A picture of Cartan subalgebras in twisted k-graph algebras

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Support from: BIRS (funded also by AWM ADVANCE); MSRI (NSA grant H98230-19-1-0119, NSF grant 1440140, the Lyda Hill Foundation, the McGovern Foundation, and Microsoft Research); Fitchburg University; Simons Foundation Collaboration Grant #360563 (SR); and NSF grant DMS-1600749 (EG)

CMS 75$^{\text{th}}$ Summer Meeting, Ottawa

Cartan subalgebras: bridging operator algebras and dynamics

Some history

Cartan subalgebras of von Neumann algebras

- introduced by Vershik in 1971
- characterize vN algebras arising from measured equivalence relations (Feldman-Moore, 1977)

Cartan C^*-subalgebras

- Introduced by Renault in 1980
- Characterize C^*-algebras that can be modeled via topologically principal groupoids (Renault, 2008)
- Are present only when the ambient algebra satisfies the UCT (Barlak-Li, 2017)
- Are present in every classifiable simple C^*-algebra (Li, 2019)

C^* Diagonal subalgebras

- Introduced by Kumjian, 1986
- characterize C^*-algebras that can be modeled via principal groupoids (Kumjian, 1986)

Variants

- Noncommutative Cartan subalgebras – correspond to Fell bundles over inverse semigroups (Exel, 2008)
- Regular and virtual Cartan inclusions – possess pseudo-expectations (Pitts, 2012)
- Γ-Cartan subalgebras – correspond to graded topologically principal groupoids (Brown-Fuller-Pitts-R, 2018)

Definition (Renault, 1980)

A maximal abelian C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is *Cartan* if

(i) $\mathcal{B}$ contains an approximate unit of $\mathcal{A}$,

(ii) The normalizers of $\mathcal{B}$ in $\mathcal{A}$ generate $\mathcal{A}$, and

(iii) There is a faithful conditional expectation $\mathcal{A} \to \mathcal{B}$.

Examples

1. The set of diagonal $n \times n$ matrices $\mathcal{B}$ is Cartan in $\mathcal{A} = M_n(\mathbb{C})$:

(i) $\mathcal{B}$ contains the identity,

(ii) The matrix units $\varepsilon_{i,j}$ are normalizers of B and generate $\mathcal{A}$, and

(iii) $E : \mathcal{A} \to \mathcal{B}$ given by $E(M)_{i,j} = \delta_i(j) M_{i,j}$ is a conditional expectation.

2. If a directed graph E has no cycles without entry (Condition (L)) then the diagonal subalgebra $\mathcal{D} := C^*(\{s_\alpha s_\alpha^* \mid \alpha \text{ a path in } E\})$ is Cartan in $C^*(E)$.

3. In a general graph algebra, the *cycline subalgebra* $\mathcal{M}$, which is generated by the diagonal along with diagonal-like elements with detour through a cycle without entry, is Cartan (Nagy-R, 2012). In a higher-rank graph algebra, the analogous subalgebra is often Cartan (Brown-Nagy-R, 2013 and Brown-Nagy-R-Sims-Williams, 2018).

Notions:

A *groupoid* is a small category G in which every morphism has an inverse. Denote by $G^{(0)}$ the objects (unit space of identity morphisms), and denote the range and source maps $r, s : G \to G^{(0)}$.

The *isotropy subgroupoid* $\mathrm{Iso}(G) = \{g \in G \mid r(g) = s(g)\}$.

A *topological groupoid* is a groupoid endowed with a topology with respect to which inversion, multiplication, and composition are continuous. G is *étale* if r and s are local homeomorphisms.

A groupoid is *topologically principal* if the units at which the isotropy is trivial form a dense set in $G^{(0)}$.

A ($\mathbb{T}$-valued) *2-cocycle* on a groupoid $\mathcal{G}$ is a function $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ such that

$$\sigma(g, s(g)) = \sigma(r(g), g)) = 1 \text{ for all } g \in \mathcal{G}, \text{ and}$$

$$\sigma(g, hk)\sigma(h, k) = \sigma(gh, k)\sigma(g, h) \text{ whenever } (g, h), (h, k) \in \mathcal{G}^{(2)}.$$

$C^*(\mathcal{G}, \sigma)$ and $C_r^*(\mathcal{G}, \sigma)$ are defined to be completions of $C_c(\mathcal{G})$, a $*$-algebra with $f * h(\gamma) = \sum_\eta f(\gamma\eta)h(\eta^{-1})\sigma(\gamma\eta, \eta^{-1})$ and $f^*(\gamma) = \overline{f(\gamma^{-1})\sigma(\gamma, \gamma^{-1})}$.

Theorem (Renault, 2008): If Σ is a twist over a 2^{nd} countable, étale, locally compact Hausdorff, topologically principal groupoid G, then $C_0(G^{(0)})$ is Cartan in $C_r^*(\Sigma; G)$.

