Partial isometric representations of semigroups

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June 8, 2021

Semigroups

P countable semigroup (associative multiplication)

Cancellative: $\begin{array}{ll} ps = pq & \Rightarrow s = q \\ sp = qp & \Rightarrow s = q \end{array}$

Principal right ideal: $rP = \{rq \mid q \in P\}$
Principal left ideal: $Pr = \{qr \mid q \in P\}$

Assume $1 \in P$ (ie, P is a monoid)

Semigroups

Study P by representing on a Hilbert space, similar to groups.

$\ell^2(P)$ – square-summable complex functions on P.

δ_x – point mass at $x \in P$. Orthonormal basis of $\ell^2(P)$.

$v_p : \ell^2(P) \to \ell^2(P)$ bounded operator $v_p(\delta_x) = \delta_{px}$ (necessarily isometries)

$\{v_p\}_{p \in P}$ generate the reduced C*-algebra of P, $C_r^*(P)$

$v : P \to C_r^*(P)$ is called the left regular representation

Unlike the group case, considering all representations turns out to be a disaster

Nica, Li: we have to care for ideals.

The solution

For $X \subset P$, then $e_X : \ell^2(P) \to \ell^2(P)$ is defined by

$$(e_X\xi)(p) = \begin{cases} \xi(p) & \text{if } p \in X \\ 0 & \text{otherwise.} \end{cases}$$

Note: $v_1 = e_P$

Note that in $\mathcal{B}(\ell^2(P))$,

$$v_p e_X v_p^* = e_{pX} \qquad v_p^* e_X v_p = e_{p^{-1}X}$$

If $p \in P$ and X is a right ideal, then

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

are right ideals too.

The solution

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

$\mathcal{J}(P)$ – smallest set of right ideals containing P, $\emptyset$, and closed under finite intersection and the above operations – constructible ideals.

These are the ideals which are "constructible" inside $C_r^*(P)$.

1. $e_X e_Y = e_{X \cap Y}$
2. $e_P = 1$, $e_\emptyset = 0$
3. $v_p e_X v_p^* = e_{pX}$ and $v_p^* e_X v_p = e_{p^{-1}X}$

The solution

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Definition (Li)

$C^*(P)$ is the universal C-algebra generated by isometries $\{v_p \mid p \in P\}$ and projections $\{e_X \mid X \in \mathcal{J}(P)\}$ satisfying the above (and $v_p v_q = v_{pq}$).*

Partial isometric representations

$\pi : P \to A$ is a partial isometric representation if $\pi(qr) = \pi(q)\pi(r)$ and $\pi(q)$ is a partial isometry for all $q \in P$.

$\implies \pi(p)^n$ is a partial isometry for all n (power partial isometry).

Hancock–Raeburn (1990) $P = \mathbb{N}$ (a single power isometry)

$$J_n : \mathbb{C}^n \to \mathbb{C}^n$$

$$J_n(\delta_i) = \begin{cases} \delta_{i+1} & i < n \\ 0 & i = n \end{cases} \qquad \text{truncated shift}$$

They defined $J := \bigoplus_{n=2}^{\infty} J_n : \bigoplus_{n=2}^{\infty} \mathbb{C}^n \to \bigoplus_{n=2}^{\infty} \mathbb{C}^n$

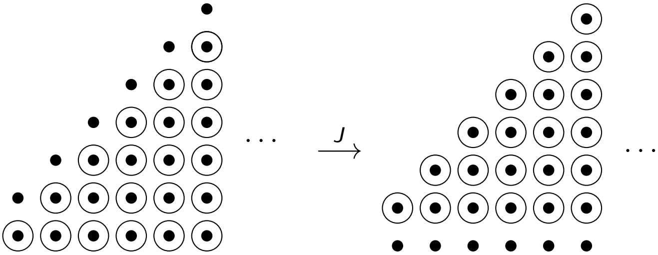
They showed $C^*(J)$ is universal for partial isometric representations of $\mathbb{N}$

Partial isometric representations

$\bigoplus_{n=2}^{\infty} \mathbb{C}^n \cong \ell^2(\Delta)$, where $\Delta = \{(n,i) \in \mathbb{N} \times \mathbb{N} : n \geq 2, i \leq n\}$

$$J : \ell^2(\Delta) \to \ell^2(\Delta)$$

$$J(\delta_{(n,i)}) = \begin{cases} \delta_{(n,i+1)} & i < n \\ 0 & i = n \end{cases}$$



Partial isometric representations

$P = \mathbb{N}$: $\Delta = \{(n,i) : i \le n\}$. Note that $i \le n \iff i + \mathbb{N} \supseteq n + \mathbb{N}$

For P a general cancellative monoid, let

$$\Delta = \{(a,x) \in P \times P : Px \supseteq Pa\}$$

$$J : P \to \mathcal{B}(\ell^2(\Delta))$$

$$J_p(\delta_{(a,x)}) = \begin{cases} \delta_{(a,px)} & Ppx \supseteq Pa \\ 0 & \text{otherwise} \end{cases}$$

We define $C^*_{\text{ts}}(P, P^{\text{op}})$ to be the C*-algebra generated by the J_p.

