Supercritical equilibrium states on a C*-algebra from number theory

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Let A be a C*-algebra, σ_t a strongly-continuous $\mathbb{R}$-action, and $\beta \in \mathbb{R}$.

Definition

A KMS_β *state* on (A, σ_t) is a state ϕ such that

$$\phi(xy) = \phi(y\sigma_{i\beta}(x))$$

for all x, y in a dense subalgebra of A.

β is the inverse temperature. The set of KMS_β states is a Choquet simplex.

Example

$$A = M_n(\mathbb{C}), \quad \sigma_t(x) = e^{itH} x e^{-itH}, \quad H \geq 0,$$

$$\phi(x) = \frac{\operatorname{Tr}(xe^{-\beta H})}{\operatorname{Tr}(e^{-\beta H})}.$$

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$$\begin{aligned}
\mathrm{Tr}(xy\ e^{-\beta H}) &= \mathrm{Tr}(y\ e^{-\beta H}\ x) \\
&= \mathrm{Tr}(y\ (e^{-\beta H}\ x\ e^{\beta H})\ e^{-\beta H}) \\
&= \mathrm{Tr}(y \sigma_{i\beta}(x)\ e^{-\beta H}).
\end{aligned}$$

- U a unitary, $\{V_a : a \in \mathbb{N}^\times\}$ commuting isometries satisfying

$$\begin{aligned} UV_a &= V_a U^a, \\ V_a V_b &= V_{ab}, \\ V_a^* V_b &= V_b V_a^* \text{ when } \gcd(a,b) = 1. \end{aligned}$$

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- Our algebra:

$$\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) = C^*_{univeral}(V_a, U \ : \ a \in \mathbb{N}^\times)$$
$$= \overline{\text{span}}\{V_a U^n V_b^* \ : \ a, b \in \mathbb{N}^\times, n \in \mathbb{Z}\}.$$

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- $\mathbb{R}$-action:

$$\sigma_t(V_a) = a^{it} V_a, \quad \sigma_t(U) = U.$$

Applying the Fourier transform to U allows us to substitute $\sum \lambda_n V_a U^n V_b^*$ with $V_a f V_b^*$, $f \in C(\mathbb{T})$.

The relation $UV_a = V_a U^a$ becomes $fV_a = V_a f \circ \omega_a$, where

$$\omega_a : \mathbb{T} \to \mathbb{T}, \quad \omega_a(z) = z^a.$$

Low temperature equilibrium

Low temperature KMS_β states ($\beta > 1$) can all be computed using zeta functions as follows (an Huef-Laca-Raeburn):

- For η a probability measure on the circle $\mathbb{T}$, the function

$$\phi_{\eta,\beta}(V_a U^n V_b^*) = \delta_{a,b} \frac{a^{-\beta}}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} \int_{\mathbb{T}} z^{cn} d\eta$$

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  extends to a KMS_β state.

- Equivalently,

$$\psi_{\nu,\beta}(V_a U^n V_b^*) = \delta_{a,b} a^{-\beta} \int_{\mathbb{T}} z^n d\nu,$$

  where

$$\nu = T_\beta \eta = \frac{1}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} \omega_{c*}\eta.$$

The formula for $\psi_{\nu,\beta}$ is well-defined for all $\beta > 0$, but may not extend to a state.

Theorem

The map $\nu \mapsto \psi_{\nu,\beta}$ is an affine isomorphism between the KMS_β states on $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma_t)$ and probability measures ν on $\mathbb{T}$ satisfying

$$\sum_{d|n} \mu(d) d^{-\beta} \omega_{d*}\nu \geq 0 \tag{1}$$

for all $n \in \mathbb{N}^\times$, where μ is the Möbius function.

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We call a measure satisfying condition (1) β-*subconformal*. A more general form of this was investigated by Afsar, Larsen, and Neshveyev.

Subconformal measures

When $\beta > 1$, ν is β-subconformal if and only if $\nu = T_\beta \eta$ for some probability measure η.

$$T_\beta \eta = \frac{1}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} \omega_{c*} \eta = \frac{1}{\zeta(\beta)} \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-\beta} \omega_{p*}} \right) \eta$$

$$\sum_{d|n} \mu(d) d^{-\beta} \omega_{d*} \nu = \left(\prod_{p|n} 1 - p^{-\beta} \omega_{p*} \right) \nu.$$

These are inverse operations, up to scaling.

Formulas

Some functions from number theory:

$$\varphi(n) = \text{ of integers } 1 \leq k < n \text{ with } \gcd(k, n) = 1$$

$$= n \prod_{p|n} 1 - p^{-1},$$

$$\varphi_\beta(n) = n^\beta \prod_{p|n} 1 - p^{-\beta},$$

$$\mathrm{ord}(z) = \inf\{n : \ z^n = 1\}.$$

Theorem

For $\beta \leq 1$, the simplex of β-subconformal measures is affinely isomorphic to the simplex of probability measures on $\mathbb{N}^\times \cup \{\infty\}$. This isomorphism sends δ_∞ to Haar measure and δ_n to the finitely-supported measure defined by

$$\nu_{n,\beta}(\{z\}) = \begin{cases} n^{-\beta} \frac{\varphi_\beta(\mathrm{ord}(z))}{\varphi(\mathrm{ord}(z))} & \text{if } \mathrm{ord}(z) \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary

Haar measure is the unique non-atomic β-subconformal measure for $\beta \leq 1$.

The phase transition

The extremal β-subconformal measures are parameterized by $\mathbb{T}$ for $\beta > 1$ and by $\mathbb{N}^\times \cup \{\infty\}$ for $\beta \leq 1$.

How do we transition from a connected space to one which is disconnected?

If $\operatorname{ord}(z) = \infty$, then integrating:

$$\int_{\mathbb{T}} z^n d\left(T_\beta \delta_z\right) = \frac{\mathrm{Li}_\beta(z^n)}{\zeta(\beta)}, \quad \text{where } \mathrm{Li}_\beta(z) = \sum_{c=1}^{\infty} z^c c^{-\beta}.$$

This tends to $\delta_{n,0}$ as $\beta \to 1^+$, which is the integral with Haar measure.

The phase transition

If $\operatorname{ord}(z) = n < \infty$, then

$$T_\beta \delta_z = \sum_{k=1}^{n} n^{-\beta} \cdot \frac{\zeta(\beta, \frac{k}{n})}{\zeta(\beta)} \delta_{z^k}, \quad \text{where } \zeta(\beta, s) = \sum_{c=0}^{\infty} (c+s)^{-\beta}.$$

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Compare this to the formula

$$\nu_{n,\beta} = \sum_{k=1}^{n} n^{-\beta} \cdot \frac{\varphi_\beta\left(\mathrm{ord}(z^k)\right)}{\varphi\left(\mathrm{ord}(z^k)\right)} \delta_{z^k} \quad (\mathrm{ord}(z^k) = n/\gcd(n,k)).$$

These agree in the limit $\beta \to 1$.

The phase transition

Putting this together, we can now describe the phase-transition:

$$\lim_{\beta \to 1^+} T_\beta \delta_z = \begin{cases} \nu_{n,1} & \text{if } \mathrm{ord}(z) = n \in \mathbb{N}^\times, \\ \text{Haar measure} & \text{if } \mathrm{ord}(z) = \infty. \end{cases}$$

The topology of $\mathbb{T}$ is completely forgotten in the phase transition!

Future work

This is the first non-trivial example of phase transition with non-unique supercritical equilibrium from number theory that we are aware of.

Future work will examine the phase transitions for more general number fields than $\mathbb{Q}$, where we expect similar behaviour.