$\mathcal{Z}$-stability and decomposition rank

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Noncommutative dimension

Desirable to extend the theory of dimension to the noncommutative case (C^*-algebras).

Some older measures of dimension: stable rank (Rieffel '83), real rank (Brown-Pedersen '91).

Decomposition rank and nuclear dimension are more recent measures of dimension for C^*-algebras.
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Decomposition rank (Kirchberg-Winter '04

A C^*-alg. A has decomposition rank $\leq n$ if

For any finite subset $\{a_1, \dots, a_k\} \subset A$ and any $\epsilon > 0$, there exist f.d. algebras $F_0, \dots, F_n$ and c.p.c. maps

$$A \xrightarrow{\psi} F_0 \oplus \cdots \oplus F_n \xrightarrow{\phi} A$$

such that $\|\phi\psi(a_i) - a_i\| < \epsilon$ for all i, and

$\phi|_{F_i}$ is order 0 (orthogonality preserving, $ab = 0 \Rightarrow \phi(a)\phi(b) = 0$.)

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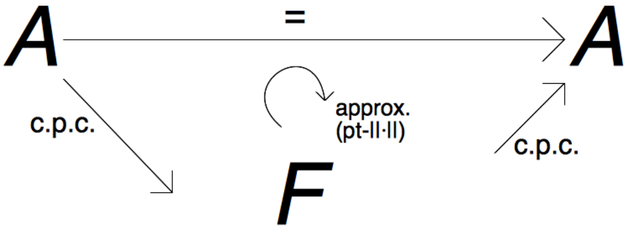

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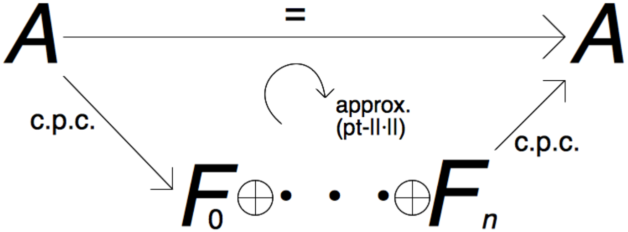

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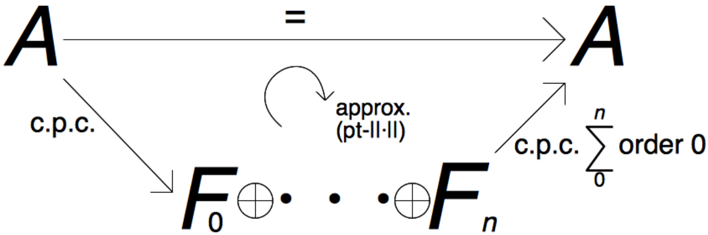

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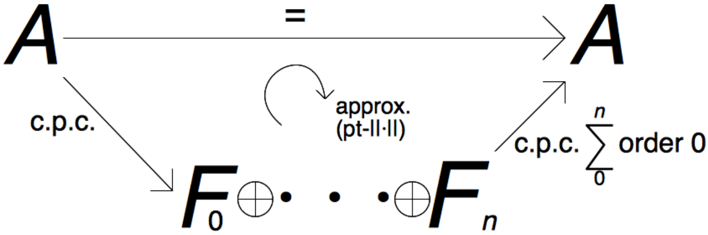

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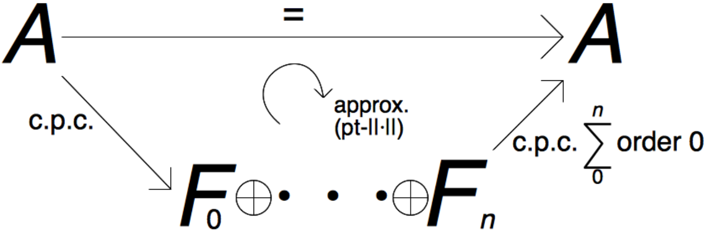