Observation: Some C^*-algebras of *non*-topologically principal groupoids do indeed have a Cartan subalgebra.

Theorem Brown-Nagy-R-Sims-Williams (2016): If $\mathcal{G}$ is a l.c. Hausdorff étale groupoid with $\mathrm{Iso}(\mathcal{G})^\circ$ abelian and closed, then $C^*(\mathrm{Iso}(\mathcal{G})^\circ)$ is Cartan in $C^*(\mathcal{G})$.

Goal: find an analogous result – a description of subgroupoids that generate Cartan subalgebras – for groupoid algebras with a 2-cocycle twist.

Motivating example: The irrational rotation algebra

A_θ can be realized as the twisted group C^*-algebra $C_r^*(\mathbb{Z}^2, c_\theta)$ with the 2-cocycle $c_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ given by

$$c_\theta((n_1, n_2), (m_1, m_2)) = e^{2\pi i \theta n_2 m_1}.$$

Then c_θ restricted to $S = \mathbb{Z} \times \{0\}$ is trivial, and $C_r^*(S, c_\theta) \cong C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ is Cartan in $C_r^*(\mathbb{Z}^2, c_\theta)$.

Theorem (Duwenig-Gillaspy-Norton-R-Wright, JFA 2020)

Let $\mathcal{G}$ be a second countable, locally compact Hausdorff, étale groupoid, and let $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ be a 2-cocycle. Suppose $\mathcal{S}$ is maximal among abelian subgroupoids of $\mathrm{Iso}(\mathcal{G})$ on which σ is symmetric. If $\mathcal{S}$ is clopen, normal, and immediately centralizing, then $C_r^*(\mathcal{S}, \sigma)$ is Cartan in $C_r^*(\mathcal{G}, \sigma)$.

A subgroupoid $\mathcal{S} \subseteq \mathrm{Iso}(\mathcal{G})$ is *immediately centralizing* if for all $g \in \mathrm{Iso}(\mathcal{G})$,

$$(\forall \alpha \in \mathcal{S}\, \{\alpha^k g \alpha^{-k} \mid k \in \mathbb{N}\} \text{ is finite}) \Rightarrow g \in \mathcal{S}'.$$

Sketch of how the hypotheses are used:

$C_r^*(\mathcal{S}, \sigma)$ is a subalgebra because $\mathcal{S}$ open, so $g \mapsto \chi_{\mathcal{S}} g$ embeds $C_c(\mathcal{S}, \sigma)$ in $C_c(\mathcal{G}, \sigma)$.

$C_r^*(\mathcal{S}, \sigma)$ is abelian because $\mathcal{S}$ is abelian and σ is symmetric on it.

$C_r^*(\mathcal{S}, \sigma)$ is *maximal* abelian because $\mathcal{S}$ is maximal among abelian subgroupoids on which σ is symmetric, and moreover $\mathcal{S}$ is immediately centralizing.

There is a conditional expectation $\Phi : C_r^*(\mathcal{G}, \sigma) \to C_r^*(\mathcal{S}, \sigma)$ because $\mathcal{S}$ is closed, so $f \mapsto f_{\mathcal{S}}$ can be extended.

The normalizer of $C_r^*(\mathcal{S})$ in $C_r^*(\mathcal{G}, \sigma)$ generates $C_r^*(\mathcal{G}, \sigma)$ because S is normal, so $N_{C_r^*(\mathcal{G},\sigma)}(C_r^*(\mathcal{S}, \sigma))$ contains functions supported in bisections.

$C_r^*(\mathcal{S}, \sigma)$ contains an approximate unit for $C_r^*(\mathcal{G}, \sigma)$ because $C_0(\mathcal{G}^{(0)}) \subseteq C_r^*(\mathcal{S}, \sigma)$.

Theorem (Renault, 2008): If $\mathcal{B} \subseteq \mathcal{A}$ is a Cartan subalgebra, then there is an étale, 2^{nd} countable, locally compact Hausdorff, topologically principal twisted groupoid $G_{\mathcal{A},\mathcal{B}}$ and a groupoid twist, $\mathbb{T} \times G^{(0)}_{\mathcal{A},\mathcal{B}} \longrightarrow \Sigma \longrightarrow G_{\mathcal{A},\mathcal{B}}$, such that

$$(\mathcal{A}, \mathcal{B}) \cong (C^*_r(\Sigma; G_{\mathcal{A},\mathcal{B}}), C_0(G^{(0)}_{\mathcal{A},\mathcal{B}}))$$

.

What is the corresponding Weyl groupoid guaranteed by Renault's theorem?

