Dynamical tilings and $\mathscr{Z}$-stability

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Orbit equivalence and hyperfiniteness

Theorem (Dye)

There is a unique hyperfinite II_1 equivalence relation.

In the setting of p.m.p. actions one can establish hyperfiniteness via a **Rokhlin lemma**, as was done first by Dye in the polynomial growth case then more generally by Ornstein and Weiss, yielding:

Theorem (Ornstein–Weiss)

Any two free ergodic p.m.p. actions of countably infinite amenable groups are orbit equivalent.

Bypassing the Rokhlin lemma, a Namioka-type argument modeled on Connes's *injectivity* $\Rightarrow$ *hyperfiniteness* for v.N. algebras shows:

Theorem (Connes–Feldman–Weiss)

Every countable nonsingular amenable equiv. relation is hyperfinite.

Rohklin lemma for $\mathbb{Z}$

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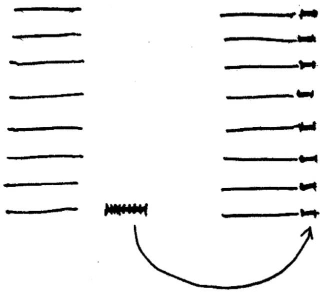
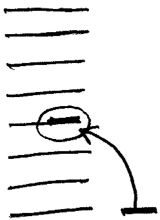


P.m.p. action



Continuous action

Classification of C*-algebras

Classification theorem[1]

The class of infinite-dim'l simple separable unital nuclear $\mathscr{Z}$-stable C*-algebras satisfying the UCT is classified by the Elliott invariant.

For free minimal actions $G \curvearrowright X$ of countably infinite amenable groups on compact metric spaces, the issue of whether $C(X) \rtimes G$ is classifiable boils down to the problem of whether it is $\mathscr{Z}$-stable, as the UCT is automatic (Tu).

[1] Combines results of Gong–Lin–Niu, Elliott–Gong–Lin–Niu, Tikuisis–White–Winter, Castillejos–Evington–Tikuisis–White–Winter, Kirchberg, and Phillips, completing a long line of development.

$\mathscr{Z}$-stability

The Jiang–Su algebra $\mathscr{Z}$ is a non-type I version of the complex numbers (with the same Elliott invariant) and can be defined as an inductive limit of dimension drop algebras over the unit interval. A C*-algebra A is $\mathscr{Z}$**-stable** if $A \otimes \mathscr{Z} \cong A$.

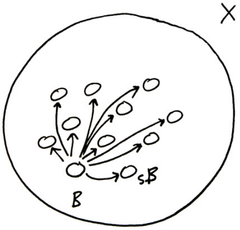
By a theorem of Hirshberg–Orovitz, a simple separable unital nuclear C*-algebra $A \ncong \mathbb{C}$ is $\mathscr{Z}$-stable iff for every $n \in \mathbb{N}$, $\Omega \Subset A$, $\varepsilon > 0$, and nonzero $a \in A_+$ there is an order-zero c.p.c. map $\varphi : M_n \to A$ such that

1. $1 - \varphi(1) \precsim a$,
2. $\|[b, \varphi(c)]\| < \varepsilon$ for all $b \in \Omega$ and norm-one $c \in M_n$.

Towers

Let $G \curvearrowright X$ be an action on a compact metric space.

- A **tower** is a pair (S, B) where the **base** $B \subseteq X$ and the **shape** $S \Subset G$ are such that the **levels** sB for $s \in S$ are pairwise disjoint.



- A **castle** is a finite collection $\{(S_i, B_i)\}_{i \in I}$ of towers such that the sets $S_i B_i$ are pairwise disjoint. It is **open** if each B_i is open, and **clopen** if each B_i is clopen.

Dynamics and classification

For free p.m.p. actions of amenable groups, matrix models for the crossed product can be produced using the Ornstein–Weiss quasitower theorem. In the topological setting, the presence of dimensionality forces the matrix models to be either

(1) **continuous** instead of discrete, or
(2) conceptualized as **order-zero maps** from matrices into the algebra.

