Structure and Classification of Real C^*-algebras

(A brief survey and some open questions)

Andrew Dean

Real C*-algebras

The natural definition for a real C*-algebra is that it a real Banach *-algebra that is isomorphic to a norm closed self adjoint algebra of operators on a real Hilbert space. (By *-algebra we mean that it has an involution * that is real linear and satisfies (ab)*=b*a*.) This is then analogous to the definition of complex C*-algebra. One would then like to find an abstract set of axioms, like in the complex case. It turns out that one requires one more axiom: One must assume that $x^*x + 1$ is always invertible in the unitisation. One can then form the complexification $A \otimes \mathbb{C}$ of a real C^*-algebra A and extend the norm of A to a C^*-norm on $A \otimes \mathbb{C}$. On the complexification we then get a map $\varphi : A \otimes \mathbb{C} \to A \otimes \mathbb{C}$ defined by $\varphi(x + iy) = x^* + iy^*$ (note the $+$, which makes it different from just the adjoint).

This map satisfies $\varphi(a+\lambda b) = \varphi(a) + \lambda\varphi(b)$ for all $a, b \in A \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}$, $\varphi(ab) = \varphi(b)\varphi(a)$, $\varphi(a^*) = \varphi(a)^*$, and $\varphi^2 = \mathit{identity}$. In words, it is an involutive *-antiautomorphism. We can identify A inside of $A \otimes \mathbb{C}$ as $\{a \in A \otimes \mathbb{C} \mid \varphi(a) = a^*\}$. Conversely, if we are given a complex C^*-algebra, and an involutive *-antiautomorphism φ on it, the subset above is a real C^*-algebra whose complexification is the given one. We thus have two ways of viewing real C^*-algebras, as real Banach algebras themselves, or via involutive *-antiautomorphisms (henceforth called real structures) on complex C^*-algebras. We shall write (A, τ) for a complex C^*-algebra with real structure τ.

Example: Group C^*-algebras

If G is a finite group, we get a real structure on $C^*(G)$ defined by $\tau(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g g^{-1}$. The real form may give additional information. For example, for the dihedral group D_8 and quaternion group Q_8 we have $C^*(D_8) \cong C^*(Q_8) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})$ but the real form for D_8 is $\mathbb{R}^4 \oplus M_2(\mathbb{R})$ and the real form for Q_8 is $\mathbb{R}^4 \oplus \mathbb{H}$.

Commutative Real C^*-algebras

If A is a commutative real C^*-algebra, then there exists a locally compact Hausdorff space X and a homeomorphism τ of X with $\tau^2 = id$ such that

$$A \cong C_0(X, \tau) = \{f \in C_0(C) \mid f(\tau(x)) = \overline{f(x)} \text{ for all } x \in X\}.$$

Finite Dimensional Real C^*-algebras

The most familiar non-trivial real structure on a C^*-algebra is probably the transpose operation on $M_n(\mathbb{C})$. In this case, $\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\}$ is just $M_n(\mathbb{R})$.
On the 2×2 matrices there is another real structure, usually denoted with a $\#$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\#} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In this case, $\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\}$ is $\mathbb{H}$. On $M_{2n}(\mathbb{C})$, we get an extension of $\#$ by $(x \otimes y)^{\#} = x^{tr} \otimes y^{\#}$.

Up to unitary equivalence, these are the only real structures on $M_q(\mathbb{C})$. On $M_q(\mathbb{C}) \oplus M_q(\mathbb{C})$ we also have $\varphi(x, y) = (y^{tr}, x^{tr})$. In this case,
$\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\} = \{(x, \bar{x}) \,|\, a \in M_q(\mathbb{C})\} \cong M_q(\mathbb{C})$.
Any finite dimensional real C^*-algebra is isomorphic to a finite direct sum of full matrix algebras, each of which is of the form $M_n(\mathbb{C})$, $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$.

Real AF Algebras

We say a real C^*-algebra is AF if it is an inductive limit of finite dimensional real C^*-algebras. Real AF algebras were classified by Giordano using an invariant consisting of $K_0(A_\varphi)$, $K_2(A_\varphi)$,$K_4(A_\varphi)$, and an order structure on $K_0(A_\varphi) \oplus K_2(A_\varphi)$, and by Stacey using a diagram

$$K_0(A_\varphi) \to K_0(A) \to K_0(A_\varphi \otimes \mathbb{H}).$$

The range of invariant problem for this invariant has also been solved. What other kinds of real structures a complex AF algebra can have is open.

