On operator amenability of Fourier-Stieltjes algebras

Nico Spronk, U. of Waterloo

CMS 2021
Operator algebras and applications
June 8, 2021

(Operator) amenability and weak amenability

$\mathcal{A}$ – Banach algebra (and operator space)

[Johnson '72 & '73] $\mathcal{A}$ is (op.) amenable if it satisfies equiv. conds.

(i) (b.a.d.) $\exists$ (com.) bdd. net $(d_\alpha) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ ((op.) proj. t.p.) s.t.

$$a \otimes 1\, d_\alpha - d_\alpha\, 1 \otimes a \xrightarrow{\alpha} 0, \quad \mathrm{mult}(d_\alpha) \text{ approx. id.}$$

Let $\mathrm{AM}_{(op)}(\mathcal{A}) = \inf\{\sup_\alpha \|d_\alpha\| : (d_\alpha) \text{ b.a.d}\}$.

(ii) $\forall$ (com.) bdd. hom'm & anti-hom'm $\lambda, \rho : \mathcal{A} \to (\mathcal{C})\mathcal{B}(\mathcal{X})$ w. $[\lambda(\mathcal{A}), \rho(\mathcal{A})] = \{0\}$, & $\forall\ D \in (\mathcal{C})\mathcal{B}(\mathcal{X}^*)$

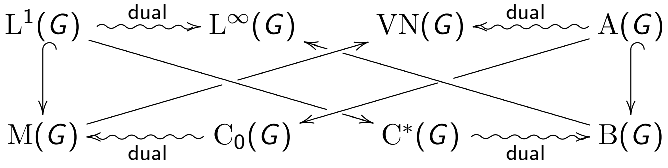
$$D(ab) = \rho(a)^* D(b) + \lambda(b)^* D(a) \quad \Rightarrow \quad D(a) = \rho(a)^* f - \lambda(a)^* f.$$

for some $f \in \mathcal{X}^*$.

$\mathcal{A}$ is weakly (op.) amenable if (ii) holds for $\mathcal{X} = \mathcal{A}$, λ/ρ r./l. acts.

Group, measure, Fourier, and Fourier-Stietjes algebras

G – locally compact group



G convolution "$\widehat{G}$" functions

Coproducts:

$\mathrm{W}^*(G) = \mathrm{C}^*(G)^{**}$, generated by universal rep'n ϖ

$\mathrm{W}^*(G) \to \mathrm{W}^*(G)\overline{\otimes}\mathrm{W}^*(G) : \varpi(s) \mapsto \varpi(s) \otimes \varpi(s).$

$\mathrm{VN}(G) \to \mathrm{W}^*(G)\overline{\otimes}\mathrm{VN}(G) : \lambda(s) \mapsto \varpi(s) \otimes \lambda(s) \cong \lambda(s)^{\dim \mathcal{H}_\varpi}$

On amenability of group and Fourier algebras

Theorem [Johnson '73, Ruan '95]

TFAE

(i) G is amenable

(ii) $\mathrm{L}^1(G)$ is (op.) amenable

(iii) $\mathrm{A}(G)$ is op. amenable

Theorem [Johnson '91, S. '02], [Despić-Ghahramani '94, Samei '05]

(i) $\mathrm{L}^1(G)$ always (op.) weakly amenable

(ii) $\mathrm{A}(G)$ always op. weakly amenable

Theorem [Forrest-Runde '05, Losert '19]

(i) $\mathrm{A}(G)$ is amenable $\Leftrightarrow$ G virtually abelian

(ii) $\mathrm{A}(G)$ is weakly amenable $\Leftrightarrow$ G_e abelian

On amenability of measure and Fourier-Stieltjes algebras

Theorem [Dales-Ghahramani-Helemskiĭ '02]

(i) $\mathrm{M}(G)$ is (op.) weakly amenable $\Leftrightarrow$ G is discrete

(ii) $\mathrm{M}(G)$ is (op.) amenable $\Leftrightarrow$ G is discrete and amenable

Proposition (after [Ghandehari '12])

$\mathrm{B}(G)$ amenable $\Leftrightarrow$ G is virtually abelian and compact.

Naïve Conjecture

$\mathrm{B}(G)$ operator (weakly) amenable $\Leftrightarrow$ G is compact

Naïve Conjecture false

Fell group: $G_p = \mathbb{Q}_p \rtimes \mathbb{O}_p^\times$, $q : G \to \mathbb{O}_p^\times$ quotient map

$$\mathrm{B}(G_p) = \mathrm{A}(G_p) \oplus_{\ell_1} \mathrm{A}(\mathbb{O}_p^\times) \circ q$$

Theorem [Runde-S. '04 & '07]

(i) $\mathrm{AM}_{op}(\mathrm{B}(G)) < 5 \Leftrightarrow G$ compact.

(ii) $\mathrm{AM}_{op}(\mathrm{B}(G_p)) = 5$.

(ii) $\mathrm{B}(G_p)$ weakly amenable.

