Finite dimensional approximations in operator algebras

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Non-selfadjoint operator algebras

A unital operator algebra is a norm closed subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ such that $\mathrm{id}_{\mathcal{K}} \in \mathcal{A}$.

A representation of $\mathcal{A}$ is a completely contractive homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Example

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f\big|_{\mathbb{D}} \text{ is holomorphic}\}$.

Theorem (von Neumann, Sz.-Nagy)

If $T \in B(\mathcal{H})$ with $\|T\| \le 1$, then there exists a unital representation

$$\pi : A(\mathbb{D}) \to B(\mathcal{H}), \quad p \mapsto p(T) \quad (p \in \mathbb{C}[z]).$$

This gives $1-1$ correspondence between unital representations of $A(\mathbb{D})$ and contractions on Hilbert space.

Residual finite dimensionality (RFD)

Definition

An operator algebra $\mathcal{A}$ is residually finite dimensional (RFD) if for all $n \in \mathbb{N}$ and all $a \in M_n(\mathcal{A})$, we have

$$\|a\| = \sup\{\|\pi^{(n)}(a)\| : \pi : A \to B(\mathcal{H}) \text{ rep. with } \dim(\mathcal{H}) < \infty\}.$$

Equivalently, there exist a family $\{\mathcal{H}_\lambda : \lambda \in \Lambda\}$ of finite dimensional Hilbert spaces and a completely isometric homomorphism

$$\pi : \mathcal{A} \to \prod_{\lambda \in \Lambda} B(\mathcal{H}_\lambda).$$

Introduced by Mittal–Paulsen. Systematically studied by Clouâtre–Marcoux, Clouâtre–Ramsey, Clouâtre–Dor-On, Thompson, ...

Examples

- A C^*-algebra is RFD in the C^*-sense iff it is RFD in the non-selfadjoint sense.
- Every finite dimensional operator algebra is RFD.
- Every uniform algebra (subalgebra of commutative C^*-algebra) is RFD. In particular, $A(\mathbb{D})$ is RFD.
- $\{T \in B(\ell^2) : T$ is upper triangular$\}$ is RFD.
- Multiplier algebras of reproducing kernel Hilbert spaces are RFD (Mittal-Paulsen).
- The universal operator algebra generated by d commuting contractions is RFD (Agler, Mittal–Paulsen).

The Exel–Loring theorem

A state φ on a unital C^*-algebra $\mathfrak{A}$ is finite dimensional if the GNS representation associated with φ acts on a finite dimensional Hilbert space.

A representation $\pi : \mathfrak{A} \to B(\mathcal{H})$ is finite dimensional if $\dim(\pi(\mathfrak{A})\mathcal{H}) < \infty$.

Theorem (Exel–Loring)

The following assertions are equivalent for a unital C^*-algebra $\mathfrak{A}$:

(i) $\mathfrak{A}$ is RFD;

(ii) the finite dimensional states are weak-$*$ dense in the state space of $\mathfrak{A}$;

(iii) for every representation $\pi : \mathfrak{A} \to B(\mathcal{H})$, there exists a net (π_λ) of finite dimensional representations such that $\pi_\lambda(a) \to \pi(a)$ in SOT for all $a \in \mathfrak{A}$.

Question (Clouâtre–Dor-On)

Is there a non-selfadjoint version of this result?

A non-selfadjoint Exel–Loring theorem

The matrix state space

Let $\mathcal{A}$ be a unital operator algebra. For $n \in \mathbb{N}$, let

$$X_n = \{\varphi : \mathcal{A} \to M_n : \varphi \text{ is linear and u.c.c.}\}.$$

The matrix state space of $\mathcal{A}$ is $S(\mathcal{A}) = (X_n)_{n=1}^{\infty}$. (matrix convex set)

Theorem (Arveson, Stinespring)

If $\varphi : \mathcal{A} \to M_n$ is a matrix state, then there exist a Hilbert space $\mathcal{H}$, an isometry $w : \mathbb{C}^n \to \mathcal{H}$ and a u.c.c. homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ with

$$\varphi(a) = w^* \pi(a) w \quad \text{for all } a \in \mathcal{A}.$$

Definition

A matrix state $\varphi : \mathcal{A} \to M_n$ is finite dimensional if $\mathcal{H}$ can be chosen to be finite dimensional.

