The isomorphism problem for tensor algebras of multivariable dynamical systems

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C*-dynamical systems and tensor algebras

A (multivariable) **C*-dynamical system** $(\mathcal{A}, \alpha)$ consists of a unital C*-algebra $\mathcal{A}$ and a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of unital $*$-endomorphisms $\alpha_i : \mathcal{A} \to \mathcal{A}$

A **row isometric representation** of $(\mathcal{A}, \alpha)$ consists of a non-degenerate $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a row isometry $V = (V_1, \ldots, V_n)$ acting on $\mathcal{H}^{(n)}$ such that

$$\pi(a)V_i = V_i\pi(\alpha_i(a)), \quad a \in \mathcal{A}$$

The **tensor algebra** $\mathcal{T}^+(\mathcal{A}, \alpha)$ is the universal operator algebra for these representations, so $\mathcal{T}^+(\mathcal{A}, \alpha) = \overline{\mathrm{Alg}(\mathcal{A}, V_1, \ldots, V_n)}$.

Structure and examples

Every element $a \in \mathcal{T}^+(\mathcal{A}, \alpha)$ admits a formal Fourier series description that converges in the Cesaro means

$$a = \lim_{m \to \infty} \sum_{k=0}^{m} \sum_{w \in \mathbb{F}_n^+, |w|=k} \left(1 - \frac{k}{m+1}\right) a_w V_w, \quad a_w \in \mathcal{A}$$

where $w = w_1 \cdots w_k$ and $V_w = V_{w_1} \cdots V_{w_k}$.

The so-called constant term $E_0(a) = a_0$ is important as E_0 is an expectation of $T^+(\mathcal{A}, \alpha)$ onto $\mathcal{A}$.

Examples:

- $\mathcal{T}^+(\mathbb{C}, \mathrm{id}) = A(\mathbb{D})$
- $\mathcal{T}^+(\mathbb{C}, (\mathrm{id}, \ldots, \mathrm{id})) = \mathcal{A}_d$, the noncommutative disk algebra
- $\mathcal{T}^+(C(X), C_\varphi) = C(X) \times_{C_\varphi} \mathbb{Z}^+$

Isomorphism problem

Problem: When are two tensor algebras isomorphic topologically, isometrically or completely isometrically?

Equivalences of C*-dynamical systems:

$(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are said to be **unitarily equivalent after a conjugation** if there exists a $*$-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ and a unitary matrix U with entries in $\mathcal{A}$ such that

$$\mathrm{diag}(\alpha_1(c), \ldots, \alpha_n(c)) = U \mathrm{diag}(\gamma^{-1} \circ \beta_1 \circ \gamma(c), \ldots, \gamma^{-1} \circ \beta_m \circ \gamma(c)) U^*,$$

If $n = m$ and U is a permutation matrix this is called **conjugation** and in the case where $n = m = 1$ this is called **outer conjugation**.

History: Single variable systems

Arveson (1967), ..., Davidson & Katsoulis (2008)
Two single variable commutative C*-dynamical systems $(C(X), \alpha)$ and $(C(Y), \beta)$ are conjugate if and only if their tensor algebras are isomorphic (topologically, isometrically or completely isometrically).

Davidson & Kakariadis (2014) studied outer conjugacy and showed that it is equivalent to isometrically isomorphic tensor algebras in the injective map case and other situations.

History: Multivariable systems

Davidson & Katsoulis (2011) studied the multivariable commutative case and showed that

u. equiv. after conj. $\Rightarrow$ c.i.i tensor algebras

and

top. isomorphic tensor algebras $\Rightarrow$ piecewise conjugate dynamical systems (open cover of conjugate pieces).

Kakariadis & Katsoulis (2014) studied the general multivariable case with $*$-automorphisms and showed that u. equiv. after conj. if and only if isometrically isomorphic tensor algebras.

Main result

Theorem: (Katsoulis & R. (2020)
Two C*-dynamical systems are unitarily equivalent after a conjugation if and only if their tensor algebras are completely isometrically isomorphic.

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By universality the forward direction is straightforward: suppose $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ have a $*$-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ and unitary U in $M_{n,m}(\mathcal{A})$ then

$$\gamma^{-1}, VU = \left(\sum_{i=1}^{n} V_i U_{i1}, \ldots, \sum_{i=1}^{n} V_i U_{im} \right)$$

is an isometric representation of $(\mathcal{B}, \beta)$. This induces a completely contractive homomorphism of $T^+(\mathcal{B}, \beta)$ onto $\mathcal{T}^+(\mathcal{A}, \alpha)$. The other direction is similar.

Simple simplifications

For the converse suppose $\mathcal{T}^+(\mathcal{A}, \alpha)$ and $\mathcal{T}^+(\mathcal{B}, \beta)$ are completely isometrically isomorphic.

