Phase transitions of C*-dynamical systems from number theory

Marcelo Laca
Victoria

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C*-dynamical systems and states

C*-dynamical system = pair (A, σ)

- A $=$ C$*$-algebra
- A^{sa} = *observables*
- $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$ $=$ time evolution, or *dynamics* on A:

$\sigma_0 = \mathrm{id}, \quad \sigma_s \circ \sigma_t = \sigma_{s+t},$ and $t \mapsto \sigma_t(a)$ norm continuous

states of A = linear functionals $\varphi : A \to \mathbb{C}$ such that

$$\varphi(a^* a) \geqslant 0 \quad \text{and} \quad \|\varphi\| = \varphi(1) = 1$$

$\varphi(\sigma_t(a))$ = *expectation value* of $a \in A^{sa}$ at $t \in \mathbb{R}$ in state φ.

These systems model the evolution of quantum physical systems (Heisenberg picture: states are fixed, observables evolve)

Example 0: finite quantum systems

Take $A = \mathrm{Mat}_n(\mathbb{C})$ $\qquad \sigma_t(a) = e^{itH} a e^{-itH}$ $\qquad \varphi(\cdot) = \mathrm{Tr}(\cdot\, Q_\varphi)$,

Hamiltonian $H \in A^{sa}$, $\quad$ density matrix $Q_\varphi \in A_1^+$

TFAE

1) $Q_\varphi = \frac{1}{\mathrm{Tr}(e^{-\beta H})} e^{-\beta H} \quad =: Q_{Gibbs}$

2) $\mathrm{Tr}(abQ_\varphi) = \mathrm{Tr}(be^{-\beta H} a e^{\beta H} Q_\varphi) \quad \forall\, a, b \in \mathrm{Mat}_n(\mathbb{C})$

(1 $\Longrightarrow$ 2) obvious,
(2 $\Longrightarrow$ 1) by linear algebra

Rewrite 2) as

2') $\varphi(ab) = \varphi(b\sigma_{i\beta}(a)) \quad \forall a, b \in \mathrm{Mat}_n(\mathbb{C})$.

This characterization of Gibbs (equilibrium) states for finite systems becomes the definition for general C*-algebraic systems

KMS equilibrium condition

(A, σ) $=$ C*-algebraic dynamical system

Definition [Haag-Hugenholtz-Winnink, 1967]:
A state φ on A is a σ- KMS$_\beta$ state (equiv. satisfies the KMS condition with respect to σ at inverse temperature $\beta \neq 0$) if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a)) \qquad \forall a, b \in A, \quad a \ \sigma\text{-analytic}$$

1) twisted-tracial condition, *‘twisted by σ along imaginary time’*

2) $a \in A$ is σ-analytic if $t \to \sigma_t(a)$ extends to an entire function

3) σ-analytic elements form a dense *-subalgebra.

Phase transition with spontaneous symmetry-breaking

Phase transition $=$ abrupt change of physical properties

e.g. think of water and magnets as temperature increases.

often the symmetry group of a pure phase becomes smaller as temperature decreases.

- a snowflake is less symmetric than a spherical drop of water.
- a ferromagnet exhibits spontaneous magnetization

In C*-algebraic terms:
the group of automorphisms of A that commute with σ acts on KMS$_\beta$-states with β-dependent action.

Example I: The Bost–Connes system $(\mathcal{C}_{\mathbb{Q}}, \sigma)$

Algebra $\mathcal{C}_{\mathbb{Q}} \cong C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^{\times} \big(\cong C(\prod_p \mathbb{Z}_p) \rtimes \mathbb{N}^{\times} \big)$

$$\text{generated by} \begin{cases} \text{semigroup of isometries} & \{\mu_n : n \in \mathbb{N}^{\times}\} \\ \text{group of unitaries} & \{e(r) : r \in \mathbb{Q}/\mathbb{Z}\} \end{cases}$$

$$\text{subject to} \qquad \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

Dynamics σ_t, $\begin{cases} \sigma_t(\mu_n) = n^{it} \mu_n & n \in \mathbb{N}^{\times} \\ \sigma_t(e(r)) = e(r) & r \in \mathbb{Q}/\mathbb{Z} \end{cases} \qquad t \in \mathbb{R}$

Symmetries θ_χ, $\begin{cases} \theta_\chi(\mu_n) = \mu_n & n \in \mathbb{N}^{\times} \\ \theta_\chi(e(r)) = e(\chi(r)) & r \in \mathbb{Q}/\mathbb{Z} \end{cases} \qquad \chi \in \mathrm{Aut}\, \mathbb{Q}/\mathbb{Z}$