A non-selfadjoint Exel–Loring theorem

Let $\mathcal{A}$ be a unital operator algebra. A representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is finite dimensional if $\dim(C^*(\pi(\mathcal{A}))\mathcal{H}) < \infty$.

Theorem (H.)

The following assertions are equivalent for a unital operator algebra $\mathcal{A}$:

(i) $\mathcal{A}$ is RFD;

(ii) the finite dimensional matrix states are weak-$*$ dense in the matrix state space $S(\mathcal{A})$;

(iii) for every representation $\pi : \mathcal{A} \to B(\mathcal{H})$, there exists a net (π_λ) of finite dimensional representations such that $\pi_\lambda(a) \to \pi(a)$ in WOT for all $a \in \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{H}$ are separable, the net in (iii) can be replaced with a sequence.

Sketch of proof

(i) RFD $\Rightarrow$ (ii) density of finite dimensional matrix states:

Matrix convex adaptation of Exel–Loring proof; uses Hahn–Banach separation theorem of Effros–Winkler.

(ii) density of f.d. matrix states states $\Rightarrow$ (iii) WOT-approximation by f.d. representations:

Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be unital representation. Let $M \subset \mathcal{H}$ be f.d. and

$$\varphi : \mathcal{A} \to B(M), \quad a \mapsto P_M \pi(a)\big|_M.$$

Approximate φ by f.d. matrix state ψ. Then dilate ψ to f.d. representation σ of $\mathcal{A}$.

Get $P_M \pi(a) P_M \approx P_M \sigma(a) P_M$.

WOT vs. SOT

Easy observation

If $\mathfrak{A}$ is a C^*-algebra and $\pi_\lambda, \pi : \mathfrak{A} \to B(\mathcal{H})$ are representations, then

$$\pi_\lambda(a) \to \pi(a) \text{ WOT for all } a \in \mathfrak{A} \Leftrightarrow \pi_\lambda(a) \to \pi(a) \text{ SOT for all } a \in \mathfrak{A}.$$

Proof: If (A_λ) is a net in $B(\mathcal{H})$ with $A_\lambda \to A$ and $A_\lambda^* A_\lambda \to A^* A$ in WOT, then $A_\lambda \to A$ in SOT.

Question (Clouâtre–Dor-On)

Let $\mathcal{A}$ be an RFD operator algebra and let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a representation. Is there a net (π_λ) of f.d. representations such that

(a) $\pi_\lambda(a) \to \pi(a)$ in SOT for all $a \in \mathcal{A}$;

(b) $\pi_\lambda(a) \to \pi(a)$ in SOT-$*$ for all $a \in \mathcal{A}$?

A counterexample

A counterexample

$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f\big|_{\mathbb{D}}$ is holomorphic$\}$. Let $\mathbb{T} = \partial\mathbb{D}$ and regard $A(\mathbb{D}) \subset C(\mathbb{T})$ by maximum modulus principle.

Theorem (H.)

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} : f, g \in A(\mathbb{D}), h \in C(\mathbb{T}) \right\} \subset M_2(C(\mathbb{T})).$$

Then:

(a) $\mathcal{B}$ is a unital operator algebra that is RFD;

(b) there exists a representation $\pi : \mathcal{B} \to B(\mathcal{H})$ that is not the point SOT-limit of a net of finite dimensional representations of $\mathcal{B}$.

A non-approximable representation

If $f \in L^2(\mathbb{T})$, let $\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} \, dm(z)$ be the Fourier coefficients of f. The Hardy space is $H^2 = \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for all } n < 0\}$.

If $h \in C(\mathbb{T})$, the Toeplitz operator with symbol h is

$$T_h : H^2 \to H^2, \quad f \mapsto P_{H^2}(h \cdot f).$$

Theorem

$$\pi : \left\{ \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} : \begin{array}{c} f, g \in A(\mathbb{D}) \\ h \in C(\mathbb{T}) \end{array} \right\} \to B(H^2 \oplus H^2), \quad \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} \mapsto \begin{bmatrix} T_f & 0 \\ T_h & T_{\overline{g}} \end{bmatrix},$$

is a representation that is not the point SOT-limit of a net of finite dimensional representations.

π is multiplicative since $T_{\overline{g}} T_h = T_{\overline{g}h}$ and $T_h T_f = T_{hf}$.

RFD C^*-covers

C^*-covers

Let $\mathcal{A}$ be a unital operator algebra.