Theorem (DGNRW, '20)
With the assumptions of the previous theorem, and assuming in addition that $\mathcal{G}$ is a group, $\mathcal{Q} := \mathcal{G}/\mathcal{S}$ acts on the Gelfand dual $\widehat{C^*_r(\mathcal{S}, \sigma)}$, and $\mathcal{Q} \ltimes \widehat{C^*_r(\mathcal{S}, \sigma)} = \mathcal{W}$, where $\mathcal{W} = G_{C^*_r(\mathcal{G},\sigma), C^*_r(\mathcal{S},\sigma)}$.

Theorem (Duwenig-Gillaspy-Norton, '21): analogous result for groupoids.

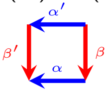
For $k \in \mathbb{N}^+$, a ***k*-graph** is a countable category Λ with a "degree" map $d : \Lambda \to \mathbb{N}^k$

$$d(\mu\nu) = d(\mu) + d(\nu) \qquad \Lambda^n := d^{-1}\{n\}$$

satisfying the *Unique Factorization Property:*

If $\lambda \in \Lambda^{m+n}$ then there are unique $\mu \in \Lambda^m,\ \nu \in \Lambda^n$ s.t. $\lambda = \mu\nu$.

- Think of elements of degree ε_i as edges of color i.
- A morphism of degree $\varepsilon_{\boldsymbol{i}} + \varepsilon_{\boldsymbol{j}} = \varepsilon_{\boldsymbol{j}} + \varepsilon_{\boldsymbol{i}}$ has two factorizations: $\alpha\beta = \beta'\alpha'$, where $d(\alpha') = d(\alpha) = \varepsilon_j$ and $d(\beta) = d(\beta') = \varepsilon_i$



These "commuting squares" determine all factorization rules of the k-graph.

Example: $\Omega_k = \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k \,|\, \forall i\ m_i \le n_i\}$ is a k-graph with $d(m,n) = n - m$ (componentwise), $s(m,n) = n$, $r(m,n) = m$ and $(m,n)(n,l) = (m,l)$.

Path groupoid of a k-graph Λ

An *infinite path* in Λ is a degree-preserving functor $x : \Omega_k \to \Lambda$, i.e., a filling of $\mathbb{N}^k$ with compatible elements of Λ. Denote by Λ^∞ the infinite path space of Λ.

The path groupoid:

$$\mathcal{G}_\Lambda = \{(\overset{r}{\alpha y}, d(\alpha) - d(\beta), \overset{s}{\beta y}) \mid y \in \Lambda^\infty,\ s(\alpha) = s(\beta)\}$$

- range and source of an element as shown.
- Composition: $(x, d, y)(y, d', z) = (x, d + d', z)$.
- Inverse: $(x, d, y)^{-1} = (y, -d, x)$
- The isotropy subgroupoid: $\mathrm{Iso}(\mathcal{G}_\Lambda) = \{(\alpha y, d, \beta y) \in \mathcal{G}_\Lambda \mid \alpha y = \beta y\}$
- The unit space: $\mathcal{G}_\Lambda^{(0)} := \{(x, 0, x) \mid x \in \Lambda^\infty\}$

The cylinder sets $Z(\alpha, \beta) = \{(\alpha y, d(\alpha) - d(\beta), \beta y)\}$, $s(\alpha) = s(\beta)$, form a basis for a locally compact, amenable, Hausdorff, étale topology.

Goal: identify which k-graphs give rise to groupoids with subgroupoids satisfying the hypotheses of the theorem. We use the cocycle σ described by Kumjian-Pask-Sims, which is induced by an assignment of constants to commuting squares.

We require: S to be maximal among abelian subgroupoids of $\mathrm{Iso}(\mathcal{G})$ on which σ is symmetric. Then if S is clopen, normal, and immediately centralizing, $C_r^*(\mathcal{S}, \sigma)$ is Cartan in $C_r^*(\mathcal{G}, \sigma)$.

Theorem (Norton-R-Wright, '21) Conditions on Λ and procedure for identifying $\mathcal{S} \subseteq \mathrm{Iso}(\mathcal{G}_\Lambda)$ in the case $k = 2$.

Observations:

- To ensure σ is symmetric and $\mathcal{S}$ is maximal, for each infinite path $x \in \Lambda^\infty$ the set $\{d \in \mathbb{N}^k \mid (x, d, x) \in \mathcal{S}\}$ must be a maximal one-dimensional subgroup.
- To ensure $\mathcal{S}$ is normal, the subgroups corresponding to paths x, y in the same shift orbit (some $\sigma^n(x) = \sigma^m(y)$) must coincide.

Remark: Condition (L) makes it easy to find a Cartan subalgebra of a graph algebra, even when there is lots of periodicity. (The subalgebra is the diagonal.). To find a Cartan subalgebra of a twisted k-graph algebra using our theorem we need all the periodicity to arise from cycles without entry.

Thank you!

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