Tracially complete C*-algebras

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Overview

- New concept: tracially complete C*-algebras.
- Origins in tracial ultrapowers, Matui–Sato.
- Many unanswered questions.

Plan:

- Tracial ultrapowers
- Their use in structure and classification of C*-algebras
- Tracial completions of C*-algebras
- Tracially complete C*-algebras
- Property Γ
- Questions
- Questions

Tracial ultrapowers

Let A be a C*-algebra, and let $T(A)$ denote its set of traces. Assume $T(A) \neq \emptyset$.

Define the *uniform 2-seminorm* $\|\cdot\|_{2,u}$ on A by

$$\|a\|_{2,u} := \sup_{\tau \in T(A)} \tau(a^*a)^{1/2}.$$

For a free ultrafilter ω, define the *tracial ultrapower* of A by

$$A^\omega := l^\infty(\mathbb{N}, A)/\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\|_{2,u} = 0\}.$$

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E.g. Unique trace case, with $T(A) = \{\tau\}$. Set $M := \pi_\tau(A)''$ where π_τ is the GNS representation associated to τ.

Then $M^\omega := l^\infty(\mathbb{N}, M)/\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\|_{2,u} = 0\}$ is a vN algebra, and using the Kaplansky Density Theorem, $A^\omega \cong M^\omega$.

Conclusion:

- A^ω is very tractable (a vN algebra).
- A^ω forgets a lot about A.

Tracial ultrapowers

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A more common ultrapower in C*-algebras is the *norm ultrapower*,
$$A_\omega := l^\infty(\mathbb{N}, A)/\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\| = 0\}.$$
I won't talk much about this in this talk.

Buzzword: A^ω is a quotient of A_ω, and the kernel is called the *trace kernel ideal*.

Tracial ultrapowers: origins and applications

Origins in Matui–Sato's work on Toms–Winter conjecture:

Theorem (Matui–Sato, '12)

If A is a simple nuclear C*-algebra with unique trace, which has strict comparison of positive elements, then A is $\mathcal{Z}$-stable.

(i) Unique trace gives structural properties of A^ω (property Γ — we'll come back to this).

(ii) Strict comparison leads to *"property (SI)"*, used to relate properties of $A^\omega \cap A'$ to properties of $A_\omega \cap A'$.

This framework:

(i) Establish structural properties of A^ω, and

(ii) Use extra information to transfer results to A_ω

was used many times.

Tracial ultrapowers: origins and applications

Theorem (Matui–Sato, '12)

If A is a simple nuclear C*-algebra with unique trace, which has strict comparison of positive elements, then A is $\mathcal{Z}$-stable.

This framework was used many times, e.g.:

- Generalizing to the case that $T(A)$ has compact, finite dimensional boundary (KR, S, TWW).
- If A is simple, unital, nuclear, quasidiagonal C*-algebra with unique trace which is $\mathcal{Z}$-stable, then $\mathrm{dr}(A) < \infty$ (MS).
- If A is simple, nuclear, and $\mathcal{Z}$-stable, then $\dim_{\mathrm{nuc}}(A) < \infty$ (SWW, BBSTWW, CETWW, CE).
- A new proof that every faithful trace on a nuclear UCT C*-algebra is quasidiagonal (Schafhauser).
- Every exact UCT C*-algebra with a faithful trace is AF-embeddable (Schafhauser).
- A new proof of classification (CGSTW, work in progress — see Chris' talk).

Tracial completions

Recall: when A has unique trace, can describe A^ω via $\pi_\tau(A)''$.
Beyond the unique trace case, the *tracial completion* $\overline{A}^{T(A)}$ plays the role of $\pi_\tau(A)''$.

Definition

Given a C*-algebra A and a nonempty subset $X \subseteq T(A)$, define

$$\|a\|_{2,X} := \sup_{\tau \in X} \sqrt{\tau(a^*a)}, \quad a \in A.$$

Define $\overline{A}^X :=$ the C*-algebra formed by making $\|\cdot\|_{2,X}$ a norm and adding limit points of bounded, $\|\cdot\|_{2,X}$-Cauchy sequences.

