Quasidiagonality and amenability

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Joint work with Stuart White and Wilhelm Winter

Quasidiagonality

Definition

A (separable) C^*-algebra A is *quasidiagonal* if there exists a sequence of c.p.c. maps $\phi_n : A \to M_{k_n} = M_{k_n}(\mathbb{C})$ that are:
(i) approximately multiplicative ($\|\phi_n(a)\phi_n(b) - \phi_n(ab)\| \to 0$, for $a, b \in A$); and
(ii) approximately isometric ($\|\phi_n(a)\| \to \|a\|$, for $a \in A$).

Fact: A is quasidiagonal if there exists an injective *-homomorphism $\phi : A \to \prod_\omega M_{k_n}$ with a c.p.c. lift $A \to \prod_{n=1}^\infty M_{k_n}$ where $\prod_\omega M_{k_n}$ denotes the ultraproduct with respect to the free ultrafilter ω.

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Quasidiagonality: examples

A is quasidiagonal if there exists an injective *-homomorphism $\phi : A \to \prod_\omega M_{k_n}$ with a c.p.c. lift $A \to \prod_{n=1}^\infty M_{k_n}$.

If A is subhomogeneous ($A \subseteq C(X, M_m)$) then A is quasidiagonal.

If A is approximately subhomogeneous (ASH) then A is quasidiagonal.

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Quasidiagonality: non-examples

If A contains an infinite projection, then A is not quasidiagonal.

This is straightforward: $\prod_{\omega} M_{k_n}$ contains no infinite projections.

Theorem (Rosenberg)

Let G be a discrete group. If $C_r^*(G)$ is quasidiagonal, then G must be amenable (equivalently, $C_r^*(G)$ must be amenable).

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Amenable traces

Definition (N. Brown)

A unital separable C*-algebra A has an *amenable trace* if there is a sequence of u.c.p. maps $\phi_n : A \to M_{k_n}$ that are $\|\cdot\|_2$-approximately multiplicative, i.e., $\|\phi_n(a)\phi_n(b) - \phi_n(ab)\|_2 \to 0$, for $a, b \in A$, where $\|x\|_2 := \tau_{M_{k_n}}(x^*x)^{1/2}$.

(An amenable trace is $\lim_{n\to\omega} \tau_{M_{k_n}} \circ \phi_n$.)

Fact: If A is amenable then every trace is amenable.

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Quasidiagonality: obstructions

In summary, if A is quasidiagonal then:

(i) A is stably finite, and

(ii) if A is unital then it has an amenable trace.

These are the only known obstructions to quasidiagonality.

Question (Blackadar–Kirchberg)

Is every amenable, stably finite C^*-algebra quasidiagonal?

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Is the hyperfinite II_1 factor $\mathcal{R}$ quasidiagonal?

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Quasidiagonal traces: applications

Decomposition rank (Kirchberg–Winter) and nuclear dimension (Winter–Zacharias) are noncommutative versions of covering dimension that have featured prominently in the recent study of structure of C*-algebras.

Conjecture

A simple C*-algebra has finite decomposition rank if and only if it has finite nuclear dimension, it is quasidiagonal, and all traces are quasidiagonal.

This conjecture is known to hold for many simple C*-algebras (Bosa–Brown–Sato–T–White–Winter). (Namely, for A provided $\partial_e T(A)$ is compact. The unique trace case is due to Matui–Sato.)

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Classification predicts quasidiagonality.

Suppose that $\mathcal{C}$ is a class of stably finite C*-algebras that are classified, and the range of invariant is exhausted by approximately subhomogeneous C*-algebras. Then every trace on every C*-algebra in $\mathcal{C}$ is quasidiagonal.

In fact, if A is such that $A \otimes \mathcal{Z} \in \mathcal{C}$, then every trace on A is quasidiagonal.

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Classification Theorem (Gong–Lin–Niu ’15, Elliott–Gong–Lin–Niu ’15)

Let A and B be simple, separable, unital C*-algebras with finite nuclear dimension, which satisfy the Universal Coefficient Theorem (UCT). Suppose that all traces on A and B are quasidiagonal. Then

$$A \cong B \text{ if and only } \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

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Main result

Theorem (T–White–Winter)

Let A be an amenable C^*-algebra which satisfies the UCT and let $\tau \in T(A)$ be a faithful trace. Then τ is quasidiagonal.

Hence A is quasidiagonal.

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$$0 \to \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

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Question

Does every amenable C*-algebra satisfy the UCT?

Among amenable C*-algebras, the UCT is preserved by Morita equivalence, extensions, inductive limits, and $\mathbb{Z}$- and $\mathbb{R}$-crossed products,... .

Tu '99: If $\mathcal{G}$ is an amenable étale groupoid, then $C^*(\mathcal{G})$ satisfies the UCT.

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Definition (Rosenberg–Schochet)

A satisfies the UCT if, for every σ-unital C^*-algebra B,

$$0 \to \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

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Does every amenable C*-algebra satisfy the UCT?

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Main result

Theorem (T–White–Winter)

Let A be an amenable C^*-algebra which satisfies the UCT and let $\tau \in T(A)$ be a faithful trace. Then A and τ are quasidiagonal.

Corollary (with Rosenberg, Tu)

A group G is amenable if and only if $C_r^*(G)$ is quasidiagonal.

Proof of $\Rightarrow$. The canonical trace is faithful.

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