Non-Commutative Stochastic Processes and Bi-Free Probability

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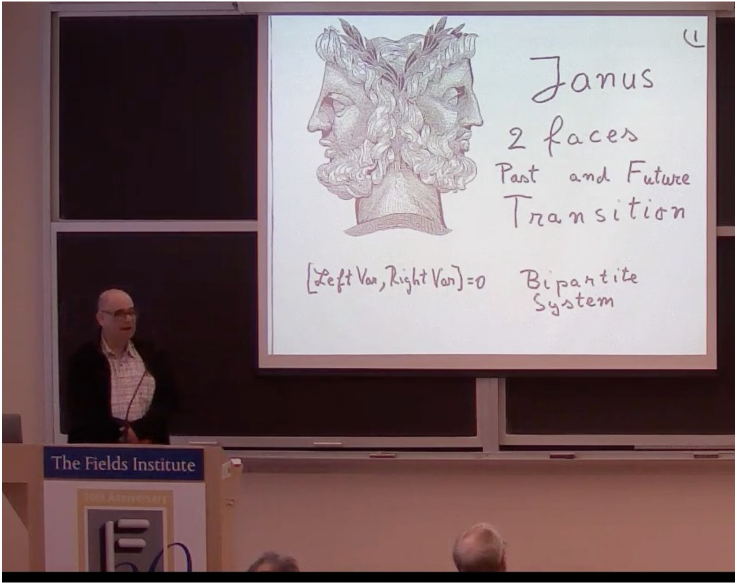
Janus



Janus
2 faces
Past and Future
Transition
[Left Var, Right Var] = 0
Bipartite System
The Fields Institute

Non-Commutative Stochastic Processes

Definition

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space (NCPS); that is, $\mathcal{A}$ is a unital C*-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital positive linear functional. A *self-adjoint non-commutative stochastic process* (SA-NC-SP) is a collection $(X_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$. The index set T is considered a time parameter.

Free Gaussian Markov Processes

Example, (Bożejko, Kummerer, Speicher; 1997)

- $\mathcal{H}$ a real Hilbert space, $\mathcal{H}_{\mathbb{C}}$ the complexification of $\mathcal{H}$.
- $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ the Fock space associated to $\mathcal{H}_{\mathbb{C}}$.
- $\tau : \mathcal{B}(\mathcal{F}(\mathcal{H}_{\mathbb{C}})) \to \mathbb{C}$ the vacuum vector state.

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- $(f_t)_{t\in T}$ a set of vectors in $\mathcal{H}$ with index set T.
- A *free (centred) Gaussian Markov process* is $(X_t)_{t\in T}$ where

$$X_t = l(f_t) + l^*(f_t)$$

where l and l^* are the left creation and annihilation operators.

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  - free Brownian bridge: $c(\ell, r) = \ell(1 - r)$ for $\ell \leq r$ with $T = [0, 1]$.
  - free Ornstein-Uhlenbeck process: $c(\ell, r) = e^{-|\ell - r|}$ with $T = \mathbb{R}$.

Transition Operators

Bożejko, Kummerer, and Speicher compute the *transition operators* of such processes.

Definition

Let $(X_t)_{t\in T}$ be a SA-NC-SP in a tracial von Neumann algebra $(\mathfrak{M}, \tau)$. For $t \in T$, let μ_t be the distribution of X_t. Note $W^*(X_t)$ is isomorphic to $L_\infty(\mu_t)$.

For $\ell, r \in T$ with $\ell \leq r$, an operator $K_{\ell,r} : L_\infty(\mu_r) \to L_\infty(\mu_\ell)$ where

$$E_{W^*(X_\ell)}(h(X_r)) = (K_{\ell,r}(h))(X_\ell)$$

for all Borel $h \in L_\infty(\mu_r)$ is called a *transition operator* of the process $(X_t)_{t\in T}$.