The Real Structure on the CAR Algebra

It was shown by Blackadar, in his paper on symmetries on the CAR algebra, that the K-theory of any real structure on the CAR algebra is completely determined by homological considerations. Stacey has since shown that up to isomorphism there is a unique real structure on the CAR algebra, so the obvious AF one is the only one. (Very different from the case of $\mathbb{Z}_2$ actions.)

Inductive limit type actions on AF algebras

Handelman and Rossmann showed that locally representable actions of a compact group G on an AF algebra A could be classified by $K_0(A \rtimes_\alpha G)$ viewed as an ordered module over $K_0(C^*(G))$, with distinguished elements. An analogous classification for inductive limit type actions on real C^*-algebras can be given using as invariant a diagram:

$$K_0(A_\varphi \rtimes_\alpha^{\mathbb{R}} G) \to K_0(A \rtimes_\alpha G) \to K_0((A_\varphi \rtimes_\alpha^{\mathbb{R}} G) \otimes_{\mathbb{R}} \mathbb{H})$$

Elliott and Su showed that inductive limit type actions of $\mathbb{Z}_2$ could be classified by K-theory invariants without the local representability assumption. This result also has a real AF analogue.

Real Structures on Factors

It was shown by Størmer, and independently by Giordano and Jones, that there is a unique real structure, up to conjugacy, on the hyperfinite II_1 factor R. There is also a unique real structure on the injective II_∞ factor. (This in spite of there being two distinct real structures on $B(H)$. Notice that $R_{\mathbb{R}} \otimes \mathbb{H} \cong R_{\mathbb{R}}$.)

Purely Infinite Real C^*-algebras

Theorem (Boersema, Ruiz, Stacey)

Two real stable Kirchberg algebras A and B are isomorphic if, and only if, $K^{CRT}(A) \cong K^{CRT}(B)$. *Two real unital Kirchberg algebras A and B are isomorphic if, and only if,* $(K^{CRT}(A), [1_A]) \cong (K^{CRT}(B), [1_B])$.

(Here a real Kirchberg algebra is one whose complexification is a Kirchberg algebra)

Real Structures on the Jiang-Su Algebra

Theorem (P. J. Stacey)

There is a real structure ρ on the Jiang-Su algebra Z such that $K^{CRT}(Z_\rho) \cong K^{CRT}(\mathbb{R})$, and $Z_\rho \otimes Z_\rho \cong Z_\rho$.

It is not known if the real structure with these properties is unique.

Real Interval Algebras

There are the following five basic real forms for interval algebras:

$$A(n,\mathbb{R}) = \{f \in C([0,1], M_n(\mathbb{C})) \mid f(1) \in M_n(\mathbb{R})\}$$

$$A(n,\mathbb{H}) = \{f \in C([0,1], M_{2n}(\mathbb{C})) \mid f(1) \in M_n(\mathbb{H})\}$$

$$M_n(C_{\mathbb{F}}[0,1]) = M_n(\{f : [0,1] \to \mathbb{F} \mid f \text{ is continuous}\})$$

for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Simple Real AI algebras

Theorem (P. J. Stacey)

Let A and B be two unital real C^*-algebras each arising as an inductive limit of finite direct sums of real interval algebras. Suppose there exist isomorphisms $\phi_T : T(B \otimes_{\mathbb{R}} \mathbb{C}) \to T(A \otimes_{\mathbb{R}} \mathbb{C})$ and $(\phi_K^1, \phi_K^2, \phi_K^3)$ of
$(K_0(A), [1]) \to (K_0(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (K_0(A \otimes_{\mathbb{R}} \mathbb{H}), [1])$ *with*
$(K_0(B), [1]) \to (K_0(B \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (K_0(B \otimes_{\mathbb{R}} \mathbb{H}), [1])$ *such that* ϕ_T *is compatible with* ϕ_K^2 *in the usual way. Then there exists a *-isomorphism $\varphi : A \to B$ giving rise to these maps on the invariant.*

Cuntz Equivalence

Definition

Let A be a C^*-algebra, either real or complex, and let a, b be positive elements of A. We say that a is Cuntz sub-equivalent to b, and write $a \preccurlyeq b$ if there exists a sequence $d_n \in A$ such that $d_n b d_n^* \to a$. We write $a \sim b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$. Then $\sim$ is an equivalence relation on the set of positive elements of A, called Cuntz equivalence.

