# Dilation theory for right LCM semigroup dynamical systems

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# Background and Motivation

**Theorem** (Sz. Nagy 1953)

*For a contraction* $T \in \mathcal{B}(\mathcal{H})$ *(that is* $TT^* \leq I$*), there exists an isometry* $V \in \mathcal{B}(\mathcal{K})$ *on* $\mathcal{K} \supset \mathcal{H}$ *such that* $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ *for all* $n \geq 1$.

**Theorem** (Brehmer 1961)

*For commuting contractions* $T_1, \cdots, T_n \in \mathcal{B}(\mathcal{H})$*, if for each* $F \subset \{1, \cdots, n\}$*, we have*

$$\sum_{U \subset F} (-1)^{|U|} T_U T_U^* \geq 0.$$

*Then* $T_i$ *can be dilated to commuting isometries* $V_i$.

Note: $V_i$ can be chosen to be doubly commuting (that is, Nica-covariant).

# Background and Motivation

**Theorem** (Frazho-Bunce-Popescu 1980's)

*For non-commuting contractions* $T_1, \cdots T_n$*, if* $\sum_{i=1}^n T_i T_i^* \leq I$*, then* $T_i$ *dilate to isometries* $V_i$ *with orthogonal ranges.*

*Moreover, if* $\sum_{i=1}^n T_i T_i^* = I$*, the minimal dilations* $V_i$ *also satisfy* $\sum_{i=1}^n V_i V_i^* = I$*.*

**Theorem** (L. 2019)

*Let* $P$ *be a right LCM semigroup. Then a contractive representation* $T : P \to \mathcal{B}(\mathcal{H})$ *has an isometric Nica-covariant dilation if and only if for any* $F \subset P$*,*

$$\sum_{U \subset F} (-1)^{|U|} T_U T_U^* \geq 0.$$

# Right LCM Semigroup Dynamical Systems

Recall a semigroup $P$ is called right LCM if for any $p, q \in P$,

$$pP \cap qP = \begin{cases} rP, & \text{if } pP \cap qP \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

An isometric representation $V$ of $P$ is called Nica-covariant if for any $p, q \in P$,

$$V_p V_p^* V_q V_q^* = \begin{cases} V_r V_r^*, & \text{if } pP \cap qP = rP; \\ 0, & \text{otherwise.} \end{cases}$$

The universal C*-algebra for isometric Nica-covariant representations is called the semigroup C*-algebra of $P$.

# Right LCM Semigroup Dynamical Systems

A semigroup dynamical system is a triple $(\mathcal{A}, P, \alpha)$ where $\mathcal{A}$ is a unital C*-algebra, $P$ is a semigroup, and $\alpha$ is a $P$-action on $\mathcal{A}$ by injective $*$-endomorphisms. One may notice that $\alpha$ only encodes the multiplicative structure on $P$ but not the right LCM structure.

**Definition**

*Let $P$ be a right LCM semigroup. A semigroup dynamical system $(\mathcal{A}, P, \alpha)$ is called a right LCM semigroup dynamical system if each $\alpha_p(\mathcal{A})$ is an ideal in $\mathcal{A}$ and for any $p, q \in P$,*

$$\alpha_p(1)\alpha_q(1) = \begin{cases} \alpha_r(1), & \text{if } pP \cap qP = rP; \\ 0, & \text{if } pP \cap qP = \emptyset. \end{cases}$$

# Right LCM Semigroup Dynamical Systems

## Example

*The semigroup $C^*$-algebra for a right LCM semigroup $P$ has a natural right LCM semigroup dynamical system. Let $\mathcal{D}_P = \overline{\text{span}}\{V_p V_p^*\}$. Let $\alpha_p(x) = V_p x V_p^*$. Then $(\mathcal{D}_P, P, \alpha)$ is a right LCM semigroup dynamical system.*

*By the Nica-covariance condition, $\mathcal{D}_P = C(\Omega_P)$ is a commutative $C^*$-algebra. If $K \subset \Omega_P$ is a compact subset such that $K$ and $K^c$ are invariant under $\alpha$, then we get a right LCM semigroup dynamical system $(C(K), \alpha, P)$.*

*One such $K$ gives the boundary quotient $(C(\partial\Omega_P), P, \alpha)$. The boundary quotient is generated by isometric representations $V$ such that for all foundation set $F \subset P$, $\prod_{i \in F}(I - V_i V_i^*) = 0$.*

# Right LCM Semigroup Dynamical Systems

Fix a right LCM semigroup dynamical system $(\mathcal{A}, P, \alpha)$

## Definition

*An isometric covariant representation is a pair* $(\pi, V)$ *where:*

1. $\pi$ *is a unital* $*$*-homomorphism of* $\mathcal{A}$*;*
2. $V$ *is an isometric representation of* $P$*;*
3. *For all* $p \in P$ *and* $a \in \mathcal{A}$*,* $V_p \pi(a) V_p^* = \pi(\alpha_p(a))$*.*

Note: from the right LCM condition, $V$ is Nica-covariant.