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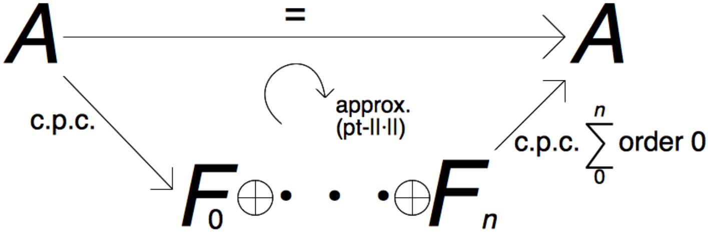

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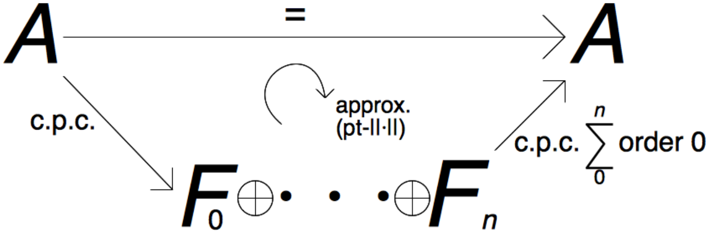

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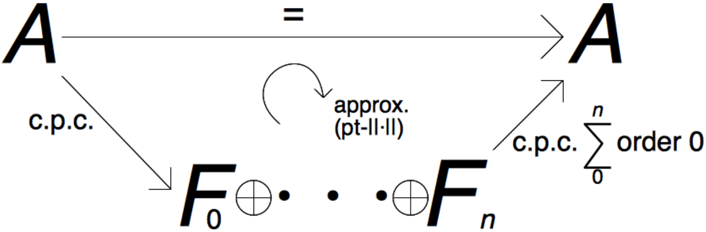

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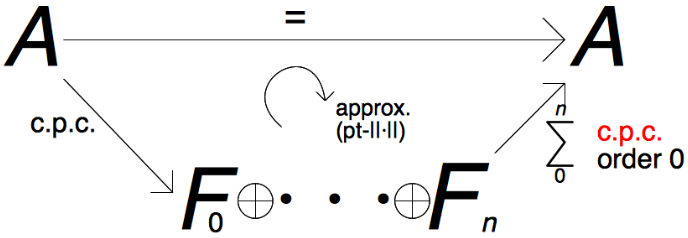

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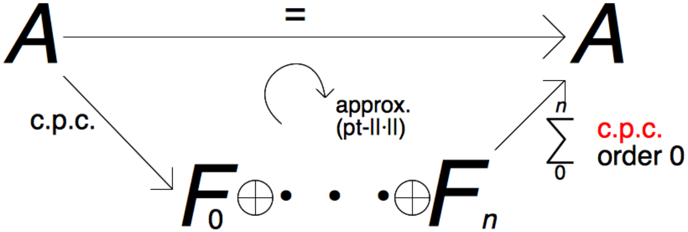

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While $\mathrm{dr}\, C(X, M_n) = \mathrm{dr}\, C(X) = \dim X$ for every compact metrizable X (doesn't depend on n),

$$\mathrm{dr}\, \varinjlim C(X_i, M_{n_i}) \leq 2$$

if $\lim \frac{\dim X_i}{n_i} = 0$ (SDG = slow dimension growth) and the limit is simple.

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For any simple, separable, finite C^*-algebra A such that $K_0(A)$ is unperforated,

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What is the decomposition rank of $C(X, M_{n^\infty}) = \lim_{k\to\infty} C(X, M_{n^k})$?

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(Like tensoring with M_{n^∞}.)

$\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ and $K_*(\mathcal{Z}) \cong K_*(\mathbb{C})$, so being $\mathcal{Z}$-stable (of the form $B \otimes \mathcal{Z}$) is unrestrictive.
(Unlike M_{n^∞}-stability.)

Theorem (T-Winter)

The decomposition rank of $C_0(X, \mathcal{Z})$ is always at most 2.

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We will outline the proof – but with $\mathcal{Z}$ replaced by M_{n^∞}.