The Cuntz Semigroup

Definition

Let A be a separable C^*-algebra, either real or complex. Let $Cu(A)$ denote the set of Cuntz equivalence classes of positive elements of $A \otimes_{\mathbb{R}} K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ is the real C^*-algebra of compact operators on a separable real Hilbert space. Fix an isomorphism of $K_{\mathbb{R}}$ with $M_2(K_{\mathbb{R}})$, and define addition on $Cu(A)$ by $[a] + [b] = [\begin{pmatrix} a & 0 \\ o & b \end{pmatrix}]$ (this does not depend on the choice of isomorphism). Define a partial order on $Cu(A)$ by $[a] \leq [b]$ if, and only if, $a \preccurlyeq b$ (this does not depend on choice of representatives). With these definitions, $Cu(A)$ becomes a partially ordered abelian semigroup with neutral element.

An Invariant for Nonsimple Real AI algebras

Given a unital real C^*-algebra A, our invariant, denoted $Inv(A)$, consists of the triple $(Cu(A), [1]) \to (Cu(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (Cu(A \otimes_{\mathbb{R}} \mathbb{H}), [1])$ of Cuntz semigroups with distinguished elements, where the connecting maps are induced by the inclusions. A morphism of invariants $\eta : Inv(A) \to Inv(B)$ consists of a triple (η_r, η_c, η_h) of unital homomorphisms of ordered abelian partial semigroups preserving suprema of increasing sequences, zero elements, and compact containment such that the following diagram commutes:

$$\begin{array}{ccccc} (Cu(A), [1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{H}), [1]) \\ \downarrow \eta_r & & \downarrow \eta_c & & \downarrow \eta_h \\ (Cu(B), [1]) & \longrightarrow & (Cu(B \otimes_{\mathbb{R}} \mathbb{C}), [1]) & \longrightarrow & (Cu(B \otimes_{\mathbb{R}} \mathbb{H}), [1]). \end{array}$$

Existence and Uniqueness for Interval Algebras

Theorem (A.D. and Luis Santiago)

*Let A be a real interval algebra and let B be a unital real AI algebra. Then if η is a morphism of invariants from $Inv(A)$ to $Inv(B)$, there exists a unital *-homomorphism $\varphi : A \to B$ such that $\eta = Inv(\varphi)$.*

Theorem (A.D. and Luis Santiago)

Let A be a real interval algebra and let B be a real AI algebra. If $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv(\varphi) = Inv(\psi)$, then φ and ψ are approximately unitarily equivalent (via unitaries in the real C^*-algebra B).

Classification of Real AI Algebras

Theorem (A.D. and Luis Santiago)

Let A and B be unital real AI algebras. Then if $(\eta_r, \eta_c, \eta_h) : Inv(A) \to Inv(B)$ is a morphism of invariants, there exists a unital $*$-homomorphism $\varphi : A \to B$ such that $Cu(\varphi) = \eta_r$, $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = \eta_c$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = \eta_h$. Moreover, if $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv(\varphi) = Inv(\psi)$, then φ and ψ are approximately unitarily equivalent.

Stable Rank for Real C^*-Algebras

For the real interval algebras we have $tsr(A(n,\mathbb{R})) = tsr(A(n,\mathbb{H})) = tst(M_n(C_{\mathbb{C}}[0,1]) = tsr(M_n(C_{\mathbb{H}}[0,1]) = 1$, but $tsr((M_n(C_{\mathbb{R}}[0,1])) = 2$. In the commutative case, we have the familiar formulas

$$tsr(C_{\mathbb{C}}(X)) = \lfloor dim(X)/2 \rfloor + 1$$

and

$$tsr(C_{\mathbb{R}}(X)) = dim(X) + 1.$$

Question: What pairs (n,m) arise as $(tsr(A), tsr(A \otimes \mathbb{C}))$ for a simple real C^*-algebra A?

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