Relative Cohomology for Operator Modules

joint work with Martin Mathieu

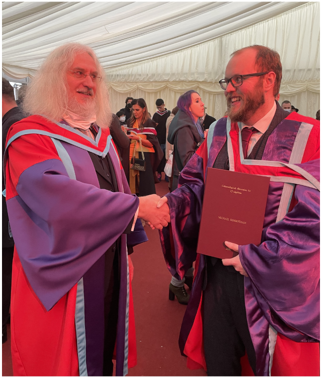
COSy 2022

Michael Rosbotham

Queen's University Belfast
& Carleton University

2022 MAY 31

This is joint work with Martin Mathieu at Queen's University Belfast.



Exact structures for operator modules. Canadian Journal of Mathematics (2022).

I have been able to work at Carleton University thanks to a *London Mathematical Society Early Career Fellowship* with support from Heilbronn Institute for Mathematical Research.

Definitions

Let A (unital, Banach algebra) be equipped with an operator space structure (an assignment of norms $\|\cdot\|_n$ on $M_n(A)$ satisfying Ruan's axioms)

- A is a (unital) operator algebra if

$$\|ab\|_n \leq \|a\|_n \|b\|_n$$

$\forall a, b \in M_n(A)$ and $n \in \mathbb{N}$.
(Equivalently, $m\colon A \times A \to A, (a, b) \mapsto ab$ induces a completely contractive linear map $A \otimes_h A \to A$).

- A is a (unital) completely contractive Banach algebra if

$$\|[a_{ij} b_{k\ell}]_{(ij)(k\ell)}\|_{nm} \leq \|[a_{ij}]\|_n \|[b_{k\ell}]\|_m$$

$\forall [a_{ij}] \in M_n(A), [b_{k\ell}] \in M_m(A)$ and $n, m \in \mathbb{N}$.
(Equivalently, $m\colon A \times A \to A, (a, b) \mapsto ab$ induces a completely contractive linear map $A \widehat{\otimes} A \to A$).

Operator (space) modules

- A - completely contractive Banach algebra;
- E - unital right A-module equipped with (complete) operator space structure;

E is a (right, unital) hA-module if

$$m\colon E \times A \to E, \quad (x, a) \mapsto x \cdot a$$

induces a completely contractive linear map $E \otimes_h A \to E$.

$h\mathscr{M}od_A^\infty$: category of right (unital) hA-modules with completely bounded A-module maps.

E is a (right, unital) matrix normed (m.n.) A-module over A if

$$m\colon E \times A \to E, \quad (x, a) \mapsto x \cdot a$$

induces a completely contractive linear map $E \widehat{\otimes} A \to E$.

$mn\mathscr{M}od_A^\infty$: category of right $m.n.A$-modules with completely bounded A-module maps.

Examples: right unital hA-modules

For A unital operator algebra, E operator space.

- A with $a \cdot a' = aa'$.
- $E \otimes_h A$ with $(x \otimes a) \cdot a' := x \otimes (aa')$.

Examples: right unital $m.n.A$-modules

For A unital completely contractive Banach algebra, E operator space.

- A with $a \cdot a' = aa'$.
- $E \widehat{\otimes} A$ with $(x \otimes a) \cdot a' := x \otimes (aa')$.
- $CB(A, E)$ with $(T \cdot a)(b) = T(ab)$.

Examples: right unital hA-modules

For A unital operator algebra, E operator space.

- A with $a \cdot a' = aa'$.
- $E \otimes_h A$ with $(x \otimes a) \cdot a' := x \otimes (aa')$.

Examples: right unital $m.n.A$-modules

For A unital completely contractive Banach algebra, E operator space.

- A with $a \cdot a' = aa'$.
- $E \widehat{\otimes} A$ with $(x \otimes a) \cdot a' := x \otimes (aa')$.
- $CB(A, E)$ with $(T \cdot a)(b) = T(ab)$.

CES Representation Theorem [Christensen–Effros–Sinclair (1987)]

Let A be a unital operator algebra, E and operator space and $E \in \mathscr{Mod}_A$.
If $E \in h\mathscr{Mod}_A^\infty$, there exist Hilbert space H, a complete isometry $\phi\colon E \to B(H)$ and a completely isometric unital algebra homomorphism $\pi\colon A \to B(H)$ such that

$$\phi(x \cdot a) = \phi(x)\pi(a), \quad \forall a \in A, x \in E.$$

Let $\mathscr{A}$ be a category, $\mathscr{M}$ a class of monomorphisms, and $\mathscr{P}$ a class of epimorphisms.