Some insight: unitary dynamics

$$\varpi = \bigoplus_{u \in \mathrm{P}_1(G)} \pi_u$$ – universal representation

$$\mathrm{C}^*(G) = \overline{\varpi(\mathrm{L}^1(G))}, \quad \mathrm{W}^*(G) = \varpi(G)'' \cong \mathrm{B}(G)^*$$

Unitary compactification:

$$G^{\varpi} = \overline{\varpi(G)}^{w*} \subset \mathrm{W}^*(G)$$

– semitopological semigroup.

$$\mathrm{ZE}(G^{\varpi}) = \left\{ p \in G^{\varpi} : p^2 = p,\ [p, \varpi(G)] = \{0\} \right\}$$

– complete lattice.

Connection to operator amenability

Theorem [S. '20]

$\mathrm{B}(G)$ op. amenable $\Rightarrow |\mathrm{ZE}(G^{\varpi})| < \infty$.

Idea. Let $S \subseteq \mathrm{ZE}(G^{\varpi})$ any finite meet-semilattice.

$\chi \in \widehat{S}$, let $p_S(\chi) = \prod_{\{p \in S : \chi(p)=1\}} p \prod_{\{p' \in S : \chi(p')=0\}} (I - p') \in \mathrm{ZW}^*(G)$.

$Q_L : \mathrm{B}(G) \to \ell^1(\widehat{S})$, $Q_L(u) = \sum_{\chi \in \widehat{S}} \langle u, p_S(\chi) \rangle \delta_\chi$ com. quot. homo'm.

$\mathrm{AM}_{op}(\mathrm{B}(G)) \geq \mathrm{AM}(\ell^1(\widehat{S})) \geq 2|S| + 1$ [Ghandehari-Hatami-S. '09].

$G_p = \mathbb{Q}_p \rtimes \mathbb{O}_p^{\times}$. $G_p^{\varpi} \cong G_p \sqcup \mathbb{O}_p^{\times}$, $\mathrm{ZE}(G_p) = \{e_{G_p}, 1_{\mathbb{O}_p}\}$.

Connected groups ...

Theorem [Bouziad-Lemańczyk-Mentzen '01], [Elgün '16]

$G = \mathbb{R}, \mathbb{Z}$, $|\mathrm{ZE}(G^\varpi)| \geq \mathfrak{c}$.

Theorem (after [Mayer '97])

If G is connected with $|\mathrm{ZE}(G^\varpi)| < \infty$, then

(i) G admits no quotient with non-compact centre;

(ii) There is compact L with Lie quotient $G/L = N \rtimes R$ w. N nilpotent, R reductive, $R \curvearrowright N$ no non-triv. fixed points.

Ex. $G = \mathbb{C}^2 \rtimes \mathbb{T}$, $(x, y, z)(x', y', z') = (x + zx', y + zy', zz')$.
Here $G^\varpi \cong G \sqcup \bigsqcup_{\substack{L \leq \mathbb{C}^2 \\ \dim_{\mathbb{C}} L = 1}} G/L \sqcup \mathbb{T}$, $|\mathrm{ZE}(G^\varpi)| = \mathfrak{c}$.

… satisfy the naïve conjecture on operator amenability

Theorem [S. '20]

G connected. Then $\mathrm{B}(G)$ op. amenable $\Leftrightarrow$ G is compact.

Idea. $(\Leftarrow)$ $\mathrm{B}(G) = \mathrm{A}(G)$ op. amenable [Ruan '95].

$(\Rightarrow)$ $\mathrm{A}(G)$ op. amenable so G amenable [Ruan '95].

Hence $G/M = N \rtimes K$, K compact. Say N non-trivial.

$\mathrm{B}_0(G)$ must have b.a.i., so too does $\mathrm{B}_0(N \rtimes K)$, hence
$\mathrm{B}_0(Z \rtimes K)$, hence $\mathrm{B}_0^K(Z) = \{u \in \mathrm{B}_0(Z) : k \mapsto u(k\cdot) \text{ cts.}\}$
[Cowling-Rodeway '79], hence $\mathrm{B}_0(Z)^K = \{u \in \mathrm{B}_0(Z) : u(k\cdot) = u\}$.

But, this contradicts [Ragozin '73].

[Liukkonen-Mislove '75] $\mathrm{B}(G)|_{G_e} = \mathrm{B}(G_e)$.
Hence result extends to G almost connected.

References

N.S. On operator amenability of Fourier-Stieltjes algebras. *Bull. Sci. Math.* 158 (2020), 102823, 16 pp.; `arXiv:1806.08421`

N.S. Weakly almost periodic topologies, idempotents and ideals, *Indiana U. Math. J.* (accepted); `arXiv:1805.09892`

Questions

(i) For what non-compact G is $\mathrm{B}(G)$ operator amenable?
Need G amenable & totally disconnected.

(ii) For what discrete G is $\mathrm{B}(G)$ operator amenable?
Need:

- G amenable
- no infinite abelian subgroups
- $|\mathrm{ZE}(G)^{\varpi}| < \infty$, so there are finitely many "unitarizable maximally co-compact" completions of G.

(iii) For what non-compact G is $\mathrm{B}(G)$ (op.) weakly amenable?
Can we answer this for connected G?
Obstruction: fewer results about ideals.

Thank-you

Niá:wen! Miigwech!

(Thank-you, in Mohawk, a Haudenosaunee language; and in Anishnaabemowin of the Mississaugans.)

In Waterloo, we live and work in the traditional territory of the Neutral, Anishinaabeg and Haudenosaunee peoples. We are situated in the Haldimand tract, land promised to the Six Nations (Haudenosaunee), which includes six miles along each side of the Grand River.