Note: $\|\cdot\|_{2,u} = \|\cdot\|_{2,T(A)}$.

E.g. When A has unique trace, $\overline{A}^{T(A)} \cong \pi_\tau(A)''$.

In general, $A^\omega = (\overline{A}^{T(A)})^\omega$ (appropriately defined).

Tracially complete C*-algebras

Definition

Given a C*-algebra A and a nonempty subset $X \subseteq T(A)$, define

$$\|a\|_{2,X} := \sup_{\tau \in X} \sqrt{\tau(a^*a)}, \quad a \in A.$$

Define $\overline{A}^X :=$ the C*-algebra formed by making $\|\cdot\|_{2,X}$ a norm and adding limit points of bounded, $\|\cdot\|_{2,X}$-Cauchy sequences.

Definition

A *tracially complete C*-algebra* is a pair (M, X) where:

(i) M is a unital C*-algebra,

(ii) X is a nonempty closed convex subset of $T(M)$, and

(iii) $M = \overline{M}^X$.

(M, X) is *factorial* if X is a face. $\leftarrow$ **case of interest.**

Tracially complete C*-algebras

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Condition (iii) says **both** that $\|\cdot\|_{2,X}$ is a norm and that M contains limits of bounded, $\|\cdot\|_{2,X}$-Cauchy sequences.

Tracially complete C*-algebras

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(iii) $M = \overline{M}^X$.

(M, X) is *factorial* if X is a face.

Examples

E.g. $(\overline{A}^{T(A)}, T(A))$ is a factorial tracially complete C*-algebra, provided $T(A)$ is nonempty and compact.

E.g. If M is a von Neumann algebra with faithful trace τ_M then $(M, \{\tau_M\})$ is a tracially complete C*-algebra.
It is factorial when M is a factor.

Examples

E.g. (*trivial W*-bundle*) Let N be a finite factor with trace τ_N and let X be a compact Hausdorff space. Define

$$C_\sigma(X,N) := \{f : X \to N \mid f \text{ is bounded and } \|\cdot\|_{2,\tau_N}\text{-continuous}\}.$$

Then $\mathrm{Prob}(X) \cong T(C(X)) \subseteq T(C_\sigma(X,N))$ in a canonical way: for $\mu \in \mathrm{Prob}(X)$, define a trace τ_μ on $C_\sigma(X,N)$ by

$$\tau_\mu(f) := \int_X \tau_N(f(x))\, d\mu(x).$$

$(C_\sigma(X,N), \mathrm{Prob}(X))$ is a factorial tracially complete C*-algebra.

Buzzword: *W*-bundles*

Let (M, X) be a factorial tracially complete C*-algebra.
$\partial_e X :=$ extreme boundary, i.e., the set of extreme points.

Theorem (Ozawa)

If $\partial_e X$ is compact, then there is:

(i) a canonical copy of $C(\partial_e X)$ inside $\mathcal{Z}(M)$, and

(ii) a canonical conditional expectation $E : M \to C(\partial_e X)$.

Moreover, $\|a\|_{2,X} = \|E(a)\|$ for all $a \in M$.

This case gave rise to Ozawa's notion of a *W*-bundle*.

Examples

E.g. ("*toy example*")

Fix an isomorphism $\theta : M_2(\mathcal{R}) \to \mathcal{R}$.

Define

$$M := \left\{ (a_n)_{n=1}^\infty \in l^\infty(\mathbb{N}, \mathcal{R}) : a_n \xrightarrow{\|\cdot\|_2} \theta \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right\},$$

$$X := \text{closed convex hull of } \{\tau_1, \tau_2, \dots\}$$

where $\tau_i((a_n)_{n=1}^\infty) = \tau_{\mathcal{R}}(a_i)$.