$I \in \mathscr{A}$ is *$\mathscr{M}$-injective* if any morphism whose codomain is I, can be extended along morphisms in $\mathscr{M}$

$$\begin{tikzcd} E \arrow[r, "\forall \mu \in \mathscr{M}"] \arrow[d, "\forall f"'] & F \arrow[dl, dashed, "\exists \widetilde{f}"] \\ I & \end{tikzcd}$$

$P \in \mathscr{A}$ is *$\mathscr{P}$-projective* if if any morphism whose domain is P, can be lifted over morphisms in $\mathscr{P}$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \widetilde{f}"'] \arrow[d, "\forall f"] \\ E \arrow[r, "\forall \pi \in \mathscr{P}"'] & F \end{tikzcd}$$

Relative Homological Algebra

Idea: Focus on extensions along (and liftings over) morphisms that behave well under a forgetful functor.

Algebraic Module Categories: Hochschild 1950s

Additive Categories: Eilenberg–Moore 1960s

Banach and Topological Algebras: Taylor 1970s, Johnson, Helemskii, Selivanov, ...

Operator Algebras: Paulsen 1990s, Aristov, Wood, ...

Theorem [Helemskii, 1980s]

Let A be a unital C^*-algebra.
Every right Banach A-module is relatively projective if and only if A is classically semisimple.

In additive category $\mathscr{A}$, $\quad E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$

is a kernel-cokernel pair if μ is a kernel of π, and π is a cokernel of μ.

μ a kernel:

$$
\begin{tikzcd}
K \arrow[r, "\mu"] \arrow[rr, bend left, "0"] & E \arrow[r, "f"] & F \\
& G \arrow[ul, dashed, "\exists!\overline{g}"] \arrow[u, "\forall g"'] \arrow[ur, "0"'] &
\end{tikzcd}
$$

π a cokernel:

$$
\begin{tikzcd}
E \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & F \arrow[r, "\pi"] \arrow[d, "\forall g"] & C \arrow[dl, dashed, "\exists!\overline{g}"] \\
& G &
\end{tikzcd}
$$

In additive category $\mathscr{A}$, $\quad E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$

is a kernel-cokernel pair if μ is a kernel of π, and π is a cokernel of μ.

μ a kernel:

$$
\begin{tikzcd}
K \arrow[r, "\mu"] \arrow[rr, bend left, "0"] & E \arrow[r, "f"] & F \\
& G \arrow[u, "\forall g"'] \arrow[ur, "0"'] \arrow[ul, dashed, "\exists ! \overline{g}"] &
\end{tikzcd}
$$

π a cokernel:

$$
\begin{tikzcd}
E \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & F \arrow[r, "\pi"] \arrow[d, "\forall g"] & C \arrow[dl, dashed, "\exists ! \overline{g}"] \\
& G &
\end{tikzcd}
$$

In $\mathscr{hMod}_A^\infty$ and $\mathscr{mnMod}_A^\infty$:

$E \overset{\mu}{\rightarrowtail} F$ kernel $\iff$ μ has closed range and completely bounded inverse $\mu^{-1}\colon \mu(E) \to E$

$F \overset{\pi}{\twoheadrightarrow} G$ cokernel $\iff$ π is a completely open mapping

For $\mathscr{A}$ an additive category and $(\mathscr{M}, \mathscr{P})$ a class of kernel-cokernel pairs:

$\mathscr{E}x = (\mathscr{M}, \mathscr{P})$ is an exact structure (in the sense of Quillen) if:

- [E0] For all $E \in \mathscr{A}$, $\mathrm{id}_E \in \mathscr{M}$.
- [E0$^{\mathrm{op}}$] For all $E \in \mathscr{A}$, $\mathrm{id}_E \in \mathscr{P}$.
- [E1] $\mathscr{M}$ is closed under composition.
- [E1$^{\mathrm{op}}$] $\mathscr{P}$ is closed under composition.
- [E2] The pushout of a morphism in $\mathscr{M}$ along an arbitrary morphism exists and yields a morphism in $\mathscr{M}$.
- [E2$^{\mathrm{op}}$] The pullback of a morphism in $\mathscr{P}$ along an arbitrary morphism exists and yields a morphism in $\mathscr{P}$.

In this case we say $(\mathscr{A}, \mathscr{E}x)$ is an exact category.

T. Bühler. *Exact categories.* Expo. Math., 28(1) 1-69, 2010.

Theorem [Ara-Mathieu], [Mathieu-R. 2022]

Let A be an operator algebra. The class $\mathscr{E}x_{\mathrm{max}}$ of **all** kernel-cokernel pairs forms an exact structure on $h\mathscr{M}od_A^\infty$ and $mn\mathscr{M}od_A^\infty$.

Exact categories: $(h\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathrm{max}}), (mn\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathrm{max}})$

Every additive category has the minimal exact structure $\mathscr{E}x_{\mathsf{min}}$ of the split kernel-cokernel pairs.

