The real span of a dimension group

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Dimension groups

A **dimension group** is a directed ordered group $(G, G^+ := \{g \in G : g \geq 0\})$ satisfying:

(i) Unperforation: if $g + \cdots + g \geq 0$ then $g \geq 0$

(ii) Riesz interpolation: given a_1, a_2, c_1, c_2 satisfying

$$\begin{matrix} a_1 \\ a_2 \end{matrix} \leq \begin{matrix} c_1 \\ c_2, \end{matrix}$$

$\exists b$ satisfying

$$\begin{matrix} a_1 \\ a_2 \end{matrix} \leq b \leq \begin{matrix} c_1 \\ c_2. \end{matrix}$$

Examples: lattice ordered groups (use $\max\{a_1, a_2\}$ or $\min\{c_1, c_2\}$ as an interpolant),
$C(X, \mathbb{R})$ with strict order, $f < g$ if $f(x) < g(x)\ \forall x$.

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Recall that an AF algebra is given by an inductive limit of finite-dimensional C^*-algebras.

$$A_1 \to A_2 \to \cdots \to A.$$

Ordered K_0-group computation:

Theorem (Elliott, Effros-Handelman-Shen)

The countable dimension groups are exactly the K_0-groups of AF algebras.

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A good embedding of a dimension group

Our goal: to understand dimension groups better by embedding them into real vector spaces.

Embed G into a real vector space V (s.t. G spans V).
Set $V^+ = \mathbb{R}^+ \cdot G^+$
$= $ the real cone generated by G^+.

We want:

(i) (V, V^+) to be an ordered vector space
(need $V^+ \cap -V^+ = 0$, ie. not just preordered)

(ii) to recover G^+ from V^+:
$G^+ = V^+ \cap G$

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Bad embeddings: example one

(V, V^+) may not be an ordered vector space. Let $G = \mathbb{Z}^2$, $G^+ = \{(x, y) : x + \theta y \geq 0\}$ $(\theta \notin \mathbb{Q})$.

This ordered group is denoted $\mathbb{Z} + \theta\mathbb{Z}$; indeed, it embeds into $V = \mathbb{R}$ by $(x, y) \mapsto x + \theta y$, which is a good embedding.

But $(x, y) \mapsto x + \eta y$ (where $\eta \neq \theta$) is bad, since $V^+ = \mathbb{R}$. Also $V^+ \cap G = G \neq G^+$.

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Bad embeddings: example two

(V, V^+) may not have Riesz interpolation.
Pick four $\mathbb{Q}$-linearly independent vectors $v_1, \ldots, v_4$ in one half-space of $\mathbb{R}^3$, such that none of them is in the cone generated by the other three.

Embed $G = (\mathbb{Z}^4, \mathbb{N}^4)$ into $V = \mathbb{R}^3$ by $(x_1, \ldots, x_4) \mapsto x_1 v_1 + \cdots + x_4 v_4$.

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Good embeddings

There is always a good embedding:

Theorem (Maloney-T)

If (G, G^+) is a finite-rank dimension group then there exists an embedding $G \hookrightarrow V = \mathbb{R}^n$, such that:

(i) is an ordered vector space with Riesz interpolation; and

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In fact, we may use the canonical embedding

$$\begin{aligned} G &\hookrightarrow G \otimes \mathbb{Q} \\ &\cong \mathbb{Q}^n \hookrightarrow \mathbb{R}^n \end{aligned}$$

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The proof mainly looks at the positive functionals $H \to \mathbb{R}$ for ideals H of G.

(An ideal is an order-convex, directed subgroup.)

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If $K \subseteq H$, $f : H \to \mathbb{R}$ is an extreme positive functional then $f|_K$ is either zero or an extreme positive functional.

Lemma: Restriction is one-to-one

If $K \subseteq H$, $f : K \to \mathbb{R}$ is an extreme positive functional which extends to a positive functional on H, then it has a unique extreme extension.

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there exists an extreme extension to $H_1 + H_2$.

Proof: $f(x_1 + x_2) = f_1(x_1) + f_2(x_2)$ is the unique common extension of f_1, f_2, and it is positive.

Use uniqueness of extreme extensions to conclude that f is extreme.

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Can prove the two lemmas for $K_0(\mathrm{AF})$ using operator theory.

countable dimension group	$\leftrightarrow$	AF algebra
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positive homomorphism	$\leftrightarrow$	densely finite trace
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Lemma: Restriction preserves extremeness

If $K \subseteq H$, $f : H \to \mathbb{R}$ is an extreme positive functional then $f|_K$ is either zero or an extreme positive functional.

Proof: $H = K_0(A)$, $K = K_0(I)$, f corresponds to traceable factor representation $\pi : A \to B(\mathcal{H})$.

Then $\pi(A)'' = \pi(I)'' \oplus M$, so $\pi|_I$ is either 0 or a factor rep.

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Lemma: Restriction is one-to-one

If $K \subseteq H$, $f : K \to \mathbb{R}$ is an extreme positive functional which extends to a positive functional on H, then it has a unique extreme extension.

Proof: $H = K_0(A)$, $K = K_0(I)$, and let f correspond to the trace τ on I.

There exists a rep. π of A such that $\pi(I)''$ is a factor with a faithful trace ρ, and $\tau = \rho \circ \pi|_I$.

Then $\tilde{\pi}(a) = \pi(a)1_{\pi(I)''} = \text{WOT-lim}\,\pi(ae_\alpha)$ is a factor rep. of A.

And, if π is a factor rep. then $1_{\pi(I)''} = 1_{\pi(A)''}$ so $\pi = \tilde{\pi}$.

Hence, $\tilde{\tau}(a) = \lim \tau(ae_\alpha)$ is the unique extreme densely finite trace extending τ.

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- Lemmas don't need G countable.

- Lemmas (noncountable version) used to describe explicitly all finite-dimensional ordered real vector spaces with Riesz interpolation.

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