C^*-algebras coming from a commuting k-tuple of local homeomorphisms acting on a compact metric space

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Outline of Talk:

1. Constructing the groupoid from k commuting local homeomorphisms
2. The Cocycle Condition on $[C(X,\mathbb{R})]^k$ and continuous 1-cocycles on the groupoid
3. Commuting Ruelle operators, their duals, and solving the positive eigenvalue condition
4. KMS states coming from the Radon-Nikodym problem and 1-cocycle-driven dynamics

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Commuting local homeomorphisms on compact metric space

Set-up: Let X be a compact metric space, and let $\{\sigma_i\}_{i=1}^k$ be a k-tuple of commuting surjective local homeomorphisms on X. This gives rise to an action of the semigroup $\mathbb{N}^k$ on X by endomorphisms:

$$\sigma^{\mathrm{n}}(x) \;=\; \sigma_1^{n_1}\sigma_2^{n_2}\ldots\sigma_k^{n_k}(x),\ \text{for } \mathrm{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k,\ x \in X.$$

The transformation groupoid $\mathcal{G}(X,\sigma)$, sometimes called the "semi-direct product" groupoid associated to the action of $\mathbb{N}^k$ on X, is defined by

$$\mathcal{G}(X,\sigma) \;=\; \{((x, \mathrm{m}-\mathrm{n}, y) \in X \times \mathbb{Z}^k \times X :\ \sigma^{\mathrm{m}}(x) = \sigma^{\mathrm{n}}(y)\},$$

where the unit space of $\mathcal{G}(X,\sigma)$ is identified with X via $x \to (x, 0, x)$, and then $r((x,\mathrm{n},y)) = x$ and $s((x,\mathrm{n},y)) = y$ are the range and source maps, respectively.

The associated groupoid $\mathcal{G}(X,\sigma)$ and its C^*-algebra

Recall $\mathcal{G}(X,\sigma)$ has as a basis for its topology sets of the form $U \times \{\mathrm{m}-\mathrm{n}\} \times V$, where U and V are open in X and $\sigma^{\mathrm{m}}(U) = \sigma^{\mathrm{n}}(V)$. Then $\mathcal{G}(X,\sigma)$ is an étale locally compact Hausdorff amenable groupoid, generalizing "Renault-Deaconu" groupoids ([D], [ER], [KR]).

Following the method of J. Renault ([R]) and denoting by $\mathcal{G}(X,\sigma)^{(2)}$ the set of composable pairs, there is a convolution structure on $C_C(\mathcal{G}(X,\sigma))$ as well as an adjoint operation. The groupoid C^*-algebra $C^*(\mathcal{G}(X,\sigma))$ is then constructed by completing $C_C(\mathcal{G}(X,\sigma))$ in the appropriate norm.

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The groupoid $\mathcal{G}(X,\sigma)$ is amenable, so that there is a dense embedding of $C_C(\mathcal{G}(X,\sigma))$ into

$$C_r^*(\mathcal{G}(X,\sigma)) \cong C^*(\mathcal{G}(X,\sigma)).$$

The Cocycle Condition

Definition 1: Let (X, σ) denote the compact metric space X together with a k-tuple of commuting surjective local homeomorphisms $\{\sigma_i\}_{i=1}^{k}$ acting on X. Let $\{\varphi_i\}_{i=1}^{k} \subset C(X, \mathbb{R})$.

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We say that the triple (X, σ, φ) satisfies the Cocycle Condition if:

$$\varphi_i + \varphi_j \circ \sigma_i = \varphi_j + \varphi_i \circ \sigma_j, \; 1 \le i, j \le k.$$

Example 1: If the $\{\varphi_i = r_i\}_{i=1}^k$ are all real constant functions, the cocycle condition is trivially satisfied.

Theorem on 1-cocycles on $\mathcal{G}(X, \sigma)$ taking on values in $\mathbb{R}$

Theorem 1 ([FHKP]): Let (X, σ, φ) denote a triple consisting of the compact metric space X, a k-tuple $\{\sigma_i\}_{i=1}^k$ of commuting local homeomorphisms on X, and a k-tuple of continuous real–valued functions $\{\varphi_i\}_{i=1}^k$ on X such that (X, σ, φ) satisfies the Cocycle Condition. Then defining $c_\varphi : \mathbb{N}^k \to C(X, \mathbb{R})$ by

