Tracial oscillation zero and stable rank one

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The Jiang-Su algebra $\mathcal{Z}$ is an infinite dimensional separable simple unital nuclear C^*-algebra which shares the same Elliott invariant (K-groups and trace simplex) with $\mathbb{C}$. Being $\mathcal{Z}$-stable (i.e., $A \otimes \mathcal{Z} \cong A$) is an important condition in the classification of C^*-algebras.

Theorem (By many people's work, including Kirchberg-Phillips, Elliott-Gong, Gong-Lin-Niu, Elliott-Gong-Lin-Niu, Tikuisis-White-Winter, Castillejos-Evington-Tikuisis-White-Winter,...)

Unital simple separable nuclear $\mathcal{Z}$-stable UCT C^*-algebras can be classified by their Elliott invariant.

Picture for elements in $\mathcal{Z}$-stable algebras

Let A be a $\mathcal{Z}$-stable algebra (or a tracially approximately divisible algebra, or a tracially $\mathcal{Z}$-absorbing algebra). For any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, any $n \in \mathbb{N}$,

$$x \approx_\epsilon \begin{pmatrix} \alpha(x) & 0 & \cdots & 0 \\ 0 & \alpha(x) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha(x) \end{pmatrix}_{n \times n} + (small\ \beta(x)),$$

where $x \in \mathcal{F}$.

Note that these two parts may not be orthogonal.

Being $\mathcal{Z}$-stable implies many regularity properties. One of them is the dichotomy of finiteness:

Theorem (Rørdam, 2004)

Let A be a unital simple $\mathcal{Z}$-stable C^*-algebra. Then A is either purely infinite or has stable rank one.

- C^*-algebra A is said to be has stable rank one, if $\widetilde{A} = \overline{GL(\widetilde{A})}$,

Nonunital case?

Theorem (Robert, 2015)

Let A be a simple $\mathcal{Z}$-stable projectionless C^*-algebra. Then A almost has stable rank one. In fact, every x in A can be approximated by products of two nilpotents in $\mathrm{Her}(x)$.

- A is said to be almost has stable rank one, if for all hereditary subalgebra $B \subset A$, $B \subset \overline{GL(\widetilde{B})}$.
- The subtle difference between stable rank one and almost stable rank one is, $A \subset \overline{GL(\widetilde{A})}$ may not imply $\widetilde{A} = \overline{GL(\widetilde{A})}$.
- When A is simple and has a projection, A almost has stable rank one is equivalent to A has stable rank one.

Nonunital case?

Theorem (F.-Li-Lin, 2021)

Let A be a simple $\mathcal{Z}$-stable (not necessary unital) C^*-algebra. Then A is either purely infinite or has stable rank one.

Above theorem was obtained by combining Robert's theorem and the following:

Theorem (F.-Li-Lin, 2021)

Let A be a σ-unital projectionless simple C^*-algebra with continuous scale. Suppose that, for any σ-unital hereditary subalgebra $B \subset A$, any non-invertible element in B can be approximated (in norm) by products of two nilpotent elements in B. Then A has stable rank one.

Further Development:

Oscillation Zero

&

Tracial Matricial Property

Further Result

• There are examples show that, in general (not assuming simplicity), almost stable rank one may not imply stable rank one.

Question: When does almost stable rank one implies stable rank one?

Theorem (F.-Lin, 2021)

Let A be a separable simple C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$, has strict comparison and $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective. Then A has stable rank one if and only if A has almost stable rank one.

• $\widetilde{QT}(A)$ is the set of densely defined 2-quasitrace on $A \otimes \mathcal{K}$.

Notations and terminologies

- Let τ be a 2-quasitrace on A. The dimension function induced by τ is given by $d_\tau(a) := \lim_n \tau(a^{1/n})$ for all $a \in A_+$.
- A is said to be has strict comparison, if $d_\tau(a) < d_\tau(b)$ for all 2-quasitrace τ implies $a \lesssim b$.
- $\Gamma : \mathrm{Cu(A)} \to \mathrm{LAff}_+(\widetilde{QT}(A))$, $\Gamma([a])(\tau) = d_\tau(a)$.
- Suppose that A is a σ-unital C^*-algebra with $\widetilde{QT}(A) \backslash \{0\} \neq \emptyset$. Let $T \subset \widetilde{QT}(A)$ is compact subset with $T \neq \{0\}$. Define

  $$\mathrm{I}_{_T} = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\infty} \sup\{\tau(x_n^* x_n) : \tau \in T\} = 0\}.$$

  Then $\mathrm{I}_{_T}$ is a (closed two-sided) ideal of $l^\infty(A)$.
- Let $\Pi_T : l^\infty(A) \to l^\infty(A)/\mathrm{I}_{_T}$ be the quotient map.