Fact: $\mathrm{Aut}\, \mathbb{Q}/\mathbb{Z} \cong \prod_{\text{primes}} \mathbb{Z}_p^* \cong \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$

Bost-Connes phase transition for $(\mathcal{C}_{\mathbb{Q}}, \sigma)$

Theorem [Bost-Connes '95]

1. $0 < \beta \leqslant 1 \implies \exists!$ KMS$_\beta$ state; injective type III$_1$ factor, invariant under $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \mathrm{Aut}\,\mathbb{Q}/\mathbb{Z} \cong \prod_p \mathbb{Z}_p^*$.
2. $1 < \beta \leqslant \infty \implies$ extremal KMS$_\beta$ states $\phi_{\beta,\chi}$ parametrized by embeddings $\chi : \mathbb{Q}^{ab} \to \mathbb{C}$ of the maximal abelian extension of $\mathbb{Q}$ type I factor states with free transitive action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$
3. partition function $=$ Riemann zeta function.

Remarks

The B-C system exhibits a **phase transition with spontaneous symmetry breaking** of a $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ action.

$\mathcal{C}_{\mathbb{Q}}$ has an "arithmetic $\mathbb{Q}$-subalgebra" on which the extremal KMS$_\infty$ states give the **explicit embeddings** $\mathbb{Q}^{ab} \hookrightarrow \mathbb{C}$.

Explicit class field theory: Hilbert's 12th problem asks for the explicit embeddings of $K^{ab} \hookrightarrow \mathbb{C}$ for an algebraic number field K.

Example II: “$b + ax$" systems

K = algebraic number field, $\mathcal{O}_K$ = ring of algebraic integers

semigroup $\mathcal{O}_K \rtimes \mathcal{O}_K^\times$ = “$b + ax$" semigroup of the ring $\mathcal{O}_K$

formally: $\mathcal{O}_K \times \mathcal{O}_K^\times$ with $(b, a)(d, c) := (b + ad, ac)$.

C*-algebra $A = C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times)$ (Toeplitz-type C*-algebra) generated by the l. r. r.

$$T_{(b,a)}\xi_{(y,x)} = \xi_{(b+ay,ax)} \quad \text{on} \quad \ell^2(\mathcal{O}_K \rtimes \mathcal{O}_K^\times)$$

Dynamics σ given by $\sigma_t(T_{(b,a)}) = [\mathcal{O}_K : a\mathcal{O}_K]^{it}\, T_{(b,a)} \qquad t \in \mathbb{R}$

Notation: $S^b := T_{(b,1)}$ ($B \in \mathcal{O}_K$, add.) $V_a := T_{(0,a)}$ ($a \in \mathcal{O}_K^\times$, mult.)

Phase transition for $(C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times), \sigma)$

Theorem [Cuntz-Deninger-L.] cf. [L.-Raeburn $(K = \mathbb{Q})$]

1. $0 \leqslant \beta < 1 \implies \nexists$ KMS_β states
2. $1 \leqslant \beta \leqslant 2 \implies \exists!$ KMS_β state, type III_1 factor
3. $\beta > 2 \implies$ simplex of KMS_β states is affinely isomorphic to tracial states of

$$\mathfrak{A} := \bigoplus_{\gamma \in \mathcal{C}\ell_K} C^*(J_\gamma \rtimes U_K)$$

J_γ = integral ideal representing the ideal class $\gamma \in \mathcal{C}\ell_K$

U_K = group of units in $\mathcal{O}_K \cong \mathbb{Z}^n \times W$ (free abelian $\times$ finite)

Remarks:

$C^*(J_\gamma \rtimes U_K) \cong C(\hat{J}_\gamma) \rtimes U_K \cong C(\mathbb{T}^d) \rtimes (\mathbb{Z}^n \times W)$

$\mathfrak{A}$ also gives the K-theory of A [Cuntz-Echterhoff-Li]

Invariant measures for linear toral automorphisms

extremal traces correspond to ergodic invariant measures for $\mathbb{Z}^n \times W \circlearrowright \mathbb{T}^d$ by automorphisms.

multiplication by $\rho(u) \in \mathrm{GL}_d(\mathbb{Z})$ for each $u \in U_K$ does not increase denominators, so rational points in $\mathbb{R}^d/\mathbb{Z}^d$ have finite $\mathbb{Z}^n$-orbits

not so obviously, the converse also holds (cf. cat maps)

when $\mathbb{Z}^n \circlearrowright \mathbb{T}^d$ contains a partially hyperbolic element the obvious ergodic invariant probability measures on $\mathbb{T}^d$ are

- equidistributions on finite orbits
- Haar measure

Furstenberg's question: Are these all? (open)

