Phase Transition in Dirac Ensembles

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Joint works with S. Azarfar, N. Pagliarolli, H. Hessam

Metric from Dirac operators

- Connes' distance formula: Let (M, g) be a spin Riemannian manifold with Dirac operator D Then

$$d(p, q) = \mathrm{Sup}\{|f(p) - f(q)|;\ ||[D, f]|| \leq 1\}.$$

Compare with the classical dual formula

$$d(p, q) = \mathrm{Inf} \int_p^q (g_{\mu\nu}\ dx^\mu dx^\nu)^{1/2}$$

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- Connes' notion of noncommutative Riemannian manifold, or spectral manifolds $(A, \mathcal{H}, D)$.

Discrete Dirac operators of Barrett-Glaser

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$$Z = \int_{metrics} e^{-S(g)} D(g) \Rightarrow \int_{Diracs} e^{-\mathcal{S}(D)} dD$$

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- Consider a quartic action

$$S(D) = g\mathrm{Tr}(D^2) + \mathrm{Tr}(D^4)$$

Multitrace and multimatrix unitary invariant ensembles

- Computation shows:

  $$S(D) = 2N(g \operatorname{Tr} H^2 + \operatorname{Tr} H^4) + 2g(\operatorname{Tr} H)^2 + 8 \operatorname{Tr} H \operatorname{Tr} H^3 + 6(\operatorname{Tr} H^2)^2$$

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- A unitary invariant function $F(H)$ on $\mathcal{H}_N$ is of the form

  $$F(H) = P(\operatorname{Tr}(H), \operatorname{Tr}(H^2), \dots \operatorname{Tr}(H^N))$$

  for a polynomial P in N-variables.

- Our models are all multitrace and mostly multimatrix, but unfortunately most results in RMT are for functions $F(H) = \operatorname{Tr}(V(H))$.

Genus expansion, summing over discrete surfaces/fat graphs ('t Hooft)

- Fix a polynomial $V(x) = \sum \frac{t_k}{k} x^k$. Consider the matrix integral

$$Z_N = \int_{\mathcal{H}_N} e^{-N\mathrm{Tr}(V(H))} dH,$$

- Topological expansion of $F_N = \log Z_N$

$$F_N = \sum_{g \geq 0} (N)^{2-2g} F_g\,, \qquad F_g = \sum_{[\mathcal{M}] \in \mathbb{M}_{\emptyset}^{g}} \mathrm{weight}(\mathcal{M})$$

where $\mathbb{M}_{\emptyset}^{g}$ = set of isomorphism classes of the Feynman weighted connected closed maps.

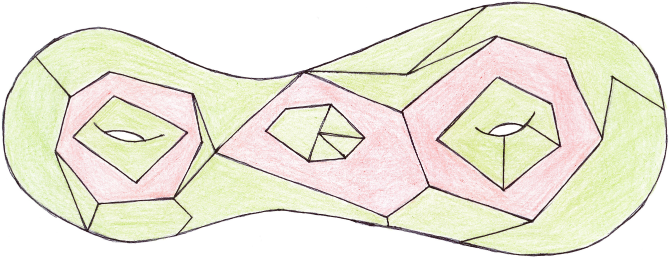

Figure: A polygonalization of a genus 2 surface

Genus expansion leads to a quick proof of the Wigner law, links with geometry of moduli spaces of curves, topological recursion (Eynard-Orantin), 2d gravity, recursion formula for volumes of moduli spaces of Riemann surfaces (Mirzakhani recursion).

Eynard-Orantin topological recursion (Tutte recursion)

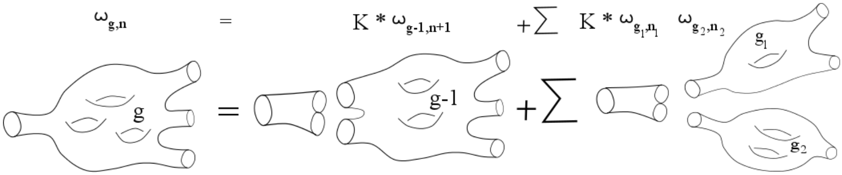

Figure: Source: Wikipedia, by B. Eynard

- $\omega_{g,n}$ is a meromorphic symmetric n-form on Σ, recursion is on $-\xi(\Sigma) = 2g - 2 + n$.
- Example: Mirzakhani recursion for Weil-Petersson volumes of moduli spaces, $\omega_{0,1} = \frac{4}{\pi} z \sin(\pi z) dz$ and $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$