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$$c_\varphi(\mathrm{n}) = \sum_{i=0}^{n_1-1} \varphi_1\circ\sigma_1^i + \sum_{i=0}^{n_2-1} \varphi_2\circ\sigma_1^{n_1}\circ\sigma_2^i + \ldots + \sum_{i=0}^{n_k-1} \varphi_k\circ\sigma_1^{n_1}\circ\ldots\circ\sigma_{k-1}^{n_{k-1}}\circ\sigma_k^i,$$

c_φ is a 1-cocycle for the action of $\mathbb{N}^k$ on $C(X,\mathbb{R})$ viewed as a $\mathbb{N}^k$-module. Moreover c_φ gives rise to a continuous groupoid 1-cocycle $c_{X,\sigma,\phi}$ on $\mathcal{G}(X,\sigma)$ taking on values in $\mathbb{R}$ defined by

$$c_{X,\sigma,\phi}(x,\mathrm{m}-\mathrm{n},y) \;=\; c_\varphi(\mathrm{m})(x) - c_\varphi(\mathrm{n})(y).$$

This map: $Z^1(\mathbb{N}^k, C(X,\mathbb{R})) \to \mathcal{Z}^1_{conts}(\mathcal{G}(X,\sigma),\mathbb{R})$ is a bijection.

Ruelle operator $\mathcal{L}$ associated to a Ruelle dynamical multi-system

Definition 2: ([E], [ER], [W]) Let X be a compact metric space, let $T : X \to X$ be a surjective local homeomorphism, and let $\psi : X \to \mathbb{R}$ be a continuous real-valued function. The Ruelle operator $\mathcal{L}_{(X,T,\psi)}$ is defined on $C(X,\mathbb{R})$ by

$$\mathcal{L}_{(X,T,\psi)}(f)(x) \;=\; \sum_{y\in T^{-1}(x)} e^{\psi(y)} f(y).$$

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Lemma 1: (Generalization of [ER, Prop 2.2]) Let X be a compact metric space, let $\{\sigma_i\}_{i=1}^k$ be a commuting family of local homeomorphism defined on X, and let $\{\varphi_i\}_{i=1}^k \subset C(X, \mathbb{R})$ be a k-tuple of continuous functions on X that satisfy the cocycle condition for the $\{\sigma_i\}_{i=1}^k$, let $c_\varphi : \mathbb{N}^k \to C(X, \mathbb{R})$ be the associated 1-cocycle. Then the map from $\mathbb{N}^k$ to $\mathrm{End}(C(X, \mathbb{R}))$ given by

$$\mathrm{n} \to \mathcal{L}_{(X,\sigma^{\mathrm{n}},c_\varphi(\mathrm{n}))}$$

is a semigroup homomorphism.

The Ruelle dual operator $\mathcal{L}^*$ acting on $M(X)$

Suppose (X, T, ψ) represents a compact metric space X, a surjective local homeomorphism $T : X \to X$ and $\psi \in C(X, \mathbb{R})$. Recall that the Ruelle operator $\mathcal{L}_{(X,T,\psi)}$ is an endomorphism of $C(X, \mathbb{R})$ to itself. Thus the dual of the Ruelle operator $\mathcal{L}^*_{(X,T,\psi)}$ maps $C(X, \mathbb{R})^*$ to $C(X, \mathbb{R})^*$. Since $C(X, \mathbb{R})^*$ can be viewed as finite signed Borel measures on X, $\mathcal{L}^*_{(X,T,\psi)}$ maps signed Borel measures on X to signed Borel measures on X as follows:

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$$\int_X f(x)\, d(\mathcal{L}^*_{(X,T,\psi)}(\mu))(x) = \int_X \mathcal{L}_{(X,T,\psi)}(f)(x) d\mu(x), \ f \in C(X, \mathbb{R}).$$

Moreover by construction, $\mathcal{L}^*_{(X,T,\psi)}$ maps positive finite Borel measures on X to positive finite Borel measures on X.

Conditions for solving the positive eigenvalue problem for $\mathcal{L}^*$

Definition 3: Let (X, T, ψ) be as in the previous slide. Then (X, T, ψ) is said to admit a unique solution to the positive eigenvalue problem if there is a unique positive number $\lambda > 0$ and a unique probability measure μ on X such that

$$\mathcal{L}^*_{(X,T,\psi)}(\mu) \;=\; \lambda\mu,$$

i.e.

$$\int_X \mathcal{L}_{(X,T,\psi)}(f)(x)d\mu(x) \;=\; \lambda \int_X f(x)d\mu(x), \;\forall f \in C(X, \mathbb{R}).$$

Remark: It is a result of P. Walters [W] that if X has a compatible metric such that T is positively expansive, T is exact, and ψ is Hölder continuous, then (X, T, ψ) satisfies the unique positive eigenvalue condition.

