Amenability, proximality and higher order syndeticity

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Theorem (Furstenberg 1973, Glasner 1976)

There is a unique universal minimal proximal flow $\partial_p G$ and a unique universal minimal strongly proximal flow $\partial_{sp} G$. For every minimal proximal flow X there is a surjective G-map $\partial_p G \to X$. Similarly for $\partial_{sp} G$.

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Statements about specific flows translate to statements about universal flows. E.g. G has a free proximal flow iff $\partial_p G$ is free.

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Theorem (K-Kalantar 2017)

The reduced C*-algebra $\mathrm{C}^*_\lambda G$ is simple iff $\partial_{sp} G$ is free.

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Key point: If $\partial_p G$ is non-trivial then it is free. This is not true for $\partial_{sp} G$. (Reminiscent of the fact that LG has a unique trace iff LG is a factor, but $C^*_\lambda G$ can have a unique trace without being simple.)

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Goal: Identify these Boolean algebras, thereby giving "concrete" descriptions of $\partial_p G$ and $\partial_{sp} G$.

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These subsets have very interesting structure!

Characterizations

Proposition

Let X be a (strongly) proximal flow. For open $U \subseteq X$ and $x \in X$, the return set $U_x = \{g \in G : gx \in U\}$ is (strongly) completely syndetic.

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Examples

Example

A subset $A \subseteq \mathbb{Z}$ is syndetic if and only if it has "bounded gaps," meaning there is $k \in \mathbb{N}$ such that for all $a \in \mathbb{Z}$,

$$\{a, a+1, \ldots, a+k\} \cap A \neq \emptyset.$$

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A subset $A \subseteq Z$ is completely syndetic if and only if for every n, A^n has "bounded diagonal gaps," meaning there is $k \in \mathbb{N}$ such that for any $(a_1, \ldots, a_n) \in \mathbb{Z}^n$,

$$\{(a_1, \ldots, a_n), (a_1+1, \ldots, a_n+1), \ldots, (a_1+k, \ldots, a_n+k)\} \cap A^n \neq \emptyset.$$

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Fact: The group $\mathbb{Z}$ does not contain disjoint completely syndetic subsets.

Example (the integers 1)

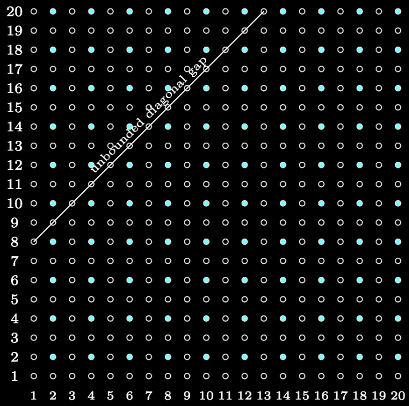

FIGURE 1. The subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$ is syndetic but not 2-syndetic since there are arbitrarily long diagonal segments in $\mathbb{Z} \times \mathbb{Z}$ that do not intersect $2\mathbb{Z} \times 2\mathbb{Z}$.

Example (the integers 2)

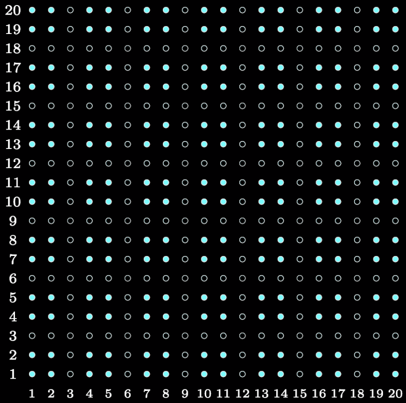

FIGURE 2. The subset $\mathbb{Z} \setminus 3\mathbb{Z} \subseteq \mathbb{Z}$ is 2-syndetic but not 3-syndetic since for $k \in \mathbb{N}$, every element in the set $\{(1,2,3),(2,3,4),(4,5,6),\ldots,(1+k,2+k,3+k)\}$ has an entry that is a multiple of 3, implying that the set does not intersect A^3.

Example (the integers 3)

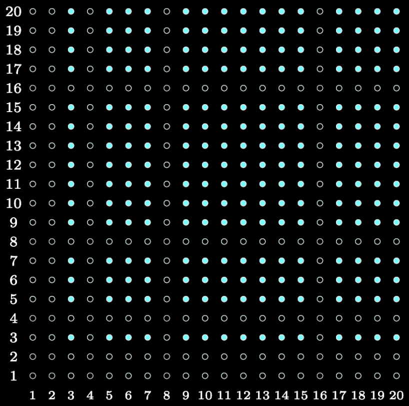

FIGURE 3. The complement of the set of powers of 2 in $\mathbb{Z}$ is completely syndetic, and in particular is 2-syndetic.

Consequences

Theorem (KRS 2020)

The group G is not strongly amenable if and only if there is a proper normal subgroup $H \leq G$ such that for every finite subset $F \subseteq G \setminus H$, there is a completely syndetic subset $A \subseteq G$ satisfying $FA \cap A = \emptyset$.

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Note: Does not seem easy to derive from existing criteria (e.g. Følner condition, paradoxicality condition).

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Consider the free group $\mathbb{F}_2 = \langle a, b \rangle$. For $w \in \mathbb{F}_2$, let

$$B_w = \{g \in G : g = wg' \text{ in reduced form}\}.$$

Can show by hand that B_a and B_b are strongly completely syndetic. Alternatively, $B_a = U_{a^\infty}$ where U is the set of infinite reduced words beginning with a in the hyperbolic boundary ∂F_2. Either way, $\mathbb{F}_2$ is non-amenable.

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A subset $A \subseteq G$ is a dense orbit set iff there is no subset $B \subseteq A^c$ with the property that for every pair of finite subsets $F_1 \subseteq B$ and $F_2 \subseteq B^c$, the following set is syndetic:

$$(\cap_{f_1 \in F_1} f_1^{-1} B) \cap (\cap_{f_2 \in F_2} f_2^{-1} B^c)$$

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Note: Proof inspired by the “topological Furstenberg correspondence.” Heaviliy utilizes semigroup structure of βG.

Thanks!