Tracial Approximate Oscillation Zero

Definition (F.-Lin, 2021)

Let A be a C^*-algebra with $\widetilde{QT}(A) \neq \emptyset$. For each $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$,

$$\omega(a) := \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in \widetilde{QT}(A)\} : c \in (\overline{a(A \otimes \mathcal{K})a})_+^1\}.$$

The number $\omega(a)$ is called the (tracial) oscillation of a.

Also define

$$\begin{aligned}\Omega^T(a) \ &= \ \inf\{\|\Pi_T(\iota(a) - \{b_n\})\| : \{b_n\} \in l^\infty(\mathrm{Her}(a))_+, \\ &\qquad \|b_n\| \leq \|a\|, \ \lim_{n\to\infty} \omega(b_n) = 0\}.\end{aligned}$$

Tracial Approximate Oscillation Zero

Definition (F.-Lin, 2021)

Let A be a σ-unital C^*-algebra with $QT(A) \neq \emptyset$. Define

$$\mathbb{O}(A) = \sup_{n\in\mathbb{N}}\{\sup\{\Omega^T(a) : a \in \mathrm{Ped}(M_n(A))_+^{\mathbf{1}}\}\}.$$

If $\mathbb{O}(A) = 0$, then we say A has T-tracial approximate oscillation zero.

Proposition

If A is a simple C^*-algebra which has T-tracial approximate oscillation zero, then every hereditary C^*-subalgebras also has T-tracial approximate oscillation zero.

Proposition

Let A be a C^*-algebra of real rank zero. Then A has T-tracial approximate oscillation zero.

An example for C^*-algebras that has T-tracial approximate oscillation zero.

Theorem (F.-Lin, 2021)

Let A be a C^*-algebra with countable $\partial_e(T_b)$ (for some $b \in \mathrm{Ped}(A)_+ \setminus \{0\}$), where $\partial_e(T_b)$ is the set of extremal points of T_b. Then

$$\Omega^T(a) = 0 \ \textit{for all} \ a \in \mathrm{Ped}(A \otimes \mathcal{K})_+.$$

In particular, A has T-tracial approximate oscillation zero.

Theorem (F.-Lin, 2021)

Let A be an algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. If A has T-tracial approximate oscillation zero, then $l^\infty(A)/\mathrm{I}_{\overline{QT(A)}^w}$ has real rank zero.

Theorem (F.-Lin, 2021)

Let A be an algebraically simple C^*-algebra with $T(A) \neq \emptyset$. Suppose that A has strict comparison and T-tracial approximate oscillation zero. Then $l^\infty(A)/\mathrm{I}_{\overline{QT(A)}^w}$ has stable rank one.

Theorem (F.-Lin, 2021)

Let A be a non-elementary and σ-unital simple C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \neq \emptyset$ and strict comparison. Suppose that A has T-tracial approximate oscillation zero. Then Γ is surjective.

Tracially Matricial Property

Definition (F.-Lin, 2021)

Let A *be a* C^**-algebra and* $S \subset \widetilde{QT}(A)\backslash\{0\} \neq \emptyset$. A *is said to have property (TM), if for any* $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, *any* $\epsilon > 0$, *any* $n \in \mathbb{N}$, *there is a c.p.c. order zero map* $\phi : M_n \to \mathrm{Her}(a)$ *such that* $\|a - \phi(1_n)a\|_{2,S} < \epsilon$.

Recall that a c.p.c. order zero map $\phi : M_n \to A$ brings a matricial structure. Hence the name Tracially Matricial Property.

Proposition

Let A *be a* C^**-algebra,* $n \in \mathbb{N}$, *and* $\phi : M_n \to A$ *be a c.p.c. order zero map. Then* $\mathrm{Her}(\phi(1_n)) \cong \mathrm{Her}(\phi(e_{1,1})) \otimes M_n$.

Theorem (F.-Lin, 2021)

Let A be a separable simple C^*-algebra which admits at least one non-trivial 2-quasitrace and has strict comparison.

Then the following are equivalent:

(1) A has tracial approximate oscillation zero;

(2) A has stable rank one;

(3) Γ is surjective and A has almost stable rank one;

(4) A has property (TM).

$$a = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix} \approx_\epsilon \begin{pmatrix} 0 & 0 & \cdots & 0 \\ x_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & y_{11} & \cdots & y_{1n} \\ 0 & 0 & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $a \in A \subset l^\infty(A)/\mathrm{I}_T$ which is not invertible. By Rørdam's method, we may assume $\{a\}^\perp \neq \{0\}$. Then the facts that A has property (TM) and $l^\infty(A)/\mathrm{I}_T$ is stable rank one imply the above in $l^\infty(A)/\mathrm{I}_T$. Then lift nilpotents and handle a tracially small part. If A is non-unital, need to use a theorem above.

Thank you!