Then $\mathcal{A} = \mathcal{T}^+(\mathcal{A}, \alpha) \cap \mathcal{T}^+(\mathcal{A}, \alpha)^*$ is $*$-isomorphic to $\mathcal{B} = \mathcal{T}^+(\mathcal{B}, \beta) \cap \mathcal{T}^+(\mathcal{B}, \beta)^*$.

Simple simplifications

For the converse suppose $\mathcal{T}^+(\mathcal{A},\alpha)$ and $\mathcal{T}^+(\mathcal{B},\beta)$ are completely isometrically isomorphic.

Then $\mathcal{A} = \mathcal{T}^+(\mathcal{A},\alpha) \cap \mathcal{T}^+(\mathcal{A},\alpha)^*$ is $*$-isomorphic to $\mathcal{B} = \mathcal{T}^+(\mathcal{B},\beta) \cap \mathcal{T}^+(\mathcal{B},\beta)^*$.

So without loss of generality we simplify the situation to

$$\mathcal{T}^+(\mathcal{A},\alpha) = \mathcal{T}^+(\mathcal{A},\beta)$$

or

$$\overline{\mathrm{Alg}(\mathcal{A}, V_1, \dots, V_n)} = \overline{\mathrm{Alg}(\mathcal{A}, W_1, \dots, W_m)}$$

for row isometries $(V_1, \dots, V_n)$ and $(W_1, \dots, W_m)$ satsifying

$$aV_i = V_i \alpha_i(a) \quad \text{and} \quad aW_j = W_j \beta_j(a)$$

Möbius transformations

Case study: suppose $A(\mathbb{D}) = \overline{\mathrm{Alg}(I, z)} = \overline{\mathrm{Alg}(I, f(z))} \subset C(\overline{\mathbb{D}})$.

There exists a Möbius transformation $\varphi_b(z) = \frac{b-z}{1-z\bar{b}}$ where $b = f(0)$ and $u \in \mathbb{T}$ such that

$$f(z) = \varphi_b(uz) \quad \text{or} \quad z = u\varphi_b(f(z))$$

since $\varphi_b \circ \varphi_b = \mathrm{id}$.

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Theorem: (Katsoulis & R. (2020))
Suppose $\mathcal{T}^+(\mathcal{A}, \alpha)$ has generating row isometry V. If $b = (b_1, \ldots, b_n)$ is a strict row contraction in $\mathcal{A}$ such that $ab_i = b_i\alpha_i(a), a \in \mathcal{A}$ then there is a completely isometric automorphism ρ_b of the tensor algebra such that

$$\rho_b|_{\mathcal{A}} = \mathrm{id}, \quad \rho_b(V) = (I - bb^*)^{1/2}(I - Vb^*)^{-1}(b - V)(I_n - b^*b)^{-1/2},$$

$\rho_b \circ \rho_b = id$, and $E_0(\rho_b(v_i)) = b_i, 1 \leq i \leq n$.

Unitarily equivalent

Thm: If $\overline{\mathrm{Alg}(\mathcal{A}, V_1, \ldots, V_n)} = \overline{\mathrm{Alg}(\mathcal{A}, W_1, \ldots, W_m)}$ then

$$b = (E_0^W(V_1), \ldots, E_0^W(V_n))$$

is a strict row contraction in $\mathcal{A}$. Moreover,

$$E_0^W\left(\rho_b(V_i)\right) = 0,\ 1 \leq i \leq n.$$

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Thm: If $\overline{\mathrm{Alg}(\mathcal{A}, V_1, \ldots, V_n)} = \overline{\mathrm{Alg}(\mathcal{A}, W_1, \ldots, W_m)}$ such that $E_0^W(V_i) = 0$ then the Fourier series expansion is

$$V_i = W_1 a_{1i} + \cdots + W_m a_{mi}.$$

Furthermore, $U = [a_{ji}]^*$ is a unitary in $M_{n,m}(\mathcal{A})$ such that $\mathrm{diag}(\alpha_1(a), \ldots, \alpha_n(a)) = U \mathrm{diag}(\beta_1(a), \ldots, \beta_m(a)) U^*$.

Conclusion: If $\overline{\mathrm{Alg}(\mathcal{A}, V_1, \ldots, V_n)} = \overline{\mathrm{Alg}(\mathcal{A}, W_1, \ldots, W_m)}$ with $b = (E_0^W(V_1), \ldots, E_0^W(V_n))$ then there exists a unitary $U \in M_{n,m}(\mathcal{A})$ such that

$$W = \rho_b(V)U$$

with $\mathrm{diag}(\alpha_1(a), \ldots, \alpha_n(a)) = U\mathrm{diag}(\beta_1(a), \ldots, \beta_m(a))U^*$.

Theorem: (Katsoulis & R. (2020)

Two C*-dynamical systems are unitarily equivalent after a conjugation if and only if their tensor algebras are completely isometrically isomorphic.