The positive eigenvalue property for commuting Ruelle families

Definition 4: Let (X, σ, φ) be a triple corresponding to a k-tuple of commuting surjective local homeomorphisms $\{\sigma_i\}_{i=1}^k$ and functions $\{\varphi_i\}_{i=1}^k \subset C(X, \mathbb{R})$ and suppose that (X, σ, φ) satisfies the Cocycle Condition. Then (X, σ, φ) is said to admit a unique solution for the positive eigenvalue problem if there is a unique k-tuple of positive numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and a unique probability measure μ on X such that

$$\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}(\mu) = \lambda_i \mu, \; 1 \leq i \leq k.$$

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$$\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}(\mu) \; = \; \lambda_i \mu, \; 1 \leq i \leq k.$$

Theorem 2 (FHKP) A triple (X, σ, φ) satisfying the Cocycle Condition admits a unique solution to the positive eigenvalues problem if there exists some $\mathrm{n} \in \mathbb{N}^k \backslash \{0\}$ such that $(X, \sigma^{\mathrm{n}}, c_\varphi(\mathrm{n}))$ satisfies the unique positive eigenvalue condition of Definition 3. Here $c_\varphi : \mathbb{N}^k \to C(X, \mathbb{R})$ is the one-cocycle associated to (X, σ, φ) by Theorem 1.

Consequences of Theorems 1 and 2

Theorems 1 and 2 taken together show that given a Ruelle triple (X, σ, φ), in order for it to admit a unique solution for the positive eigenvalue problem, it is enough to have $\mathrm{n} \in \mathbb{N}^k \backslash \{0\}$ such that σ^{n} is positively expanding and exact, and such that $c_\varphi(\mathrm{n})$ is Hölder continuous all with respect to a compatible metric d.

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Example 2: Let $X = \mathbb{T}^2$ be the 2-torus. Fix a positive integer $d \geq 2$. Define the following two commuting local homeomorphisms of $\mathbb{T}^2$:
$\sigma_1(z_1, z_2) = (z_1^d, z_2^d), \quad \sigma_2(z_1, z_2) = (z_1 z_2^{-1}, z_1 z_2),\ z_1,\ z_2 \in \mathbb{T}.$
These local homeomorphisms are both expanding and exact. It follows that choosing $r_1, r_2 \in \mathbb{R}$ and setting $\varphi_1 = r_1,\ \varphi_2 = r_2$, the triple (X, σ, φ) has a unique solution to the positive eigenvalue problem.

KMS states on $C^*(\mathcal{G}(X,\sigma))$ coming from 1-cocycles

Let α be an action of $\mathbb{R}$ on a C^*-algebra A. It extends to an analytic action of $\mathbb{C}$ on a dense $*$-subalgebra $\mathcal{A}$ of A. Then we can associate KMS states to (A,α). The elements of $\mathcal{A}$ are called the entire elements associated to α, and a state ω is called a KMS state at inverse temperature $\beta \in \mathbb{R}$ if

$$\omega(a\alpha_{i\beta}(b)) \;=\; \omega(ba),\; \forall a,b \;\in \mathcal{A}.$$

In the case where we are given any continuous 1-cocycle ρ on $\mathcal{G}(X,\sigma)$ taking values in $\mathbb{R}$, by [R] we can define $\alpha_\rho : \mathbb{R} \to \mathrm{Aut}(C^*(\mathcal{G}(X,\sigma)))$ by

$$\alpha_\rho(t)(f)(x,\mathrm{k},y) \;=\; e^{it\rho(x,\mathrm{k},y)}f(x,\mathrm{k},y),\; f \;\in \mathcal{G}(X,\sigma), t \in \mathbb{R},$$

a formula valid for $f \in C_C(\mathcal{G}(X,\sigma))$ that extends to elements of $C^*(\mathcal{G}(X,\sigma))$. Moreover the elements of $C_C(\mathcal{G}(X,\sigma))$ are entire elements for α_ρ.

The Radon-Nikodym problem for $\mathcal{G}(X,\sigma)$, 1-cocycles, and KMS states

Let (X,σ) be as before, and let μ be a Borel probability measure defined on the compact metric space X. Define the pull–back measures $s^*\mu$ and $r^*\mu$ on $\mathcal{G}(X,\sigma)$. Suppose that μ is *quasi-invariant* for $\mathcal{G}(X,\sigma)$, so that the measures $s^*\mu$ and $r^*\mu$ are equivalent to one another. The Radon-Nikodym derivative for μ is the measurable real-valued function $D = \frac{\delta r^*\mu}{\delta s^*\mu}$ defined on $\mathcal{G}(X,\sigma)$, which is a multiplicative 1-cocycle with values in $\mathbb{R}^+$. Let ρ be a 1-cocycle for $\mathcal{G}(X,\sigma)$ with values in $\mathbb{R}$, and let $\beta \in \mathbb{R}$. We say that the measure μ on X satisfies the (ρ,β)-KMS condition if it is quasi-invariant for $\mathcal{G}(X,\sigma)$, and if its corresponding Radon-Nikodym derivative $D = e^{-\beta\rho}$.

