Bi-Free Entropy with Respect to Completely Positive Maps

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Preliminaries for Free Entropy

- $(\mathfrak{M}, \tau)$ a tracial von Neumann algebra.
- $X \in \mathfrak{M}$ self-adjoint.
- B a unital von Neumann subalgebra of $\mathfrak{M}$ with expectation $E_B : \mathfrak{M} \to B$.
- $\eta : B \to B$ a completely positive map.
- $B[X]$ the $*$-algebra generated by B and X.
- $\eta_X : B[X] \to B[X]$ by $\eta_X(T) = \eta(E_B(T))$.
- Define a $B[X]$-valued inner product on $B[X] \otimes_{\mathbb{C}} B[X]$ by

$$\langle T_1 \otimes T_2, S_1 \otimes S_2 \rangle_{B[X]} = S_2^* \eta_X(S_1^* T_1) T_2.$$

- Let $\mathcal{H}(B[X], \eta_X)$ be the completion of $B[X] \otimes_{\mathbb{C}} B[X]$ with respect to the pre-inner product $\langle \cdot, \cdot \rangle = \tau(\langle \cdot, \cdot \rangle_{B[X]})$.

Conjugate Variables with Completely Positive Maps

Let $\partial_X : B[X] \to \mathcal{H}(B[X], \eta_X)$ be the linear map defined by

$$\begin{aligned} \partial_X(b) &= 0 \quad \text{for all } b \in B \\ \partial_X(X) &= 1 \otimes 1 \\ \partial_X(T_1 T_2) &= T_1 \cdot \partial_X(T_2) + \partial_X(T_1) \cdot T_2 \quad \text{for all } T_1, T_2 \in B[X]. \end{aligned}$$

Note ∂_X extends to an unbounded densely defined operator on $L_2(B[X], \tau)$.

Definition (Shlyakhtenko; 1998)

If $1 \otimes 1$ is in the domain of $\partial_X^* : \mathcal{H}(B[X], \eta_X) \to L_2(B[X], \tau)$, then the element $J(X : B, \eta) = \partial_X^*(1 \otimes 1) \in L_2(B[X], \tau)$ is said to be the *conjugate of* X *with respect to* (B, η).

Moment Formula for Conjugate Variables

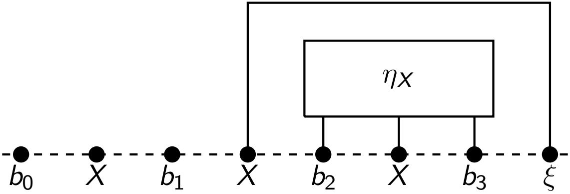
The existence of $J(X : B, \eta)$ is equivalent to the following:

Moment Condition

There exists a $\xi \in L_2(B[X], \tau)$ such that

$$\tau(b_0 X b_1 X \cdots b_{n-1} X b_n \xi) = \sum_{k=1}^{n} \tau\left(b_0 X \cdots X b_{k-1} \eta_X(b_k X \cdots X b_n)\right)$$

for all $n \in \mathbb{N}$ and $b_0, b_1, \ldots, b_n \in B$ (i.e. $\xi = J(X : B, \eta)$).



η-Semicirculars

- The *full Fock space associated to B and η* is

$$\mathcal{F}_\eta(B) = L_2(B,\tau) \oplus \left(\bigoplus_{n \geq 1} \mathcal{H}(B,\eta)^{\otimes_B n} \right).$$

- The *left η-creation operator L* is given by

$$L(\xi_1 \otimes \cdots \otimes \xi_n) = (1 \otimes 1) \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

- L is bounded and $L^* b L = \eta(b)$.
- Let $X = L + L^*$. Then $E(XbX) = \eta(b)$. We call X the *η-semicircular operator.*
- It can be shown that $J(X : B, \eta) = X$.

Fisher Information and Entropy

Let $X_1, \ldots, X_n \in \mathfrak{M}$ be self-adjoint and let $B_j = B[\{X_1, \ldots, X_n\} \setminus \{X_j\}]$.