**Definition**

*A contractive covariant representation is a pair* $(\phi, T)$ *where:*

1. $\phi$ *is a* $*$*-preserving linear map on* $\mathcal{A}$*;*
2. $T$ *is a contractive representation of* $P$*;*
3. *For all* $p \in P$ *and* $a \in \mathcal{A}$, $T_p \phi(a) T_p^* = \phi(\alpha_p(a))$.

Question: when does a contractive covariant representation $(\phi, T)$ dilate to an isometric covariant representation $(\pi, V)$?

## Example

*On $(\mathcal{D}_P, P, \alpha)$, a contractive representation $T$ of $P$ also defines a $*$-preserving linear map $\phi$ on $\mathcal{D}_P$ by $\phi(V_p V_p^*) = T_p T_p^*$. The pair $(\phi, T)$ defines a contractive covariant representation.*

*The pair $(\phi, T)$ dilates to an isometric covariant pair if and only if for any $F \subset P$,*

$$\phi\left(\prod_{i\in F}(I - V_i V_i^*)\right) = \sum_{U\subset F}(-1)^{|U|} T_U T_U^* \geq 0.$$

# Main Result

**Theorem** (Laca-L.)

*A contractive covariant representation* $(\phi, T)$ *dilates to an isometric covariant representation* $(\pi, V)$ *if and only if* $\phi$ *is unital completely positive. Moreover, the dilation* $(\pi, V)$ *can be chosen to be minimal and this minimal dilation is unique.*

# Examples and Applications

We first obtain a dilation result for the boundary quotient.

**Theorem**

*Suppose $T$ is a contractive representation of a right LCM semigroup $P$ such that for any $F \subset P$,*

$$\sum_{U \subset F} (-1)^{|U|} T_U T_U^* \geq 0,$$

*and for any foundation set $F$,*

$$\sum_{U \subset F} (-1)^{|U|} T_U T_U^* = 0.$$

*Then $T$ can be dilated to an isometric representation $V$ of the boundary quotient.*

# Examples and Applications

## Corollary

*If contractions* $T_1, \cdots, T_n$ *satisfy* $\sum_{i=1}^n T_i T_i^* = I$*, then they can be dilated to isometries* $V_1, \cdots, V_n$ *that* $\sum_{i=1}^n V_i V_i^* = I$*.*

## Corollary

*If* $T_1, \cdots, T_n$ *are commuting co-isometries, then they dilate to commuting unitaries.*

# Examples and Applications

Let $(\mathcal{A}, P, \beta)$ be an automorphic semigroup dynamical system. We can build a right LCM semigroup dynamical system $(\mathcal{D}_P \otimes \mathcal{A}, P, \tilde{\alpha})$ by

$$\tilde{\alpha}_p(f \otimes a) = \alpha_p(f) \otimes \beta_p(a).$$

A $*$-preserving linear map $\phi$ on $\mathcal{A}$ and a contractive representation $T$ of $P$ can define a contractive covariant representation $(\tilde{\phi}, T)$ by

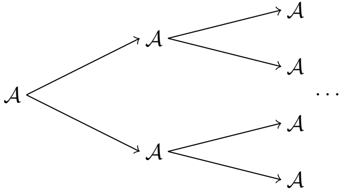
$$\tilde{\phi}(V_p V_p^* \otimes a) = T(p)\phi(\beta_p^{-1}(a))T(p)^*.$$

## Proposition

*The map $\tilde{\phi}$ is unital completely positive if and only if for each finite $F \subset P$, the map $\phi_F(a) := \sum_{U \subset F} (-1)^{|U|} T(s_U)\phi(\beta_{s_U}^{-1}(a))T(s_U)^*$ is completely positive.*

# Examples and Applications

Let $\mathcal{A}$ be a C*-algebra. Consider the direct limit $\tilde{\mathcal{A}} = C(X) \otimes \mathcal{A}$.




We have a right LCM semigroup dynamical systems $(\tilde{\mathcal{A}}, \mathbb{F}_2^+, \alpha)$ where $\alpha_{e_1}(a) = a \oplus 0$ and $\alpha_{e_2}(a) = 0 \oplus a$.

## Proposition

*If $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is unital completely positive and if there exist contractions $T_1, \cdots, T_n$ such that $\phi(a) = \sum_{i=1}^n T_i \phi(a) T_i^*$. Then we can dilate $\phi$ to a unital $*$-representation $\pi$ of $\mathcal{A}$ and $T_i$ to isometries $V_i$ with orthogonal ranges, such that $\pi(a) = \sum_{i=1}^n V_i \pi(a) V_i^*$*

Thank you