Question: does there exist a k-tuple $\{\varphi_i\}_{i=1}^k \subset C(X,\mathbb{R})$ satisfying the Cocycle Condition and a probability measure on X and $\beta \in \mathbb{R}$ such that μ satisfies the $(c_{(X,\sigma,\varphi)},\beta)$-KMS condition?

Main Theorem

Theorem 3: (FHKP) Let (X, σ, φ) be a triple admitting a unique solution for the positive eigenvalue problem for the dual of the Ruelle operators $\{\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}\}_{i=1}^k$, so that there exists unique $\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{R}^+)^k$ and and a unique Borel probability measure μ with $\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}(\mu) = \alpha_i \mu$, $1 \leq i \leq k$. Suppose $\alpha_i = 1$, $1 \leq i \leq k$. Then $\mu := \mu^{(X,\sigma,\varphi)}$ is a quasi-invariant measure for $\mathcal{G}(X, \sigma)$, with Radon-Nikodym derivative $e^{-c_{(X,\sigma,\varphi)}}$, so that $\mu^{(X,\sigma,\varphi)}$ gives rise to a KMS_1 state for the gauge dynamics $\alpha_t^{(X,\sigma,\varphi)}(f) = e^{itc_{(X,\sigma,\varphi)}} f$, $f \in C_C(\mathcal{G}(X, \sigma))$, with corresponding KMS-state ω given by

$$\omega(f) = \int_X f(x, 0, x) d\mu_{(X,\sigma,\varphi)}, \ f \in C_C(\mathcal{G}(X, \sigma)).$$

Corollary of main theorem

Corollary 1: (FHKP) Let (X, σ, φ) be triple admitting a unique solution for the positive eigenvalue problem for the Ruelle dual operators $\{\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}\}_{i=1}^k$, so that there exists a unique k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{R}^+)^k$ and and a unique Borel probability measure $\mu := \mu^{(X,\sigma,\phi)}$ such that $\mathcal{L}^*_{(X,\sigma_i,\varphi_i)}(\mu) = \alpha_i \mu$, $1 \leq \ i \ \leq k$. Fix $\beta \in \mathbb{R} \backslash \{0\}$. Then setting

$$\varsigma_i(x) \ = \ \frac{\ln(\alpha_i) - \varphi_i(x)}{\beta}, \ 1 \leq i \leq k, \ \text{and } \varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_k),$$

$\mu = \mu^{(X,\sigma,\varphi)}$ is a eigenmeasure for the Ruelle operators $\{\mathcal{L}^*_{(X,\sigma_i,\beta\varsigma_i)}\}_{i=1}^k$ with constant eigenvalue 1, so that μ corresponds to a KMS_β-state for the generalized gauge dynamics on $C^*(\mathcal{G}(X, \sigma))$ obtained from (X, σ, ς).

Another example

Example 3: We compute Ruelle eigenvalues and eigenmeasures for the 2-Ruelle dynamical system (X, σ, φ), with $X = \prod_{j \in \mathbb{N}} \{0, 1\}$, and $\sigma = \{\sigma_j\}_{j=1,2}$ defined by, for $x = \{x_n\}_{n \in \mathbb{N}}$

$$\sigma_1(x) := (x_{n+1})_{n \in \mathbb{N}}, \quad \sigma_2(x) := (x_n + 1)_{n \in \mathbb{N}}.$$

Here addition is done modulo 2 component–wise. For $a, b, c \in \mathbb{R}$ define $\varphi = \{\varphi_j\}_{j=1,2}$ by the following equation, where again addition is considered mod 2.

$$\varphi_1(x) = \begin{cases} a & \text{if } x_0 + x_1 = 0 \\ b & \text{if } x_0 + x_1 = 1 \end{cases}, \quad \varphi_2(x) = c.$$

Example 3, continued

One computes that φ_i satisfy the cocycle condition and that the eigenvalues α_1, α_2 of the associated Ruelle operators are given by $\alpha_1 = e^a + e^b$, and $\alpha_2 = e^c$, and as for the eigenmeasure, $\mu(Z[0]) = \mu(Z[1]) = \frac{1}{2}$.

Using induction, one can show that for $n \geq 1$:

$$\mu(Z[x_0 x_1 \dots x_n]) = \frac{1}{2} \prod_{j=0}^{n-1} \left[e^{\psi(x_j + x_{j+1})} / (e^a + e^b) \right],$$

where $\psi : \{0, 1\} \to \{a, b\}$ is defined by $\psi(0) = a$, and $\psi(1) = b$.

As in Corollary 1, we can modify the $\{\varphi_i\}_{i=1}^2$ to obtain a pair of functions $\{\varsigma_i\}_{i=1}^2$ such that μ corresponds to a KMS_β-state for the generalized gauge dynamics on $\mathcal{G}(X, \sigma)$ associated to the $\{\beta\varsigma_i\}_{i=1}^2$.

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