Ingredients:

- Reduce to finding suitable approx. partitions of unity inside $C(X, M_{n^\infty})$;
- An orthogonal approx. partition of unity in $C(X, M_{n^\infty})$ – approximation in trace, not norm;
- Quasidiagonality to fill the tracial holes with $C_0(Z, \mathcal{O}_2)$;
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We will outline the proof – but with $\mathcal{Z}$ replaced by M_{n^∞}.

Ingredients:

- Reduce to finding suitable approx. partitions of unity inside $C(X, M_{n^\infty})$;
- An orthogonal approx. partition of unity in $C(X, M_{n^\infty})$ – approximation in trace, not norm;
- Quasidiagonality to fill the tracial holes with $C_0(Z, \mathcal{O}_2)$;
- (Kirchberg-Rørdam '05) $\dim_{nuc} C_0(Z, \mathcal{O}_2) \leq 3$.

Relative decomposition rank

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Find that $\mathrm{dr}\, C(X, M_{n^\infty})$ depends only on approximating $C(X)$ inside $C(X, M_{n^\infty})$.

Can rephrase the problem in terms of finding an approximate partition of unity.

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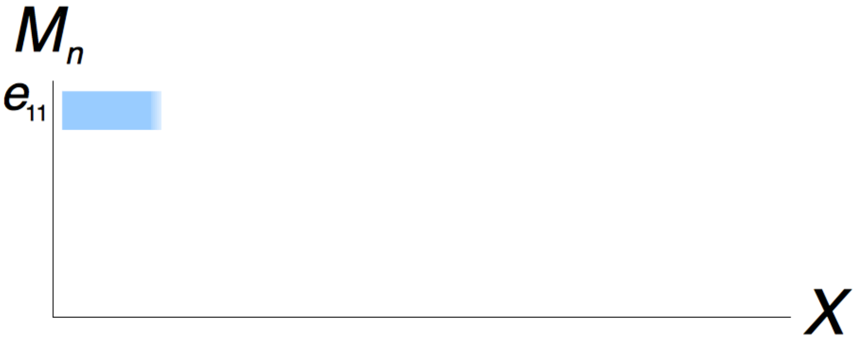
“Tracially” approximate orthogonal partition of unity



Orthogonal positive elements $a_1, \ldots, a_k \in C(X, M_n)$ with small support such that $\sum a_i \approx 1$ **in trace, though not in norm.** (Note: the error in approximation depends on n.)

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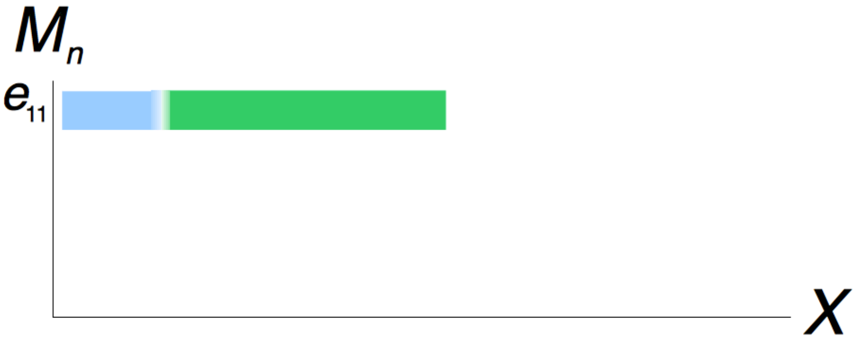

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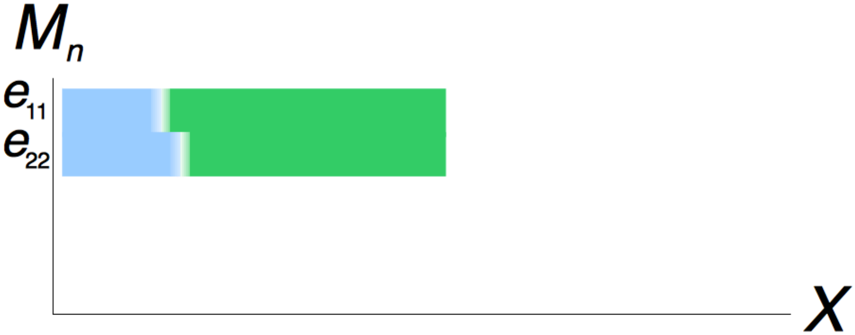

