Structures of crossed product C*-algebras

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Canadian Operator Symposium, 50th anniversary
Ottawa, June 2, 2022

Consider a free and minimal topological dynamical system (X, Γ), where

- X: compact metric space
- Γ: a countable discrete amenable group acting on X.

The crossed product C*-algebra

$$\mathrm{C}(X) \rtimes \Gamma$$

is simple unital separable stably finite C^*-algebra. Let us consider its structures such as

Classifiability, Comparison, and Stable Rank.

Uniform Rokhlin Property (URP)

Definition

A Rokhlin tower of (X, Γ) consists of a set B and a finite set $\Gamma_0 \subseteq \Gamma$ such that

$$B\gamma, \quad \gamma \in \Gamma_0,$$

are disjoint.

B is called the base set, and if B is open (closed), then (B, Γ_0) is called an open (closed) tower.

Remark

$$(B, \Gamma_0) \quad \sim \quad \mathrm{M}_{|\Gamma_0|}(\mathrm{C}_0(B))$$

Uniform Rohklin Property (URP)

A topological dynamical system (X, Γ) is said to have Uniform Rohklin Property (URP) if for any $\varepsilon > 0$ and any finite set $\mathcal{F} \subseteq \Gamma$, there are mutually disjoint open towers

$$(B_1, \Gamma_1), ..., (B_S, \Gamma_S)$$

such that

1. Γ_s, $s = 1, ..., S$, are $(\mathcal{F}, \varepsilon)$-invariant,
2. $$\mu(X \setminus \bigcup_{s=1}^{S} \bigcup_{\gamma \in \Gamma_s} B_s\gamma) < \varepsilon, \quad \mu \in \mathcal{M}_1(X, \Gamma).$$

Cuntz Comparison of Open sets (COS)

A topological dynamical system (X, Γ) is said to have (λ, m)-Cuntz-comparison of open sets, where $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

one has

$$[E] < m[F] \quad \text{in } \mathrm{C}(X) \rtimes \Gamma,$$

where $[E]$ and $[F]$ are the Cuntz class of the open sets E and F respectively.
The dynamical system (X, Γ) is said to have Cuntz Comparison of Open sets (COS) if it has (λ, m)-Cuntz-comparison of open sets for some λ and m.

Theorem

Any free and minimal dynamical system $(X, \mathbb{Z}^d)$ *has the (URP) and (COS).*

Theorem

Any free and minimal dynamical system (X, Γ) *has the (URP) and (COS) if* Γ *has subexponential growth and* (X, Γ) *has a Cantor factor.*

Theorem

Let (X, Γ) be a free and minimal dynamical system with the (URP) and (COS). Then

1. *if (X, Γ) has zero mean dimension (or, equivalently, Small Boundary Property), then*

$$(\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \Gamma.$$

   In particular, $\mathrm{C}(X) \rtimes \Gamma$ is classifiable if (X, Γ) is uniquely ergodic.

2. $$\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) \leq \frac{1}{2}\mathrm{mdim}(X, \Gamma).$$

3. *(Li-N) The stable rank of $\mathrm{C}(X) \rtimes \Gamma$, classifiable or not, is always one. And $\mathrm{C}(X) \rtimes \Gamma$ is $\mathcal{Z}$-stable if, and only if, $\mathrm{C}(X) \rtimes \Gamma$ has strict comparison of positive elements (so it satisfies the Toms-Winter conjecture).*

Corollary

1. *If* $(X, \mathbb{Z}^d)$ *has zero mean dimension (or, equivalently, Small Boundary Property), then*

$$(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \mathbb{Z}^d.$$

2. $$\mathrm{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, \mathbb{Z}^d).$$

3. *(Li-N) The stable rank of* $\mathrm{C}(X) \rtimes \mathbb{Z}^d$*, classifiable or not, is always one. And* $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ *is* $\mathcal{Z}$*-stable if, and only if,* $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ *has strict comparison of positive elements (so* $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ *satisfies the Toms-Winter conjecture).*

Proof: Tracial Approximation

Theorem

If (X, Γ) has the (URP), then, for any finite set $\mathcal{F} \subseteq \mathrm{C}(X) \rtimes \Gamma$ and any $\varepsilon > 0$, there are $p : X \to [0, 1]$ and $C \subseteq \mathrm{C}(X) \rtimes \Gamma$ such that

$$C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{|\Gamma_s|}(\mathrm{C}_0(B_s)),$$

and

1. $\|[p, f]\| < \varepsilon$, $f \in \mathcal{F}$,
2. $pfp \in_{\varepsilon} C$, $f \in \mathcal{F}$,
3. $\mathrm{d}_\tau(1 - p) < \varepsilon$, $\tau \in \mathrm{T}(\mathrm{C}(X) \rtimes \Gamma)$,

$\vdots$

Remark

In general, the cutting function p is not a projection.

C*-dynamical systems

Definition

Let (A, Γ) be a C*-dynamical system, where A is a unital separable C*-algebra such that $\mathrm{T}(A) \neq \emptyset$, and Γ is a countable discrete amenable group. Then (A, Γ) is said to have the weak Rokhlin property (WRP) if for any exact tiling $\mathcal{T}$ of Γ with shapes $K_1, \ldots, K_S$, there are positive contractions $c_s \in A' \cap A_\omega, s = 1, ..., S$ such that

1. $\gamma(c_s)$, $\gamma \in \Gamma_s$, $s = 1, 2, ..., S$, are mutually orthogonal, and
2. $$1 - \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \gamma(c_s) \in J_{\mathcal{M}_1(A,\Gamma)},$$

   where $\mathcal{M}_1(A, \Gamma)$ is the simplex of invariant tracial states, and $J_{\mathcal{M}_1(A,\Gamma)}$ is the trace-kernel with respect to invariant traces $\mathcal{M}_1(A, \Gamma)$.

C*-dynamical systems

Definition

Let (A, Γ) be a C*-dynamical system, where Γ is discrete and amenable. It has the property (COS) if there exist $\gamma \in (0, 1]$ and $m \in \mathbb{N}$ such that for any $a, b \in A^+$ satisfying

$$\mathrm{d}_\tau(a) < \gamma \mathrm{d}_\tau(b), \quad \tau \in \mathcal{M}_1(A, \Gamma),$$

one has

$$a \precsim \underbrace{b \oplus \cdots \oplus b}_{m} \quad \text{in } A \rtimes \Gamma.$$

Theorem (Li-N-Wang)

Assume (A, Γ) has the (WRP) and (COS). Then $A \rtimes \Gamma$ can be weakly tracially approximated by algebras $\mathrm{M}_n(hAh)$.

Theorem (LNW)

Let (A, Γ) be a C-dynamical system with the (WRP), where A is a unital simple C*-algebra which is tracially $\mathcal{Z}$-absorbing and Γ is a countable discrete amenable group. Then $A \rtimes \Gamma$ is tracially $\mathcal{Z}$-stable.*

Theorem (LNW)

Let (A, Γ) be a minimal C-dynamical system with (COS) and (WRP), where A is a unital C*-algebra with $\mathrm{T}(A) \neq \emptyset$ and Γ is a discrete amenable group. Assume that $|\Gamma| = \infty$. Then $A \rtimes \Gamma$ has stable rank one.*

Remark

The (URP) and (COS) can also be studied for groupoid C*-algebras (work in progress).

Thank you!