Resolvent technique

- Green function (Stieltjes transform)

$$G(z) = \frac{1}{N}\langle \mathrm{Tr}\,\frac{1}{z-H}\rangle = \int \rho_N(\lambda)\frac{d\lambda}{z-\lambda}$$

Is a holomorphic function on $\mathbb{C} \setminus \mathrm{supp}(\rho_N)$.

- Can recover the eigenvalue probability density $\rho(x)$ from $G(x)$:

$$-2\pi i \rho_N(x) = \lim_{\varepsilon \to 0^+} \left[G(x+i\varepsilon) - G(x-i\varepsilon)\right]$$

- For single trace models with potential $V(x)$, It satisfies a differential equation

$$G^2(x) - V'(x)G(x) + \frac{1}{N}G'(x) + P(x) = 0.$$

- Can drop N for large N limit

$$G(x) = \frac{1}{2}(V'(x) - \sqrt{V'(x)^2 - 4P(x)})$$

  (Since $G(x) \sim \frac{1}{x}$ for large x, negative sign is chosen).
- For $V(x) = \frac{1}{2}x^2$ this gives the semicircle law. For small enough variations support of $\rho(x)$ is an interval, but in genral will be a union of intervals (multi cut regime).

The saddle point equation

- Our models are unitary invariant but not single trace, so a slightly different approach is needed.

$$Z = \int_{\mathcal{H}_N} e^{F(H)} dH,$$

$$Z = C_N \int_{\mathbb{R}^N} e^{-N \sum_{i=1}^N V(\lambda_i) - \sum_{i,j=1}^N U(\lambda_i,\lambda_j)} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 d\lambda_1 \ldots \lambda_N,$$

$$V(s) = 2gs^2 + 2s^4,$$
$$U(s,t) = 2gst + 8st^3 + 6s^2t^2,$$

Euler-Lagrange equations

- The measure $\mu = \rho(x)dx$ is referred to as the equilibrium measure and it is the Borel probability measure that minimizes the energy functional $I(\mu) =$

$$\int V(s)d\mu(s) + \int\int U(s,t)d\mu(s)d\mu(t) - \int\int \log|s-t|d\mu(s)d\mu(t).$$

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- Euler-Lagrange equations (fully justified by results of Percy Deift) shows:

$$\text{P.V.} \int_{\text{supp}\rho} \frac{\rho(s)}{s-x} ds = 2gx + 4x^3 + gm_1 + 4m_3 + 6m_2 x.$$

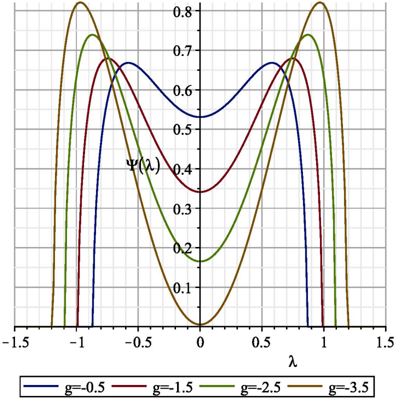

Figure: The equilibrium measure from the single cut analysis.

A precise critical value is found by setting $\rho(x) = 0$ at $x = 0$:

$$g_c = -\frac{5\sqrt{2}}{2} \approx -3.5,$$

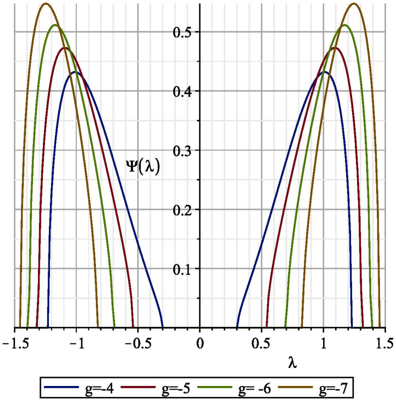

Figure: The equilibrium measure from the double cut analysis, $g < g_c$

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