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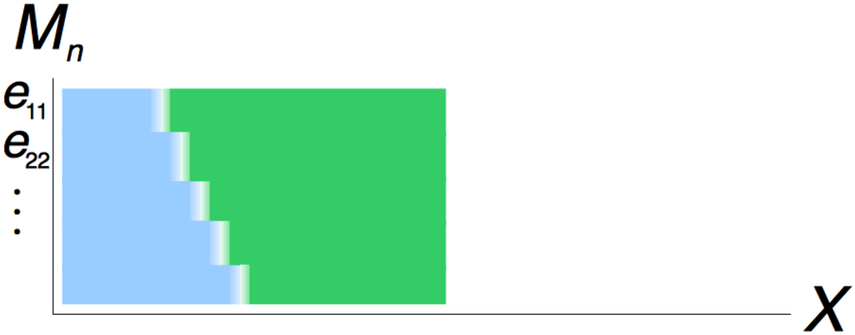

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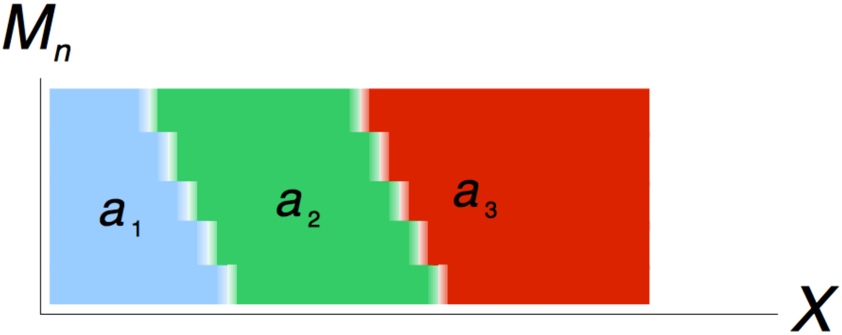

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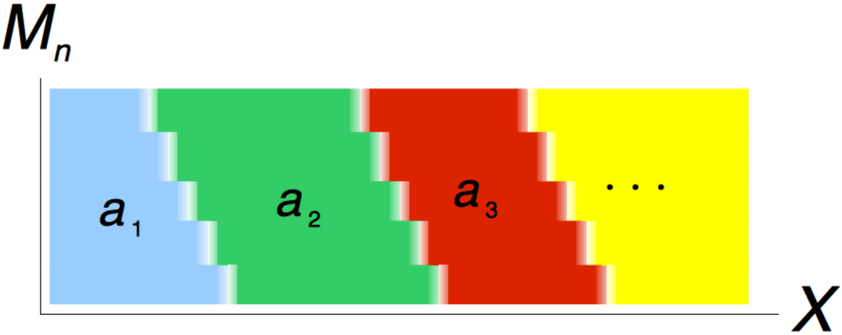

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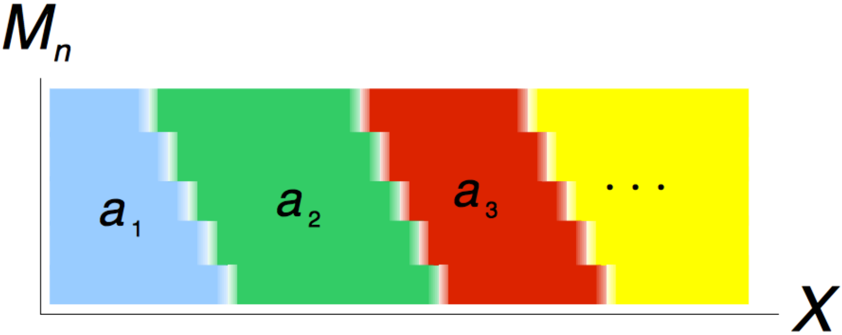