Definition (Shlyakhtenko; 1998)

The *relative free Fisher information of* $X_1, \ldots, X_n$ *with respect to* (B, η) is

$$\Phi^*(X_1, \ldots, X_n : B, \eta) = \sum_{1 \leq k \leq n} \|J(X_j : B_j, \eta)\|^2_{L_2(\mathfrak{M}, \tau)}$$

and *relative free entropy of* $X_1, \ldots, X_n$ *with respect to* (B, η) is

$$\chi^*(X_1, \ldots, X_n : B, \eta) = \frac{1}{2}\ln(2\pi e) + \frac{1}{2}\int_0^\infty \left(\frac{n\tau(\eta(1))}{1+t} - g(t)\right) dt$$

where

$$g(t) = \Phi^*\left(X_1 + \sqrt{t}S_1, \ldots, X_n + \sqrt{t}S_n : B, \eta\right)$$

where $S_1, \ldots, S_n$ are η-semicircular operators such that $\{X_1, \ldots, X_n\}$, $\{S_1\}$, ..., $\{S_n\}$ are free with amalgamation over B.

Applications of Free Entropy

Definition (Shlyakhtenko; 1998)

Let $\mu : B \to B$ be another normal, self-adjoint, completely positive map. The *free Fisher information* $\Phi^*(\mu : \eta)$ is defined to be equal to $\Phi^*(X : B, \eta)$ where X is a μ-semicircular operator over B.

Theorem (Shlyakhtenko; 1998)

If A is a subfactor of B with finite Jones index $[B : A]$ and $E : B \to A$ is the unique trace-preserving conditional expectation onto A, then $\Phi^*(E : B, \mathrm{id}) = [B : A]$.

Applications of Free Entropy

Theorem (Nica, Shlyakhtenko, Spicher; 1999)

If ν *is a probability measure with compact support on* $[0, \infty)$ *and* μ *is the symmetric probability measure on* $\mathbb{R}$ *defined such that* $\mu(U) = \nu(U^2)$ *for every symmetric Borel set* $U \subseteq \mathbb{R}$*, then*

$$\min\{\Phi^*(a, a^*) \,|\, a^*a \text{ has distribution } \nu\} = 2\Phi^*(\mu)$$

and the minimum is attained when a is R-diagonal.

Moreover, working in $M_d(\mathfrak{M})$ *with respect to* $\mathrm{tr}_d \circ \tau_d$*,*

$$\max\left\{\chi^*(\{a_{i,j}, a_{i,j}^*\}_{i,j=1}^d) \,\middle|\, \substack{A=[a_{i,j}]\in M_d(\mathfrak{M}) \text{ is such} \\ \text{that } A^*A \text{ has distribution } \nu}\right\} = 2d^2\left(\chi^*(\mu) - \frac{1}{2}\ln(d)\right)$$

and the maximum is obtained if A is R-diagonal and $\{A, A^*\}$ *is free from* $M_d(\mathbb{C})$ *in* $M_d(\mathfrak{M})$*.*

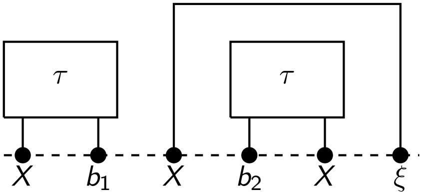
Bi-Free Entropy

The notions of conjugate variables, Fisher information, and entropy in the case $B = \mathbb{C}$ were extended to the bi-free setting (i.e. a notion of independence for pairs of algebras with actions on the left and right) in [Charlesworth, Skouf.; 2020].

