g \precsim_B y$ and $g \precsim_A x$,
5. $\|(1 - g)x(1 - g)\| > 1 - \epsilon$.

If, in addition g can be chosen so that

$$\|ga_j - a_j g\| < \epsilon,$$

then we say that B is *centrally large*.

If $B \subset A$ is a large subalgebra then

1. B is simple and infinite-dimensional.
2. A is finite $\Leftrightarrow$ B is finite, and A is purely infinite $\Leftrightarrow$ B is purely infinite.
3. The inclusion $\iota : B \to A$ induces an affine homeomorphism $T(A) \cong T(B)$.

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3. The inclusion $\iota : B \to A$ induces an affine homeomorphism $T(A) \cong T(B)$.

If, in addition B is a centrally large subalgebra of A then,

1. A is stably finite if and only if B is stably finite.
2. If A and B are both separable and nuclear then A is $\mathcal{Z}$-stable $\Leftrightarrow$ B is $\mathcal{Z}$-stable. (Archey–Buck–Phillips)

Orbit-Breaking Subalgebras of $\mathcal{O}(\Gamma(V, \alpha))$

Let X be an infinite compact metric space, $\mathcal{V} = [V, p, X]$ a vector bundle and $\alpha \colon X \to X$ a minimal homeomorphism. Let $\mathcal{E} := \Gamma(\mathcal{V}, \alpha)$.

Definition

Let $Y \subset X$ be a non-empty closed subset. The *orbit-breaking subalgebra* of $\mathcal{O}(\mathcal{E})$ at Y is $\mathcal{O}(C_0(X \setminus Y)\mathcal{E})$, that is, the Cuntz–Pimsner algebra of the C*-correspondence over $C(X)$ given by $\mathcal{E}_Y = C_0(X \setminus Y)\mathcal{E}$.

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Theorem (AAFGJSV)

Let $Y \subset X$ be a non-empty closed subset meeting each α-orbit at most once and such that for every $N \in \mathbb{Z}_{\geq 0}$ there exists an open set $W_N \supset Y$ for which $\mathcal{V}|_{\alpha^m(W_N)}$ is trivial whenever $-N \leq m \leq N$.

Then $\mathcal{O}(\mathcal{E}_Y)$ is a large subalgebra of $\mathcal{O}(\mathcal{E})$. In fact, it is stably large.

Corollary

Let $Y \subset X$ be a non-empty closed subset meeting each α-orbit at most once and such that for every $N \in \mathbb{Z}_{\geq 0}$ there exists an open set $W_N \supset Y$ for which $\mathcal{V}|_{\alpha^m(W_N)}$ is trivial whenever $-N \leq m \leq N$.

Then $\mathcal{O}(\mathcal{E}_Y)$ is simple and there is an affine homeomorphism $T(\mathcal{O}(\mathcal{E}_Y)) \cong T(\mathcal{O}(\mathcal{E}))$.

Orbit-Breaking Subalgebras of $\mathcal{O}(\Gamma(V,\alpha))$

Corollary

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Then $\mathcal{O}(\mathcal{E}_Y)$ is simple and there is an affine homeomorphism $T(\mathcal{O}(\mathcal{E}_Y)) \cong T(\mathcal{O}(\mathcal{E}))$.

Theorem (AAFGJSV)

Suppose that $\mathcal{V} = [V, p, X]$ is a line bundle over X. Assume that Y satisfies the same hypotheses as above. Then $\mathcal{O}(\mathcal{E}_Y)$ is a centrally large subalgebra of $\mathcal{O}(\mathcal{E})$.

Proposition (AAFGJSV)

Let X be an infinite compact metric space with $\dim(X) < \infty$ and $\mathcal{V} = [V, p, X]$ a line bundle. Under the same hypothesis on Y as in the previous theorem, $\mathcal{O}(\mathcal{E}_Y)$ has nuclear dimension one.

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Let X be an infinite compact metric space with $\dim(X) < \infty$ and $\mathcal{V} = [V, p, X]$ a line bundle. Under the same hypothesis on Y as in the previous theorem, $\mathcal{O}(\mathcal{E}_Y)$ has nuclear dimension one.

Therefore, $\mathcal{O}(\mathcal{E}_Y)$ is classifiable by the Elliott invariant.