(M, X) is a factorial tracially complete C*-algebra.

$\partial_e X$ is not compact. In fact, $\partial_e X = \{\tau_1, \tau_2, \dots\}$, but

$$\lim_{n\to\infty} \tau_n = \tfrac{1}{2}(\tau_1 + \tau_2) \notin \partial_e X.$$

Examples

Others?

The *Poulsen simplex* is a Choquet simplex X in which $\partial_e X$ is dense in X.

One can construct an AF algebra A with $T(A) \cong X$, and from this a factorial tracially complete C*-algebra $(\overline{A}^{T(A)}, X)$. Is there any tractable way of understanding this object?

Property Γ

Given a tracially complete C*-algebra (M, X), form
$$M^\omega := l^\infty(\mathbb{N}, M)/\{(a_n)_{n=1}^\infty : \lim_{n\to\omega} \|a_n\|_{2,X} = 0\}.$$

Definition

A tracially complete C*-algebra (M, X) has *property* Γ if $\exists$ a projection $p \in M^\omega \cap M'$ with $\tau(p) = \frac{1}{2}$ for all $\tau \in T(M^\omega)$.

Unique trace: get Dixmier's characterization of Γ for II_1 factors.

E.g. If N is a finite factor then $(C_\sigma(X, N), \mathrm{Prob}(X))$ has property Γ iff N has property Γ.

E.g. $(\overline{A}^{T(A)}, T(A))$ has property Γ whenever A is $\mathcal{Z}$-stable.

Property Γ

Property Γ turns out to be a very powerful hypothesis.

Theorem (CCEGSTW)

Let $(M, X), (N, Y)$ be separable factorial tracially complete C*-algebras with property Γ, such that every tracial GNS representation is hyperfinite. Then $(M, X) \cong (N, Y)$ iff $X \cong Y$.

Corollary

If A is a separable nuclear $\mathcal{Z}$-stable C*-algebra, then $(\overline{A}^{T(A)}, T(A))$ is approximately finite dimensional.

In the case that $\partial_e X$ is compact, this theorem is due to Ozawa, where he shows that $(M, X) \cong (C_\sigma(\partial_e X, \mathcal{R}), X)$.

Questions

Question

Is there a factorial tracially complete C*-algebra (M, X) where $X \subsetneq T(M)$?

This question is open (and interesting) even for:

- $M = \overline{A}^{T(A)}$ where A is a separable nuclear C*-algebra.
- $M = C_\sigma(X, \mathcal{R})$ where X is a compact Hausdorff space.
- $M = C_\sigma(\mathbb{N} \cup \{\infty\}, \mathcal{R})$.

Questions

Question

Is there a factorial tracially complete C*-algebra (M, X) with all tracial GNS representations hyperfinite, that does not have property Γ?

This question is open (and interesting) even for:

- $M = \overline{A}^{T(A)}$ where A is a separable nuclear C*-algebra. A negative answer would resolve the last piece of the Toms–Winter conjecture.
- W*-bundles, i.e., the case that $\partial_e X$ is compact.

However, Ozawa showed the answer is yes in the case $\partial_e X$ is compact *and finite dimensional*.

Questions

Norm-regularity questions: does every factorial tracially complete C*-algebra have real rank zero? stable rank one? comparison of projections? ...

These questions are open (and interesting) even for:

- $M = \overline{A}^{T(A)}$ where A is a separable nuclear C*-algebra.
- $M = C_\sigma(X, \mathcal{R})$ where X is a compact Hausdorff space.
- $M = C_\sigma(\mathbb{N} \cup \{\infty\}, \mathcal{R})$.

Questions

Question

If (M, X) is a tracially complete C*-algebra, is (M^ω, X^ω) as well?

Question

Is there a tracially complete C*-algebra (M, X) such that M^ω does not have comparison of projections?

References

From slide 7

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