A Comparison

With $\lambda_t = \sqrt{c(t,t)}$ for $t \in \{\ell, r\}$ and $\lambda_{\ell,r} = \frac{c(\ell,r)}{\lambda_\ell \lambda_r}$, the transition operators of free Gaussian Markov processes are integration against

$$\frac{\frac{1}{2\pi\lambda_r^2}(1-\lambda_{\ell,r}^2)\sqrt{4\lambda_r^2 - y^2}\,dy}{(1-\lambda_{\ell,r}^2)^2 - \lambda_{\ell,r}(1+\lambda_{\ell,r}^2)\left(\frac{x}{\lambda_\ell}\right)\left(\frac{y}{\lambda_r}\right) + \lambda_{\ell,r}^2\left(\left(\frac{x}{\lambda_\ell}\right)^2 + \left(\frac{y}{\lambda_r}\right)^2\right)}.$$

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The density of the bi-free central limit distribution with left covariance $c(\ell,\ell)$, right covariance $c(r,r)$, and mixed covariance $c(\ell,r)$ is

$$\frac{\frac{1}{4\pi^2\lambda_\ell^2\lambda_r^2}\left(1-\lambda_{\ell,r}^2\right)\sqrt{4\lambda_\ell^2 - x^2}\sqrt{4\lambda_r^2 - y^2}\,dx\,dy}{\left(1-\lambda_{\ell,r}^2\right)^2 - \lambda_{\ell,r}\left(1+\lambda_{\ell,r}^2\right)\left(\frac{x}{\lambda_\ell}\right)\left(\frac{y}{\lambda_r}\right) + \lambda_{\ell,r}^2\left(\left(\frac{x}{\lambda_\ell}\right)^2 + \left(\frac{y}{\lambda_r}\right)^2\right)}.$$

Free Transformations

- R-Transform: $R_X(z) = \sum_{n\geq 0} \kappa_{n+1}(X) z^n$.
- K-Transform: $K_X(z) = \frac{1}{z} + R_X(z)$.
- R-Transform for a Semicircular Operator: $R_S(z) = \varphi(S^2)z$.

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- Cauchy Transform: $G_X(z) = \varphi((z-X)^{-1}) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu_X(x)$.
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- Additivity of R-Transforms: If X and X' are freely independent, $R_{X+X'}(z) = R_X(z) + R_{X'}(z)$.

Bi-Free Transformations

[(Voiculescu; 2016), (S; 2016), (Huang, Wang; 2016)]

- Reduced Bi-Free Partial R-Transform:
  $\tilde{R}_{X,Y}(z,w) = \sum_{n,m\geq 1} \kappa_{n,m}(X,Y) z^n w^m$.
- Reduced Bi-Free Partial R-Transform for a Semicircular Pair:
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- Green's Function:
  $G_{X,Y}(z,w) = \varphi((z-X)^{-1}(w-Y)^{-1}) = \int_{\mathbb{R}^2} \frac{1}{z-x}\frac{1}{w-y}\, d\mu_{X,Y}(x,y)$.
- Cauchy Inversion:
  $d\mu_{X,Y}(x,y) = \lim_{\epsilon \searrow 0} \frac{1}{\pi^2} \Im\left(\frac{G_{X,Y}(x+i\epsilon,y+i\epsilon) - G_{X,Y}(x+i\epsilon,y-i\epsilon)}{2i}\right)$.

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- Additivity of R-Transforms: If (X,Y) and (X',Y') are bi-freely independent, $\tilde{R}_{X+X',Y+Y'}(z,w) = \tilde{R}_{X,Y}(z,w) + \tilde{R}_{X',Y'}(z,w)$.

Bi-Freeness and Transition Operators

- $(\mathfrak{M}, \tau)$ a tracial von Neumann algebra and $X_\ell, X_r \in \mathfrak{M}$ self-adjoint.
- $E : \mathfrak{M} \to W^*(X_\ell)$ trace-preserving conditional expectation.

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$$\tau(E(S)X_\ell^n) = \tau(SX_\ell^n) = \tau(X_\ell^n S).$$
- $L_2(\mathfrak{M}, \tau)$ GNS Hilbert space, $\xi = 1_{\mathfrak{M}} \in L_2(\mathfrak{M}, \tau)$.
- For $S \in \mathfrak{M}$ let $L(S)$ and $R(S)$ denote the left and right actions of S on $L_2(\mathfrak{M}, \tau)$ respectively.

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- Then $\tau(X_\ell^n X_r^m) = \langle L(X_\ell)^n R(X_r)^m \xi, \xi \rangle$.

