Regularity properties for stably projectionless C^*-algebras

Aaron Tikuisis
atiku_01@uni-muenster.de

Universität Münster

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Villadsen's counterexamples.

Reaction: start an exclusive club of good C^*-algebras.

The members of the exclusive club are:

(i) $\mathcal{Z}$-stable;
(ii) Topologically low-dimensional;
(iii) Cuntz semigroup-regular.

Hopefully these are equivalent and this is largely known for simple, unital C^*-algebras.

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What about simple, nonunital C^*-algebras?

Stably projectionless C^*-algebras?

These can be well-behaved (simply tensor with $\mathcal{Z}$) and, in fact, have (seemingly) important well-behaved examples.

Theorem (T, '11)

There exists a simple, seperable, nuclear, stably projectionless C^*-algebra A for which $\mathcal{C}u(A)$ does not have m-comparison, for any m. In particular, $A \not\cong A \otimes \mathcal{Z}$ and $\dim_{nuc} A = \infty$.

(In fact, A is ASH.)

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Nonunital complications

Theorem

Let A be the inductive limit of the system

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots$$

where each map ϕ_i^{i+1} is unital and injective. Then if $a \in A_i$ is nonzero then there exists $j \geq i$ such that $\phi_i^j(a)$ generates A_j as an ideal

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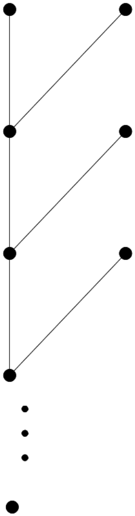

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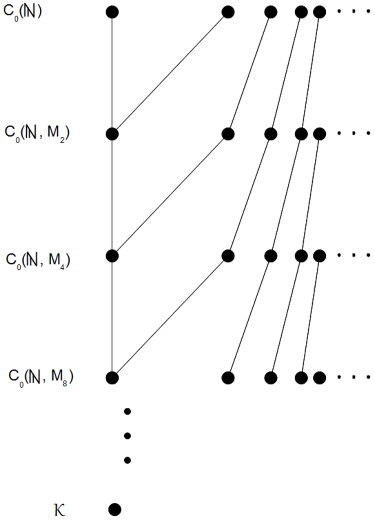

A structure theorem

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Let A be a simple seperable ASH algebra. Then A is the inductive limit of a system

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots$$

where, for all i,

(i) A_i is subhomogeneous;

(ii) $\mathrm{Prim}_n(A_i)$ is finite-dimensional for all n;

(iii) ϕ_i^{i+1} is injective and full; and

(iv) $\mathrm{Prim}(A_i)$ **is compact.**

In particular, if $a \in A_i$ is nonzero then there exists $j \geq i$ such that $\phi_i^j(a)$ generates A_j as an ideal.

Key to the proof: Blackadar-Cuntz ('82): $A \otimes \mathcal{O}_2 \otimes \mathcal{K}$ has a projection, which can be used like a unit.

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Slow dimension growth

For A subhomogeneous and $\pi : A \to M_n$ an irreducible, nondegenerate representation, set

$$d_{top}(\pi) = \dim \mathrm{Prim}_n(A).$$

For $a \in A$, define the dimension-rank ratio of a by

$$R_{d:r}(a) := \sup_{\pi} \frac{d_{top}(\pi)}{\mathrm{rank}\,\pi(a)}.$$

Think:

$$\sup_n \frac{\dim \mathrm{Prim}_n(A_i)}{n}.$$

(This is what we get if a is strictly positive.)

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Let

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be an inductive system of subhomogeneous algebras as in the structure theorem.

(A_i, ϕ_i^{i+1}) has **slow dimension growth** if

(i) There exists i and $a \in A_i$ such that

$$R_{d:r}(\phi_i^j(a)) \to 0;$$

or, equivalently,

(ii) For any i and $a \in A_i \setminus \{0\}$,

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Regularity and slow dimension growth

Let A be simple, ASH.

Proposition

$A \otimes \mathcal{Z}$ always has slow dimension growth.

Theorem (T '11)

If A has slow dimension growth then $\mathcal{C}u(A)$ has 0-comparison.

Proof: uses nonunital version of radius of comparison.

(Note: Rørdam already showed that $\mathcal{C}u(A \otimes \mathcal{Z})$ has 0-comparison.)

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To do: other generalizations

Slow dimension growth $\Rightarrow$ almost divisible (unital case: Toms '09).

Finite nuclear dimension $\Rightarrow$ m-comparison: done (Robert, '10).

Finite nuclear dimension $\Rightarrow$ m-divisible (unital case: Winter, '10).

$(m, \overline{m})$-pure $\Rightarrow$ $\mathcal{Z}$-stable (unital case: Winter, '10).

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