Furstenberg's original question is about ergodic invariant measures on $\mathbb{T}$ for the transformations $z \mapsto z^2$ and $z \mapsto z^3$

Generalized Furstenberg question for number fields

K has r real embeddings, and $2s$ complex embeddings, then

$$n := \operatorname{rank} U_K = r + s - 1 \qquad d := \deg K = r + 2s$$

[L-Warren, '20] 4 cases according to unit rank and degree:

1. $\operatorname{rank} U_K = 0$, ($K = \mathbb{Q}$ or quadratic imaginary) then $U_K = W$:
   $\{$ergodic invariant measures$\} \longleftrightarrow \hat{\mathcal{O}}_K/W$ (no F) (boring)

2. $\operatorname{rank} U_K = 1$, (real quadr., mixed cubic, complex quartic):
   $U_K \circlearrowright \hat{\mathcal{O}}_K \leftrightsquigarrow$ Bernoulli [Katznelson] (F? = no) (hopeless)

3. CM fields of degree > 4:
   $U_K \circlearrowright \hat{\mathcal{O}}_K$ has proper invariant subtori (F? = no, but...) (intriguing)
   [Katok-Spatzier](extra assumptions) zero-entropy measures on invariant sub-tori extended by Haar conditional measures on the fibers.

4. $K \neq CM$, $\quad \operatorname{rank} U_K \geqslant 2$ (ID fields) $\quad$ (F?) $\quad$ (hopeful)

   topological version: yes [Berend]

another semigroup from $+$ and $\times$ on $\mathbb{N}$

Recall: $\mathbb{N} = (\{0, 1, 2, \ldots\}, \quad +)$ and $\mathbb{N}^\times = (\{1, 2, \ldots\}, \quad \times)$

have seen: $\mathbb{N} \rtimes \mathbb{N}^\times = \mathbb{N} \times \mathbb{N}^\times$ with operation $(r, a)(s, b) = (r + as, ab)$

but there is another way of combining $+$ and $\times$:

$\mathbb{N}^\times \ltimes \mathbb{N} = \mathbb{N}^\times \times \mathbb{N}$ (as a set) with operation $(a, r)(b, s) = (ab, br + s)$

both have the quasi-lattice property:

$$xP \cap yP = \begin{cases} zP & \text{for some } z \in P \\ \emptyset & \text{otherwise} \end{cases}$$

the “$\emptyset$" can occur for $\mathbb{N} \rtimes \mathbb{N}^\times$ but never for $\mathbb{N}^\times \ltimes \mathbb{N}$ i.e. every left-quotient is a right-quotient (Ore condition), so these are very different semigroups.

$(r, a)(\mathbb{N} \rtimes \mathbb{N}^\times) = \{(r + ay, ax) \mid y \in \mathbb{N},\ x \in \mathbb{N}^\times\}$ so for instance

$$(m, a)(\mathbb{N} \rtimes \mathbb{N}^\times) \cap (n, a)(\mathbb{N} \rtimes \mathbb{N}^\times) = \emptyset \quad \text{if } m \neq n \pmod{a}$$

$(a, r)(\mathbb{N}^\times \ltimes \mathbb{N}) = \{(ax, xr + y) \mid y \in \mathbb{N},\ x \in \mathbb{N}^\times\}$

$$(a, m)(\mathbb{N}^\times \ltimes \mathbb{N}) \cap (b, n)(\mathbb{N}^\times \ltimes \mathbb{N}) = (\alpha, \rho)(\mathbb{N}^\times \ltimes \mathbb{N}) \neq \emptyset$$

$\alpha = [a, b] := \mathrm{lcm}\,(a, b) \qquad \rho = [a, b]\max(\frac{m}{a}, \frac{n}{b})$

Fact:

$$\mathbb{N}^\times \ltimes \mathbb{N} \cong (\mathbb{N} \rtimes \mathbb{N}^\times)^{opp}$$

What about $\mathcal{T}_{left}(\mathbb{N}^\times \ltimes \mathbb{N})$? [an Huef-L-Raeburn '21]

Proposition [aH-L-R]: $\mathcal{T}_{left}(\mathbb{N}^\times \ltimes \mathbb{N})$ is universal with generators s and $\{v_a : a \in \mathbb{N}^\times\}$ subject to relations

(T0) $s^*s = 1 = v_a^* v_a$

(T1) $sv_a = v_a s^a$

(T2) $v_a v_b = v_{ab}$

(T3) $v_a^* v_b = v_b v_a^*$ when $\gcd(a, b) = 1$

(T4) $s^* v_a = v_a s^{*a}$

Note: s = "plus 1" and v_a = "times a" so, e.g.