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- Then $\tau(X_\ell^n X_r^m) = \langle L(X_\ell)^n R(X_r)^m \xi, \xi \rangle$.
- Hence if $d\mu(x, y) = f_{\ell,r}(x, y)\, dx\, dy$, then the transition operator $K_{\ell,r} : L_\infty(\mu_r) \to L_\infty(\mu_\ell)$ is obtained via
$$(K_{\ell,r}(h))(x) = \int_\Omega h(y) k_{\ell,r}(x, dy)$$
where
$$k_{\ell,r}(x, dy) = \frac{f_{\ell,r}(x, y)}{f_\ell(x)}\, dy.$$

Free Poisson Process

Example

- $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra.
- $I \mapsto P_I$ a projection valued process; that is, this map is normal, projection valued, if $I, J \subseteq [0,1]$ are disjoint then $P_I P_J = 0$ and $P_I + P_J = P_{I \cup J}$, and $\tau(P_I) = |I|$ for all $I \subseteq [0,1]$ where $|I|$ denotes the Lebesgue measure of I.

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- S centred semicircular free from $\{P_I \mid I \subseteq [0,1]\}$.
- $X_t = SP_{[0,t)}S$ is called a *free Poisson process*.
- The transition operator is determined via the bi-free compound Poisson distribution (Gu, Huang, Mingo; 2016) with rate $\lambda = r$ and jump size $\nu = \frac{\ell}{r}\delta_{(1,0)} + \frac{r-\ell}{r}\delta_{(1,1)}$.

Freely Additive Increments

Definition

A SA-NC-SP $(X_t)_{t \in T}$ in a NCPS $(\mathcal{A}, \varphi)$ is said to have *freely additive increments* if for all $t_1 < t_2 < \cdots < t_n$ in T, the operators $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are freely independent.

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Theorem (S; 2022)

Let X and Y be freely independent self-adjoint operators in a tracial von Neumann algebra $(\mathfrak{M}, \tau)$. Then

$$G_{L(X),R(X+Y)}(z, w) = -\frac{G_X(z) - G_{X+Y}(w)}{z - K_X(G_{X+Y}(w))}.$$

In particular if $(X_t)_{t\in T}$ is a self-adjoint non-commutative stochastic process with freely additive increments, the above holds for $X = X_\ell$ and $Y = X_r - X_\ell$ for all $\ell < r$.

Free Cauchy Process

Example (Biane; 1998)

The *free Cauchy process* is the SA-NC-SP with freely additive increments where

$$d\mu_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}\,dx.$$

One can use the above to compute the joint density of $(L(X_\ell), R(X_r))$ to be

$$f_{\ell,r}(x,y) = \frac{1}{\pi^2}\frac{\ell}{x^2+\ell^2}\frac{r-\ell}{(x-y)^2+(r-\ell)^2}$$

and thus

$$k_{\ell,r}(x,dy) = \frac{f_{\ell,r}(x,y)}{f_\ell(x)}\,dy = \frac{1}{\pi}\frac{r-\ell}{(x-y)^2+(r-\ell)^2}dy.$$

Freely Adding NC-SP

Theorem (S; 2022)

Let X_1, X_2, Y_1, Y_2 be self-adjoint operators in a tracial von Neumann algebra $(\mathfrak{M}, \tau)$ such that $\mathrm{alg}(\{X_1, Y_1\})$ and $\mathrm{alg}(\{X_2, Y_2\})$ are freely independent. Thus $G_{X_1+X_2}(z)$ and $G_{Y_1+Y_2}(w)$ can be computed. With

$$\omega_{X_k}(z) = K_{X_k}(G_{X_1+X_2}(z)) \quad \textit{and} \quad \omega_{Y_k}(w) = K_{Y_k}(G_{Y_1+Y_2}(w))$$

for $k = 1, 2$, we have

$$\frac{1}{G_{X_1+X_2, Y_1+Y_2}(z, w)} + \frac{1}{G_{X_1+X_2}(z) G_{Y_1+Y_2}(w)} = \frac{1}{G_{X_1, Y_1}(\omega_{X_1}(z), \omega_{Y_1}(w))} + \frac{1}{G_{X_2, Y_2}(\omega_{X_2}(z), \omega_{Y_2}(w))}.$$

Thus the transition operator of $Y_1 + Y_2$ onto $X_1 + X_2$ can be computed.

Thanks for Listening!