Split:

$$\begin{tikzcd} E \arrow[r, tail, "\mu"] & F \arrow[l, dashed, bend left, "\exists\,\nu"] \arrow[r, two heads, "\pi"] & G \arrow[l, dashed, bend left, "\exists\,\theta"] \end{tikzcd}$$

such that $\nu\mu = \mathsf{id}_E$, $\pi\theta = \mathsf{id}_G$ and $\mu\nu + \theta\pi = \mathsf{id}_F$.

Exact categories: $(\hbar\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathsf{min}}), (mn\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathsf{min}})$

Proposition

Let $\mathsf{F}\colon (\mathscr{A}, \mathscr{E}x) \longrightarrow \mathscr{B}$ be the forgetful functor.
The class of kernel-cokernel pairs

$$\mathscr{E}x_{\mathsf{rel}} := \{(\mu, \pi) \in \mathscr{E}x \mid (\mathsf{F}\mu, \mathsf{F}\pi) \in \mathscr{E}x_{\mathsf{min}}\}$$

forms an exact structure on $\mathscr{B}$. The relative exact structure.
We write $\mathscr{E}x_{\mathsf{rel}} = (\mathscr{M}_{\mathsf{rel}}, \mathscr{P}_{\mathsf{rel}})$.

$\mathsf{F}_{\hbar}\colon (\hbar\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathsf{max}}) \to \mathscr{O}p^{\infty}$ and $\mathsf{F}_{mn}\colon (mn\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathsf{max}}) \to \mathscr{O}p^{\infty}$

yield exact categories $(\hbar\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathsf{rel}})$ and $(mn\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathsf{rel}})$.

An object I in an exact category with the relative exact structure is relatively injective if I is $\mathscr{M}_{\mathsf{rel}}$-injective.

An object P in an exact category with the relative exact structure is relatively projective if P is $\mathscr{P}_{\mathsf{rel}}$-projective.

Cohomological dimension in an Exact Category

$(\mathscr{A}, \mathscr{E}x)$ be an exact category with $\mathscr{E}x = (\mathscr{M}, \mathscr{P})$.
Let $E \in \mathscr{A}$. $\mathrm{Inj}_{\mathscr{M}}$-dim(E) is the smallest $n \geq 0$ such that there exists:

$$\begin{tikzcd}
E \arrow[rr, tail] \arrow[dr, two heads, "\mathrm{id}_E"'] & & I^0 \arrow[rr] \arrow[dr, two heads] & & \cdots \arrow[rr] \arrow[dr, two heads] & & I^{n-1} \arrow[rr, two heads] \arrow[dr, two heads] & & I^n \\
& K_0 \arrow[ur, tail] & & K_1 \arrow[ur, tail] & & K_{n-1} \arrow[ur, tail] & & K_n \arrow[ur, tail, "\mathrm{id}_{I^n}"'] &
\end{tikzcd}$$

where I^m all $\mathscr{M}$-injective and

$$K_m \rightarrowtail I^m \twoheadrightarrow K_{m+1} \qquad \text{in } \mathscr{E}x.$$

cohomological dimension of $(\mathscr{A}, \mathscr{E}x)$ is

$$\mathrm{cohomdim}(\mathscr{A}, \mathscr{E}x) := \sup\{\mathrm{Inj}_{\mathscr{M}}\text{-dim}(E) \mid E \in \mathscr{M}od_A\}$$

Theorem [Mathieu-R. 2022]

Let A be a unital operator algebra. The following are equivalent:

- A is classically semisimple;
- $\mathrm{cohomdim}(h\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathrm{rel}}) = 0$;
- $\mathrm{cohomdim}(mn\mathscr{M}od_A^{\infty}, \mathscr{E}x_{\mathrm{rel}}) = 0$.

Theorem [Mathieu-R. 2022]

Let A be a unital operator algebra. The following are equivalent:

- A is classically semisimple;
- $\mathrm{cohomdim}(\mathscr{hMod}_A^\infty, \mathscr{Ex}_{\mathrm{rel}}) = 0$;
- $\mathrm{cohomdim}(\mathscr{mnMod}_A^\infty, \mathscr{Ex}_{\mathrm{rel}}) = 0$.

Proof:

$$\begin{tikzcd}
A \cong_{\mathrm{cb}} \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}) \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow] & & A \text{ is classically semisimple} \\
\begin{array}{l}\text{Every object} \\ \text{in } \mathscr{mnMod}_A^\infty \\ \text{is relatively injective}\end{array} \arrow[r, Rightarrow] & \begin{array}{l}\text{Every object} \\ \text{in } \mathscr{hMod}_A^\infty \\ \text{is relatively injective}\end{array} \arrow[r, Leftrightarrow] & \begin{array}{l}\text{Every object} \\ \text{in } \mathscr{hMod}_A^\infty \\ \text{is relatively projective}\end{array} \arrow[u, Rightarrow]
\end{tikzcd}$$