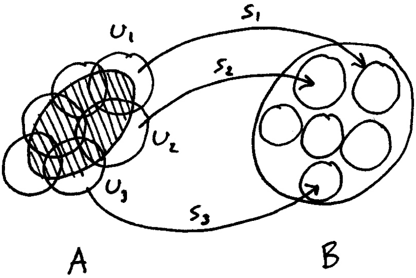
Option (1) connects to dim-rank ratio and comparison, and can be used to get $\mathscr{Z}$-stability from mean dim 0 (e.g., Elliott–Niu for $\mathbb{Z}$, Niu for $\mathbb{Z}^d$)

Option (2) can be developed in two ways:

(2a) use ideas related to **asymptotic dim** for groups to get nuclear dim estimates (e.g., Toms–Winter for $\mathbb{Z}$, Szabó for $\mathbb{Z}^d$),
(2b) use **tiling** technology to establish $\mathscr{Z}$-stability.

Subequivalence

Let $G \curvearrowright X$ as before. Let $A, B \subseteq X$. We say that A is **subequivalent** to B, written $A \prec B$, if for every closed set $C \subseteq A$ there are finitely many open sets $U_1, \ldots, U_n$ which cover C and elements $s_1, \ldots, s_n \in G$ such that the sets $s_i U_i$ for $i = 1, \ldots, n$ are pairwise disjoint and contained in B.



Amenability

Definition

A group G is **amenable** if it admits a mean (i.e., a finitely additive probability measure) which is invariant under left translation.

Følner characterization

The group G is amenable iff for every $K \Subset G$ and $\delta > 0$ there is an $F \Subset G$ such that

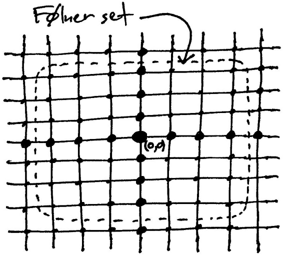
$$|gF\Delta F| < \delta|F|$$

for all $g \in K$. Such a set F is said to be (K, δ)**-invariant**.

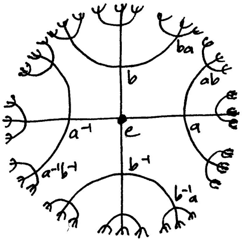
Amenability

Examples

$\mathbb{Z}^2$ is amenable, but the free group $F_2 = \langle a, b \rangle$ is not.



$\mathbb{Z}^2$



F_2

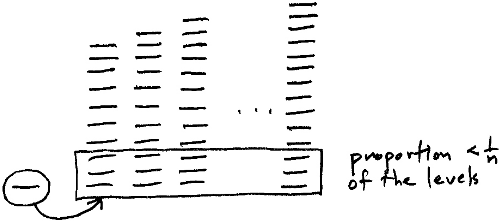
Almost finiteness

Definition

The action $G \curvearrowright X$ is **almost finite** if for every $n \in \mathbb{N}$, $K \Subset G$, and $\delta > 0$ there are

(i) an open castle $\{(V_i, S_i)\}_{i \in I}$ whose shapes are (K, δ)-invariant and whose levels have diameter $< \delta$,

(ii) sets $S_i' \subseteq S_i$ such that $|S_i'| < \frac{1}{n}|S_i|$ and $X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S_i' V_i$.



Almost finiteness

- When X is zero-dim'l, the remainder and diameter condition can be eliminated, i.e., $G \curvearrowright X$ is almost finite iff for every $K \Subset G$ and $\delta > 0$ there is a clopen castle with (K, δ)-invariant shapes.

- $G \curvearrowright X$ has the **small boundary property (SBP)** if X has a basis of open sets whose boundaries are null for every G-invariant probability measure (which happens for example if X is finite-dim'l), and has **comparison** if, for all open $A, B \subseteq X$,

$$\mu(A) < \mu(B) \quad \forall\, \mu \in M_G(X) \qquad \Longrightarrow \qquad A \prec B.$$

For free actions of countably infinite amenable G:

$$\text{almost finiteness} \iff \text{SBP + comparison}$$

Almost finiteness and $\mathcal{Z}$-stability

Theorem (K.)

Suppose that G is countably infinite. Let $G \curvearrowright X$ be a free minimal action which is almost finite. Then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

The proof uses Ornstein–Weiss quasitiling machinery and the Hirshberg–Orovitz characterization of $\mathscr{Z}$-stability in the simple unital nuclear setting in terms of approximately central order-zero c.p.c. embeddings of matrix algebras.

There exist free minimal $\mathbb{Z}$-actions whose crossed product is not $\mathscr{Z}$-stable (Giol-K.). It follows that such actions are not almost finite.