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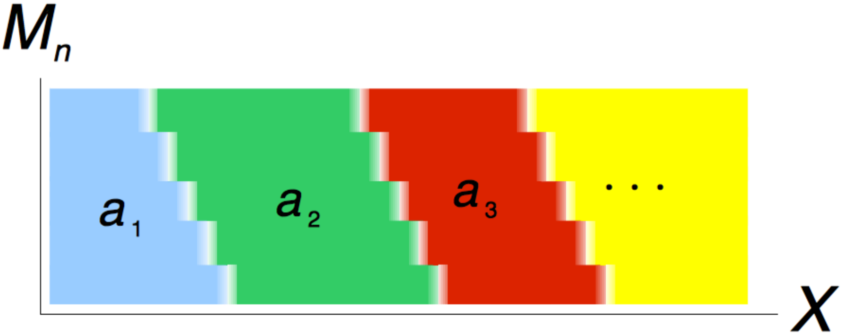

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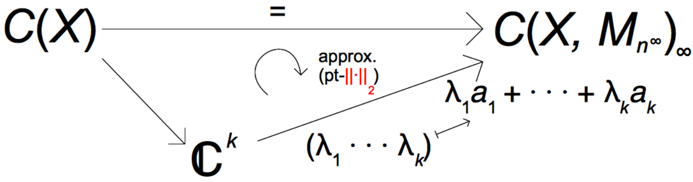

Get ϕ using quasidiagonality of $C_0((0,1],\mathcal{O}_2)$.
The space Z allows us to move a homomorphism $C_0((0,1],\mathcal{O}_2) \to (M_{n^\infty})_\infty$ around to fill holes.

$$\begin{tikzcd}
C(X) \arrow[rr, "="] \arrow[dr] & \arrow[d, phantom, "\text{approx. (pt-}\|\cdot\|\text{)}"] & C(X, M_{n^\infty})_\infty \\
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& & F_0 \oplus \cdots \oplus F_3 \arrow[urr] & &
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Kirchberg-Rørdam: $\dim_{nuc}(C_0(Z, \mathcal{O}_2)) \leq 3$.

$\therefore \dim_{nuc} C(X, M_{n^\infty}) \leq 4$.

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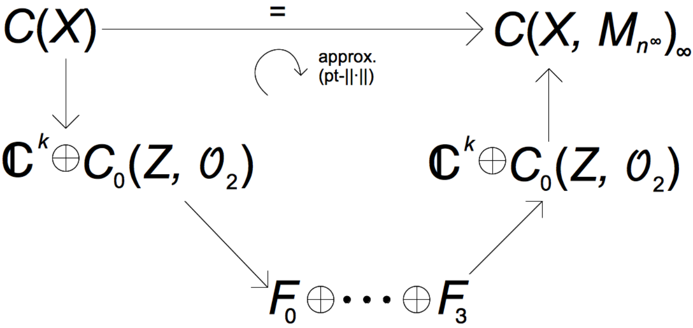

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With more care, get $\mathrm{dr} \leq 2$.

Questions

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Is $\dim_{nuc}(A \otimes \mathcal{Z}) < \infty$ for every nuclear C^*-algebra?
($\mathrm{dr}\,(A \otimes \mathcal{Z}) < \infty$ when A is sufficiently finite?)

The question is open even in the simple case.

Theorem (Winter '10)

$\dim_{nuc}(A) < \infty$ implies that $A \cong A \otimes \mathcal{Z}$ (for A simple, sep., unital, non-type I.)

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Can we approximate $C(X)$ inside $C(X, M_n)$ with a 3-decomposable system? An $(m+1)$-decomposable system, where $m < \dim X$? (Or is it necessary to put $C(X)$ into $C(X, M_{n^\infty})$?)

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