The universal algebra

As in Nica/Li, we care for the projections "constructible" inside $C^*_{ts}(P, P^{op})$.
For any subset $Y \subseteq \Delta$ and $p \in P$, let

$$\begin{aligned} Y_p &= \{(a, px) : (a, x) \in Y\} \\ Y^p &= \{(a, x) : (a, px) \in Y\} \end{aligned}$$

For the projections $e_Y \in \mathcal{B}(\ell^2(\Delta))$ we have

$$\begin{aligned} J_p e_Y J_p^* &= e_{Y_p} \\ J_p^* e_Y J_p &= e_{Y^p} \end{aligned}$$

$\mathcal{J}(P) =$ smallest subset of $\mathcal{P}(\Delta)$ closed under finite intersection, $Y \mapsto Y^p$, $Y \mapsto Y_p$, containing Δ and $\emptyset$—constructible subsets.

The universal algebra

$\mathcal{J}(P)$ = smallest subset of $\mathcal{P}(\Delta)$ closed under finite intersection, $Y \mapsto Y^p$, $Y \mapsto Y_p$, containing Δ and $\emptyset$—constructible subsets.

Definition

Let P be a cancellative monoid. Then $C^*(P, P^{\mathrm{op}})$ is the universal C*-algebra generated by partial isometries $\{S_p\}_{p \in P}$ and projections $\{e_Y\}_{Y \in \mathcal{J}(P)}$ such that

1. $S_p S_q = S_{pq}$ for all $p, q \in P$,
2. $e_Y e_Z = e_{Y \cap Z}$ for all $Y, Z \in \mathcal{J}(P)$.
3. $e_\Delta = 1, e_\emptyset = 0$,
4. $S_p e_Y S_p^* = e_{Y_p}$ for all $Y \in \mathcal{J}(P)$, $p \in P$, and
5. $S_p^* e_Y S_p = e_{Y^p}$ for all $Y \in \mathcal{J}(P)$, $p \in P$.

LCM monoids

Definition

A cancellative monoid P is called LCM if

- $pP \cap qP = rP$ for some $r \in P$, or is empty (right LCM)
- $Pp \cap Pq = Pk$ for some $k \in P$, or is empty (left LCM)

Example: Free semigroups

Example: Zappa-Szép products associated to recurrent self-similar groups

Example: Baumslag-Solitar monoids

If P embeds in an amenable group, then $C^*_{\text{ts}}(P, P^{\text{op}}) \cong C^*(P, P^{\text{op}})$

LCM monoids

In the LCM case, the set

$$\mathcal{S}_P = \{S_p S_q^* S_r : q \in Pp \cap rP\} \cup \{0\}$$

is closed under multiplication, $*$, and consists of partial isometries, and so forms an inverse semigroup.

We give an abstract characterization of this inverse semigroup and show $C^*(P, P^{\mathrm{op}}) \cong C_u^*(\mathcal{S}_P)$, Paterson's universal C*-algebra for $\mathcal{S}_P$.

In this setting, the natural boundary quotient is Exel's tight C*-algebra of $\mathcal{S}_P$, so we take this as the definition, i.e.

$$\mathcal{Q}(P, P^{\mathrm{op}}) := C^*_{\mathrm{tight}}(\mathcal{S}_P)$$

In turn, this gives our algebras groupoid models

LCM Monoids

$$S_pS_q^*S_r \mapsto \begin{cases} S_pS_{rp}^*S_r = S_pS_p^*S_r^*S_r & q = rp \\ 0 & \text{otherwise} \end{cases}$$

extends to a faithful conditional expectation.

$S_pS_p^*$, $S_r^*S_r$ commute.

$$pP \cap qP = rP \implies S_pS_p^*S_qS_q^* = S_rS_r^*$$
$$Pa \cap Pb = Pc \implies S_a^*S_aS_b^*S_b = S_c^*S_c$$

$\implies$ the spectrum of the range of the conditional expectation is a product of (ultra)filter spaces, from the two semilattices

$$\{pP : p \in P\} \cup \{\emptyset\} \qquad \{Pq : q \in P\} \cup \{\emptyset\}$$

Free semigroups

X – finite set with n elements, X^* finite words in X.

X^* is an LCM monoid under concatenation

$C^*(X^*) \cong C_r^*(X^*) \cong \mathcal{T}_n$

$\mathcal{Q}(X^*) \cong \mathcal{O}_n$

$\mathcal{Q}(X^*, X^{*\mathrm{op}}) \cong C(X^{\mathbb{Z}}) \rtimes_\sigma \mathbb{Z}$ where σ is the left shift

Here the ultrafilter space is the product $X^{\mathbb{N}} \times X^{\mathbb{N}}$, one copy each from $\{\alpha X^* : \alpha \in X^*\} \cup \{\emptyset\}$ and $\{X^*\beta : \beta \in X^*\} \cup \{\emptyset\}$