(T1) 'means' $a(x + 1) = ax + a$ or $R_{(1,1)} R_{(0,a)} = R_{(0,a)} R_{(1,a)}$

Additive boundary quotient

Let $\partial_{add}\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N}) := \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N})/\langle 1 - SS^* \rangle$

Turns S into a unitary U, alternatively, consider $\mathbb{N}^\times \ltimes \mathbb{Z}$

Proposition [aHLR 2021] $\partial_{\mathrm{add}}\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N})$ is the universal C*-algebra generated by U and $\{V_a : a \in \mathbb{N}^\times\}$ subject to

(A1) $UV_a = V_a U^a$,

(A2) $a \mapsto V_a$ is a Nica-covariant isometric representation of $\mathbb{N}^\times$, (this means $V_a V_a^* V_b V_b^* = V_{[a,b]} V_{[a,b]}^*$)

(A3) $UU^* = 1 = U^* U$.

We consider

$$A = \partial_{add}\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N}) \cong \mathcal{T}_{left}(\mathbb{N}^\times \ltimes \mathbb{Z})$$

with σ given by

$$\sigma_t(U) = U \text{ and } \sigma_t(V_a) = a^{it} V_a \qquad (t \in \mathbb{R})$$

A characterisation of KMS states of $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$

'Easy' fact: KMS$_\beta$ states factor through the additive boundary quotient, so we may replace *S* (isometry) by *U* (unitary)

Proposition [aHLR, 2021]: A state φ of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ is a KMS$_\beta$ state if and only if

$$\varphi\big(V_a U^{m-n} V_b^*\big) = \delta_{a,b} a^{-\beta} \varphi\big(U^{m-n}\big) \quad \text{for all } (a,m), (b,n) \text{ in } \mathbb{N}^\times \ltimes \mathbb{N}.$$

Note: such a state is determined by its restriction to $C^*(U) \cong C(\mathbb{T})$.

So KMS states are determined by probability measures on the circle.

But *not all probability measures on the circle arise as restrictions of KMS$_\beta$ states*, because of the underlined assumption i.e. **positivity**.

low-temperature equilibrium

Theorem [aHLR, 2021]

$1 < \beta < \infty \implies$ KMS$_\beta$ states parametrized by prob. meas. on $\mathbb{T}$

$$\psi_{\mu,\beta} \quad \longleftrightarrow \quad \mu$$

$$\psi_{\mu,\beta}(V_a U^k V_b^*) = \delta_{a,b} \frac{a^{-\beta}}{\zeta(\beta)} \sum_{c \in \mathbb{N}^\times} c^{-\beta} \mu(U^{ck})$$

Note: The formula

$$\frac{1}{\zeta(\beta)} \sum_{c \in \mathbb{N}^\times} c^{-\beta} \mu(S^{c(m-n)})$$

parametrizes the probability measures on $\mathbb{T}$ that extend to a positive linear functional (hence KMS state) as in the Proposition.

what about $\beta \leqslant 1$?

For the 'left system' of [L-R '10] the circle of extremal KMS$_\beta$ states of $(\mathbb{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ collapses to a point as $\beta \searrow 2^+$

[aH-L-R '21]: Things are different for the 'right system' $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N}), \sigma)$.

Here are three critical ($\beta = 1$) examples,
arising from three measures on $\mathbb{T}$:

1. Lebesgue measure μ:

$$\psi_{1,\mu}(V_a S^m S^{*n} V_b^*) = \delta_{a,b}\delta_{m,n}\, a^{-1}$$

2. unit point mass at 1:

$$\psi_{1,\delta_1}(V_a S^m S^{*n} V_b^*) = \delta_{a,b}\, a^{-1} \qquad (m, n \in N)$$

3. unit point mass at -1:

$$\psi_{1,\delta_{-1}}(V_a S^m S^{*n} V_b^*) = \begin{cases} a^{-1} & \text{if } a = b \text{ and } m - n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Supercritical equilibrium $T \geqslant 1$

Current joint work with Tyler Schulz:

Full description of the supercritical phase transition ($\beta \leqslant 1$)

Problem: find all the probability measures μ on $\mathbb{T}$ for which

$$\varphi(V_a U^k V_b^*) = \delta_{a,b} a^{-\beta} \int_{\mathbb{T}} z^k d\mu(z)$$

extends to a state of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ (automatically KMS$_\beta$)

Preview: $\exists$ two types of extremal solutions

As $T = \frac{1}{\beta}$ increases past $T_{critical} = 1$, the irrational points 'melt' very differently from the rational points

1) irrationals $\rightsquigarrow$ unique non nonatomic

2) rationals $\rightsquigarrow$ many atomic ones.

That’s it for now.

Thanks!