Free compression and Standard Young Tableau

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joint work with Iris Arenas Longoria, arXiv:2009.11950

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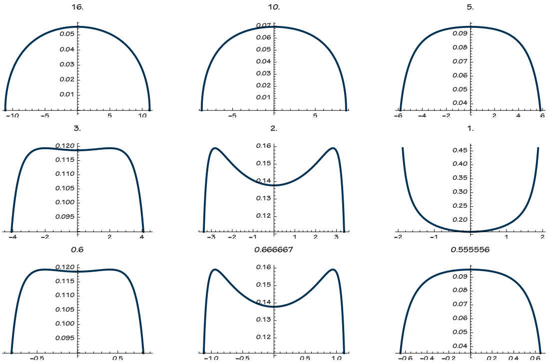
Origins

- $\mathcal{L}(\mathbb{F}_n)$ is the von Neumann algebra on $\ell^2(\mathbb{F}_n)$ of the free group on n generators: $u_1, \ldots, u_n$, φ = trace on $\mathcal{L}(\mathbb{F}_n)$
- $x = u_1 + u_1^{-1} + \cdots + u_n + u_n^{-1}$

The spectral measure of x with respect to φ for
$n = 16,\ 10,\ 5,\ 3,\ 2,\ 1,\ 3/5,\ 2/3,\ 5/9$



the bottom three measures are only sub-probability measures:
"dark matter" $= 4/5,\ 2/3,\ 8/9$

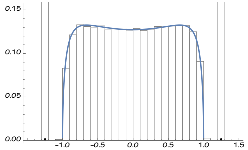
Free Compression (*Nica-Speicher*)

- x is a operator and p is a projection; pxp is the *compression* of x by p.
- if M is a finite von Neumann algebra and φ is a faithful normal trace then we can define *free independence* for elements of M:
  - unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_s$ are freely independent if whenever $a_1, \dots, a_n \in M$; $\varphi(a_i) = 0$; $a_i \in \mathcal{A}_{j_i}$ with $j_1 \neq j_2 \neq \cdots \neq j_n$ we have $\varphi(a_1 \cdots a_n) = 0$.
- if x and p are freely independent then we call pxp *the* free compression as the distribution of pxp only depends on the $*$-moments of x and $\varphi(p)$.

EX: $x = x^*$, $x^2 = 1$, $\varphi\left(\frac{1+x}{2}\right) = \varphi\left(\frac{1-x}{2}\right)$; x is a *symmetric Bernoulli random variable*, p, $\varphi(p)^{-1}pxp \overset{\mathcal{D}}{\sim} \left(\frac{\delta_{-1}+\delta_1}{2}\right)^{\boxplus \varphi(p)^{-1}}$, if p & x free

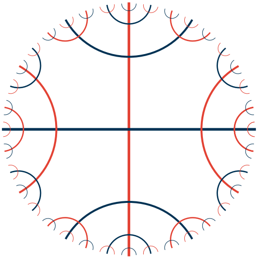
R.M. SIMULATION: $n = 3/5$

$X, P \in M_{1000}(\mathbf{C})$, $X = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$, $\mathrm{Tr}(P) = 800$, P is randomly rotated, eigenvalue distribution of $5/4\, PXP$ is plotted, $\mathrm{tr}(P) = 4/5$



Closed walks on d-regular trees (*Kesten, 1959*)

Let m_n be the number of closed walks on a d-regular tree

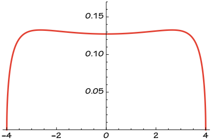


$d = 4$

n	m_n
2	d
4	$2d^2 - d$
6	$5d^3 - 6d^2 + 2d$
8	$14d^4 - 28d^3 + 20d^2 - 5d$
10	$42d^5 - 120d^4 + 135d^3 - 70d^2 + 14d$
12	$132d^6 - 495d^5 + 770d^4 - 616d^3 + 252d^2 - 42d$