Definition

A C^*-cover of $\mathcal{A}$ is a pair $(\mathfrak{A}, \iota)$, where $\mathfrak{A}$ is a unital C^*-algebra, $\iota : \mathcal{A} \to \mathfrak{A}$ is a unital completely isometric homomorphism and $\mathfrak{A} = C^*(\iota(\mathcal{A}))$.

If $(\mathfrak{A}_1, \iota_1)$ and $(\mathfrak{A}_2, \iota_2)$ are two C^*-covers, say $(\mathfrak{A}_1, \iota_1) \leq (\mathfrak{A}_2, \iota_2)$ if there is a $*$-homomorphism $\pi : \mathfrak{A}_2 \to \mathfrak{A}_1$ such that $\pi \circ \iota_2 = \iota_1$.

Theorem (Harmana, Dritschel–McCullough, Arveson, Davidson–Kennedy)

There exists a smallest C^*-cover $C^*_e(\mathcal{A})$, called the C^*-envelope.

Proposition

There exists a largest C^*-cover $C^*_{max}(\mathcal{A})$.

Example: The disc algebra

$$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f\big|_{\mathbb{D}} \text{ holomorphic}\}.$$

$$\begin{tikzcd}
 & & C^*(t : \|t\| \leq 1) \arrow[ddl, dashed] \arrow[dr, dashed] & \\
 & & & C^*(T_f : f \in A(\mathbb{D})) \arrow[ddl, dashed] \\
A(\mathbb{D}) \arrow[r] \arrow[uurr] \arrow[urrr] \arrow[drr] & C(\overline{\mathbb{D}}) \arrow[dr, dashed] & & \\
 & & C(\mathbb{T}) &
\end{tikzcd}$$

C^*-envelopes are often not RFD

Example

Let $\mathcal{A} = \{T \in B(\ell^2) : T \text{ is upper triangular}\}$. Then $\mathcal{A}$ is RFD, but $C^*_e(\mathcal{A}) = B(\ell^2)$ is not RFD.

Example

Arveson's algebra $\mathcal{A}_d$, i.e. the univeral operator algebra generated by a row contractive commuting d-tuple, is RFD. But $C^*_e(\mathcal{A}_d)$ is not RFD if $d \geq 2$ (it contains the compacts).

Theorem (Clouâtre–Ramsey)

There exists a finite dimensional operator algebra whose C^*-envelope is not RFD.

Theorem (Aleman–H.–M^cCarthy–Richter)

A unital operator algebra $\mathcal{A}$ is n-subhomogeneous if and only if $C^*_e(\mathcal{A})$ is n-subhomogeneous.

Is C^*_{max} RFD?

Universal property of C^*_{max}

Every representation of $\mathcal{A}$ extends to a $*$-representation of $C^*_{max}(\mathcal{A})$.

Question (Clouâtre–Dor-On)

Let $\mathcal{A}$ be an RFD operator algebra. Is $C^*_{max}(\mathcal{A})$ RFD?

Positive answer for certain algebras, including Arveson's algebra $\mathcal{A}_d$.

Theorem (Thompson)

If $\mathcal{A}$ is RFD, there is a maximal RFD C^*-cover $\mathfrak{R}(\mathcal{A})$ of $\mathcal{A}$. Every finite dimensional representation of $\mathcal{A}$ extends to a $*$-representation of $\mathfrak{R}(\mathcal{A})$.

Theorem (Clouâtre–Dor-On)

$C^*_{max}(\mathcal{A})$ is RFD if and only if every representation of $\mathcal{A}$ is the point SOT-$*$ limit of a net of finite dimensional representations of $\mathcal{A}$.

Necessity follows from universal property and Exel–Loring theorem.

$C^*_{max}(\mathcal{B})$ is not RFD

Recall that

$$\mathcal{B} = \left\{ \begin{bmatrix} f & 0 \\ h & \overline{g} \end{bmatrix} : f, g \in A(\mathbb{D}), h \in C(\mathbb{T}) \right\} \subset M_2(C(\mathbb{T})).$$

Corollary

The algebra $\mathcal{B}$ is RFD, but $C^*_{max}(\mathcal{B})$ is not RFD.

Summary

- RFD non-selfadjoint operator algebras can be characterized in terms of their matrix space.
- Every representation of an RFD algebra can be approximated point-WOT by finite dimensional ones.
- SOT-approximation is not possible in general. Hence C^*_{max} may fail to be RFD.

Thank you!