Diagrams for Bi-Free Entropy

If $\xi = \mathcal{J}(X : B)$, then

$$\tau(Xb_1Xb_2X\xi) = \tau(b_1Xb_2X) + \tau(Xb_1)\tau(b_2X) + \tau(Xb_1Xb_2).$$



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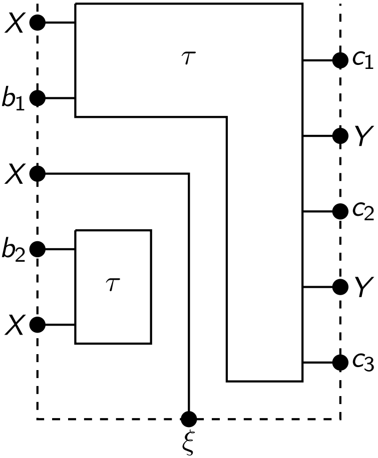

Structures for Operator-Valued Bi-Free Probability

Definition (Charlesworth, Nelson, Skouf.; 2015)

For a unital algebra B, a *B-B-non-commutative probability space* is a triple (A, E, ε) where A is a unital $*$-algebra, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is a unital $*$-homomorphism such that the restrictions $\varepsilon|_{B\otimes 1_B}$ and $\varepsilon|_{1_B \otimes B^{\mathrm{op}}}$ are both injective, and $E : A \to B$ is a unital linear map that such that

$$E(\varepsilon(b_1 \otimes b_2)a) = b_1 E(a) b_2 \qquad \text{and} \qquad E(a\varepsilon(b \otimes 1_B)) = E(a\varepsilon(1_B \otimes b)),$$

for all $b, b_1, b_2 \in B$ and $a \in A$. The unital $*$-algebras

$$A_\ell = \{a \in A \mid a\varepsilon(1_B \otimes b) = \varepsilon(1_B \otimes b)a \text{ for all } b \in B\}$$

and

$$A_r = \{a \in A \mid a\varepsilon(b \otimes 1_B) = \varepsilon(b \otimes 1_B)a \text{ for all } b \in B\}.$$

are called *left and right algebras of A* respectively.

Structures for Operator-Valued Bi-Free Probability

Definition (Katsimpas, Skouf.; 2021)

Given a unital $*$-algebra B, an *analytical B-B-non-commutative probability space* consists of a tuple $(A, E, \varepsilon, \tau)$ such that

- (A, E, ε) is a B-B-non-commutative probability space,
- $\tau : A \to \mathbb{C}$ is a state (i.e. unital and positive) that is compatible with E; that is,

  $$\tau(a) = \tau\left(\varepsilon(E(a) \otimes 1_B)\right) = \tau\left(\varepsilon(1_B \otimes E(a))\right)$$

  for all $a \in A$,
- the canonical state $\tau_B : B \to \mathbb{C}$ defined by $\tau_B(b) = \tau(\varepsilon(b \otimes 1_B))$ for all $b \in B$ is tracial,
- left multiplication of A on A/N_τ are bounded linear operators and thus extend to bounded linear operators on $L_2(A, \tau)$, and

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- left multiplication of A on A/N_τ are bounded linear operators and thus extend to bounded linear operators on $L_2(A, \tau)$, and
- $E|_{A_\ell}$ and $E|_{A_r}$ are completely positive.

Examples of Operator-Valued Structures

Example

Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra, B a unital von Neumann subalgebra of $\mathfrak{M}$, and A the algebra generated the left and right actions of $\mathfrak{M}$ on $L_2(\mathfrak{M}, \tau)$. If $P : L_2(\mathfrak{M}, \tau) \to L_2(B, \tau)$ is the orthogonal projection, $E : A \to B$ is defined by

$$E(Z) = P(Z1_{\mathfrak{M}})$$

and $\tau_A : A \to \mathbb{C}$ is defined by

$$\tau_A(T) = \langle T1_{\mathfrak{M}}, 1_{\mathfrak{M}} \rangle_{L_2(\mathfrak{M}, \tau)}$$

then $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space.

Examples of Operator-Valued Structures

Example

Let $\mathcal{A}$ and B be unital C^*-algebras, $\varphi : \mathcal{A} \to \mathbb{C}$ a state, $A = \mathcal{A} \otimes B \otimes B^{\mathrm{op}}$, $E : A \to B$ defined by

$$E(Z \otimes b_1 \otimes b_2) = \varphi(Z) b_1 b_2,$$

and $\tau_B : B \to \mathbb{C}$ a tracial state. Then $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space.