THM (Kesten 1959): Let $\rho(t) = \dfrac{d}{2\pi} \dfrac{\sqrt{4d - t^2 - 4}}{d^2 - t^2}$ for $|t| \leqslant 2\sqrt{d-1}$

then $m_{2n} = \displaystyle\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} t^{2n} \rho(t)\, dt$

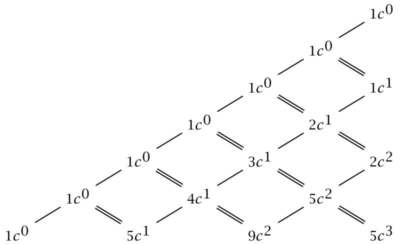


$(d = 5)$

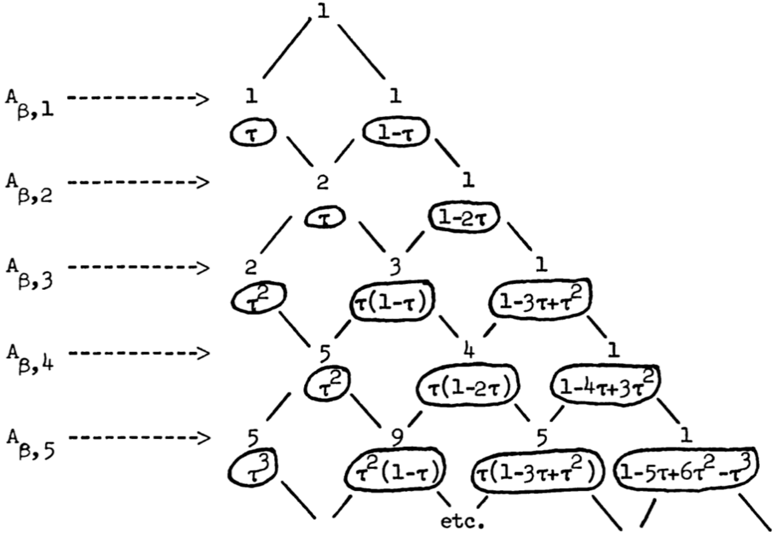
if $d\mu_d(t) = \rho(t)\, dt$ then $\mu_d = \left(\frac{\delta_{-1} + \delta_1}{2}\right)^{\boxplus d}$ for any real $d \geqslant 1$

Problem: find a simple rule to generate the moments combinatorially. Write m_n as function of $c = d - 1$ and interpolate with odd rows using a Pascal like rule

n	m_n
0	1
1	1
2	$1 + c$
3	$1 + 2c$
4	$1 + 3c + 2c^2$
5	$1 + 4c + 5c^2$
6	$1 + 5c + 9c^2 + 5c^3$
7	$1 + 6c + 14c^2 + 14c^4 + 5c^4$
8	$1 + 7c + 20c^2 + 28c^3 + 14c^4$
9	$1 + 8c + 27c^2 + 48c^2 + 42c^4$
10	$1 + 9c + 35c^2 + 75c^3 + 90c^4 + 42c^5$
11	$1 + 10c + 44c^2 + 110c^3 + 165c^4 + 132c^5$
12	$1 + 11c + 54c^2 + 154c^3 + 275c^4 + 297c^5 + 132c^6$



look familiar?



Chebyshev Polynomials of Second Kind

- $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$, Chebyshev polynomials of the second kind, $S_n(x) = U_n(x/2)$, rescaled
- $d\nu(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,dt$ on $[-2, 2]$, the semi-circle law
- $\int S_m(x) S_n(x)\, d\nu(x) = \delta_{m,n}$, orthogonality relations
-

n	S_n
0	1
1	x
2	$x^2 - 1$
3	$x^3 - 2x$
4	$x^4 - 3x^2 + 1$
5	$x^5 - 4x^3 + 3x$
6	$x^6 - 5x^4 + 6x^2 - 1$

n	x^n
0	S_0
1	S_1
2	$S_2 + S_0$
3	$S_3 + 2S_1$
4	$S_4 + 3S_2 + 2S_0$
5	$S_5 + 4S_3 + 5S_1$
6	$S_6 + 5S_4 + 9S_2 + 5S_0$

Standard Young tableaux with two rows

- Let $\lambda_1 \geqslant \lambda_2 \geqslant 1$ be integers. The Young diagram with shape (λ_1, λ_2) has λ_1 boxes in the first row and λ_2 boxes in the second row.