Almost finiteness

Examples of almost finite actions:

- a generic free minimal action of a fixed countably infinite amenable G on the Cantor set
  (Conley–Jackson–K.–Marks–Seward–Tucker-Drob)
- free actions on the Cantor set in the case that G has local subexponential growth *(Downarowicz–Zhang)*
- free actions of many solvable groups, including polycyclic groups and the lamplighter group, on the Cantor set *(C–J–M–S–TD)*

Moreover, if every free action of a given G on a zero-dim'l compact metric space is almost finite, then so is every free action of G on a finite-dim'l compact metric space (K.–Szabó).

Elementary amenable groups

The class of **elementary amenable groups** is the smallest class of groups which contains all finite groups and Abelian groups and is closed under taking (i) subgroups, (ii) quotients, (iii) extensions, and (iv) direct limits.

- includes all solvable groups, is closed under taking wreath products, and contains many groups with both exponential growth and infinite asymptotic dimension, such as $\mathbb{Z} \wr \mathbb{Z}$
- excludes finitely generated amenable groups which have intermediate growth (e.g., the Grigorchuk group) or are simple (e.g., the commutator subgroup of the topological full group of a minimal subshift).

Theorem (Chou, Osin)

The class of countable elementary amenable groups is the smallest class of groups which contains the trivial group and is closed under taking countable direct limits and extensions by $\mathbb{Z}$ and finite groups.

Almost finiteness and elementary amenability

Theorem (K.–Naryshkin)

Every free action of a countably infinite elementary amenable group on a finite-dim'l compact metric space is almost finite.

Corollary

The crossed products of free minimal actions of countably infinite elementary amenable groups on finite-dim'l compact metric spaces are

- *classified by their Elliott invariant, and*
- *expressible as inductive limits of subhomogeneous C^*-algebras whose spectra have covering dim at most two.*

Almost finiteness and elementary amenability

By the general lifting result that allows us to extend from the zero-dim'l case to the finite-dim'l case, it is enough to establish the following:

($*$) every free action of countable elementary amenable group on a zero-dim'l compact metrizable space is almost finite.

Extensions by $\mathbb{Z}$

The bulk of the effort goes into proving:

Theorem
Let $H \rtimes \mathbb{Z} \curvearrowright X$ be a free continuous action on a compact metric space. Suppose that the restricted action $H \curvearrowright X$ is almost finite. Then the action $H \rtimes \mathbb{Z} \curvearrowright X$ is almost finite.

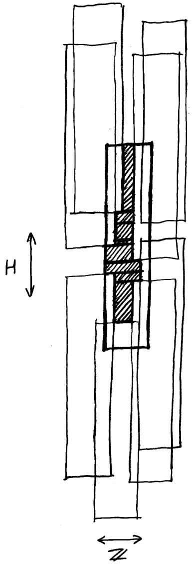
Notice that there is no finite-dimensionality assumption on X here. We need it however for the main theorem, since the strategy is to prove $(*)$ by bootstrapping our way up starting from the trivial group, for which a free action is almost finite iff the space is zero-dimensional.

Extensions by $\mathbb{Z}$

Idea of proof:

- Take an open H-castle will small levels using the almost finiteness of $H \curvearrowright X$, extend to overlapping towers for the $H \rtimes \mathbb{Z}$-action with rectangular shapes, and apply a recursive disjointification procedure involving a give-and-take in the $\mathbb{Z}$ direction.
- The rectangular shapes are chosen to be proportionally very large in the H direction, so that the multiplicity of the overlapping of the towers is small in proportion to the size of their shapes in the H direction. This allows us to disjointify into open subtowers with smaller rectangular shapes which are mostly Følner.
- The exceptional subtowers are small in H-density and can be absorbed using the fact that the H-action has comparison (a consequence of almost finiteness).
- One can also arrange that the boundaries of all of the subtower levels have zero H-density, so that they can be absorbed as well.

Extensions by $\mathbb{Z}$



Comparison

Theorem (Naryshkin)

Suppose that G is finitely generated with polynomial growth. Then every free action $G \curvearrowright X$ on a compact metric space has comparison.

Corollary

Suppose that G is finitely generated with polynomial growth. Let $G \curvearrowright X$ be a free action on a compact metric space. Then $G \curvearrowright X$ has the small boundary property iff it is almost finite.

For minimal $\mathbb{Z}^d$-actions, the small boundary property is equivalent to mean dim zero (Gutman–Lindenstrauss–Tsukamoto).

Problem

Does every free action of an elementary amenable group (or any amenable group) have comparison?