A generalized Powers averaging property for commutative crossed products

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Intuition: contains $\{\sum_{\text{finite}} a_t \lambda_t \mid t \in G, a_t \in A\}$ as a dense subset, and

$$a\lambda_s b\lambda_t = a\lambda_s b\lambda_s^* \lambda_s \lambda_t = (a(s \cdot b))\lambda_{st}.$$

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The reduced crossed product $A \rtimes_r G$ is the unique norm completion such that $E(\sum a_t \lambda_t) = a_e$ is a faithful conditional expectation.

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Dynamical characterization on the Furstenberg boundary $I_G(\mathbb{C}) = \partial_F G$	Kalantar-Kennedy, 2017
Intrinsic characterization in terms of confined subgroups of G	Kennedy, 2020
Powers' averaging property for $C_r^*(G)$	Haagerup, 2016 and Kennedy, 2020
Unique stationarity of the canonical trace	Hartman-Kalantar, 2017

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NOTE: Powers' averaging property is what Powers (1975) used to show that $C_r^*(\mathbb{F}_2)$ is simple.

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Dynamical characterization on the Furstenberg boundary $I_G(C(X))$ (spectrum)	Kawabe, 2017
Intrinsic characterization in terms of generalized residually normal subgroups	Kawabe, 2017
Powers averaging property for $C(X) \rtimes_r G$	**???**
Unique stationarity of something	**???**

Powers' averaging property for $C_r^*(G)$

Consider the reduced group C*-algebra $C_r^*(G)$ with the canonical trace τ, where $\tau(\sum_g \alpha_g \lambda_g) = \alpha_e$. Recall that $G \curvearrowright C_r^*(G)$ by $g \cdot a = \lambda_g a \lambda_g^*$.

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Theorem (Haagerup, 2016 and Kennedy, 2020)

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Theorem

A tracial von Neumann algebra (M, τ) is a factor if and only if

$$\tau(x) \in \overline{\mathrm{conv}}\,\{uxu^* \mid u \in U(M)\}.$$

Conveniently packaging up the convex combinations

Convenient way to represent convex combinations of $g \cdot a$. Consider $P(G)$, the set of probability measures on G. Given $\mu \in P(G)$, $\mu = \sum \alpha_g \delta_g$, "extend linearly" and define

$$\mu a = \sum_{g \in G} \alpha_g (g \cdot a).$$

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Then $C_r^*(G)$ is simple if and only if for any $a \in C_r^*(G)$, we have

$$\tau(a) \in \overline{\{\mu a \mid \mu \in P(G)\}}.$$

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$$\sum f_i b_i f_i, \quad b_i \in B, \;\; f_i \in C(X), \;\; f_i \geq 0, \;\; \sum f_i^2 = 1.$$

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Just like before, can define **generalized measure** $\mu \in P(G, C(X))$ to be

$$\mu = \sum_{i \in I} f_i g_i f_i, \quad \text{repetition of } g_i \in G \text{ allowed!}$$

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and an action on $C(X) \rtimes_r G$ (or any G-C*-algebra containing $C(X)$ equivariantly) by

$$\mu a = \sum_{i \in I} f_i (g_i \cdot a) f_i$$

Powers' averaging property for $C(X) \rtimes_r G$

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Theorem (Amrutam-U., 2021)

Assume $G \curvearrowright X$ is minimal. The following are equivalent.

1. *$C(X) \rtimes_r G$ is simple.*
2. *Given $a \in C(X) \rtimes_r G$ with $E(a) = 0$, we have*

$$0 \in \overline{C(X) - \text{conv}}\, \{g \cdot a \mid g \in G\} = \overline{\{\mu a \mid \mu \in P(G, C(X))\}}.$$

3. *Given $a \in C(X) \rtimes_r G$, we have $E(a) \in (\dots)$.*
4. *Given $a \in C(X) \rtimes_r G$ and $\nu \in P(X)$, we have $\nu(E(a)) \in (\dots)$.*

Unique stationarity for $C^*_r(G)$

The left action of $P(G)$ on $C^*_r(G)$ gives a right action on $S(C^*_r(G))$:

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Theorem (Hartman-Kalantar, 2017)

$C_r^*(G)$ is simple if and only if there is some measure $\mu \in P(G)$ with full support and the canonical trace $\tau \in S(C_r^*(G))$ being the unique stationary state.

Unique stationarity for $C(X) \rtimes_r G$

Theorem (Amrutam-U., 2021)

Assume $C(X) \rtimes_r G$ is simple. Then there is some $\mu \in P(G, C(X))$ (optionally full support for an appropriate notion) such that ***for all*** *$a \in C(X) \rtimes_r G$ with $E(a) = 0$, we have $\mu^n a \to 0$.*

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Corollary (Amrutam-U., 2021)

Assume $G \curvearrowright X$ is minimal. Then $C(X) \rtimes_r G$ is simple if and only if there is some full support $\mu \in P(G, C(X))$ such that any μ-stationary state $\phi \in S(C(X) \rtimes_r G)$ is of the form $\nu \circ E$ for some $\nu \in P(X)$.

Application: Simplicity of intermediate subalgebras

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Is everything $C(X) \rtimes_r G \subseteq A \subseteq C(Y) \rtimes_r G$ also simple?

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Yes, when $C(X) = \mathbb{C}$.

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Theorem (Amrutam-U., 2021)

Yes, in general.

Proof of previous result

Sketch of proof.

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So all μ-stationary states on $C(Y) \rtimes_r G$ are of the form $\nu \circ E$, where $\nu \in P(Y)$. These are faithful by minimality of Y and full support of μ.

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Assume A is not simple, with nontrivial ideal I. There exists a μ-stationary state $\phi \in S(A/I)$ by your favourite fixed point theorem.

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Composing with $A \twoheadrightarrow A/I$, we get a non-faithful μ-stationary state $\psi \in S(A)$.

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Can extend to a μ-stationary state $\widetilde{\psi} \in S(C(Y) \rtimes_r G)$ by your favourite fixed-point theorem again. Necessarily non-faithful. ■

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