Theorem (Skouf.; 2016)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\{(C_k, D_k)\}_{k \in K}$ be bi-freely independence pairs of algebras in $(\mathcal{A}, \varphi)$. Then

$$\{(C_k \otimes B \otimes 1_B, D_k \otimes 1_B \otimes B^{\mathrm{op}})\}_{k \in K}$$

are bi-free with amalgamation over B with respect to E.

Bi-Free Conjugate Variables

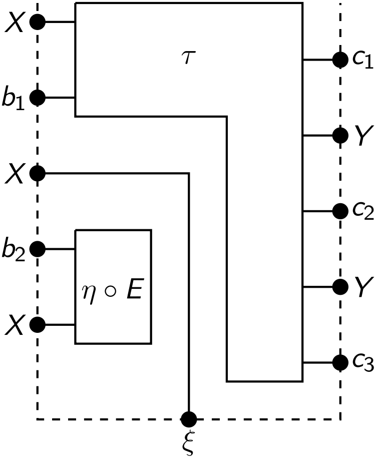
Definition (Katsimpas, Skouf.; 2021)

In an analytic B-B-non-commutative probability space $(A, E, \varepsilon, \tau)$, let (C_ℓ, C_r) be a pair of B-algebras in A, $X \in A_\ell$, and $\eta : B \to B$ a completely positive map. An element

$$\xi \in \overline{\mathrm{alg}(X, C_\ell, C_r)} \in L_2(A, \tau)$$

is said to be the *left bi-free conjugate variable relations for X with respect to η and τ in the presence of* (C_ℓ, C_r), denoted $J_\ell(X : (C_\ell, C_r), \eta)$, if ......

Bi-Free Conjugate Variables via Diagrams



Matricial Constructions for Max/Min

- $(\mathcal{A}, \varphi)$ a C*-non-commutative probability space.
- $x, y \in \mathcal{A}$ such that x^*x and xx^* have the same distribution and y^*y and yy^* have the same distribution.
- $A_2 = \mathcal{A} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\mathrm{op}}$
- $\tau_2 : A_2 \to \mathbb{C}$ by $\tau_2(T \otimes b_1 \otimes b_2) = \varphi(T)\mathrm{tr}_2(b_1 b_2)$.
- $X = x \otimes E_{1,2} \otimes I_2 + x^* \otimes E_{2,1} \otimes I_2$.
- $Y = y \otimes I_2 \otimes E_{1,2} + y^* \otimes I_2 \otimes E_{2,1}$.
- The joint moments of X and Y are 0 if of odd length and otherwise are the average of the φ-moment of a χ-alternating series of $\{x, y\}$ and $\{x^*, y^*\}$, and the series obtained via $x \leftrightarrow x^*$ and $y \leftrightarrow y^*$.
- $\Delta_{X,Y}$ all (x_0, y_0) that produce X_0 and Y_0 with the same distribution as X and Y.
- if $\{x, x^*\}$ commutes with $\{y, y^*\}$, X and Y commute and produce a distribution on $\mathbb{R}^2$.

Min/Max Bi-Free Fisher Information and Entropy

Theorem (Katsimpas, Skouf.; 2021)

Using the above notation

$$\min\left\{\Phi^*(\{x_0, x_0^*\} \sqcup \{y_0, y_0^*\} : (\mathbb{C}, \mathbb{C}), \varphi) \,|\, (x_0, y_0) \in \Delta_{X,Y}\right\} \geq 2\Phi^*(X \sqcup Y)$$

and equality holds and is achieved for any pair (x_0, y_0) that is alternating adjoint flipping and bi-R-diagonal.

Theorem (Katsimpas, Skouf.; 2021)

Using the above notation

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

and equality holds whenever the pair (x, y) is bi-R-diagonal and alternating adjoint flipping.

Thanks for Listening!