		$\cdots$			$\cdots$			λ_1 boxes
		$\cdots$			λ_2 boxes			

- put the numbers $1, 2, 3, \ldots, \lambda_1 + \lambda_2$ into the boxes so that they increase along rows and down columns, this produces a *standard Young tableau*. When $(\lambda_1, \lambda_2) = (3, 1)$ there are 3 ways of doing this:

1	2	4
3		

,

1	3	4
2		

,

1	2	3
4		

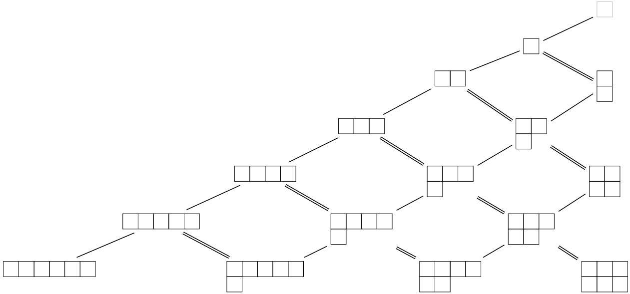
.

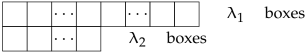
- let $f^{(\lambda_1, \lambda_2)}$ be the number of standard Young tableaux with shape (λ_1, λ_2), $f^{(3,1)} = 3$
- $f^{(\lambda)} = f^{(\lambda,0)} = 1$ because there is only 1 standard Young tableau with a row of length λ:

1	2	$\cdots$	λ

Stacking the diagrams

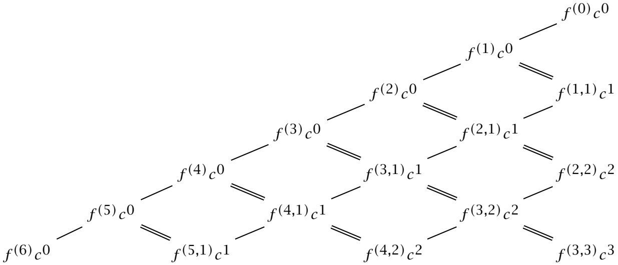
$/$ = add a box to the first row $\backslash\backslash$ = add a box to the second row



replace 

with $f^{(\lambda_1,\lambda_2)} c^{\lambda_2}$

Solving the Recurrence



$$f^{(\lambda_1,\lambda_2)} = f^{(\lambda-1,\lambda_2)} + f^{(\lambda_1,\lambda_2-1)}$$

$$f^{(\lambda_1,\lambda_2)} = \binom{\lambda_1+\lambda_2}{\lambda_1} - \binom{\lambda_1+\lambda_2}{\lambda_1+1}$$

Main Theorem

The $2n^{th}$ moment of the Kesten-McKay law with parameter $d = c + 1$ is

$$m_{2n} = \sum_{k=0}^{n} f^{(2n-k,k)} c^k$$

where

$$f^{(\lambda_1,\lambda_2)} = f^{(\lambda-1,\lambda_2)} + f^{(\lambda_1,\lambda_2-1)} \text{ and } f^{(\lambda_1,\lambda_2)} = \binom{\lambda_1+\lambda_2}{\lambda_1} - \binom{\lambda_1+\lambda_2}{\lambda_1+1}$$

Let v_1 and v_2 be two vertices one edge apart on a $d = c + 1$ regular tree. Let m_{2n-1} be the the number of walks starting at v_0 and ending at v_1. Then

$$m_{2n-1} = \sum_{k=0}^{n-1} f^{(2n-1-k,k)} c^k.$$