# Spectral triples and ergodic flows

Heath Emerson
(Joint work with N. Butler and T. Shulz)

University of Victoria

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# The quantum harmonic oscillator

$$H = -\frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{1}$$

- H is the Hamiltonian of the particle
- m is the mass
- $p = -i\hbar\frac{\partial}{\partial x}$ is momentum $\hbar$ Planck constant.
- $x =$ position.

The Schrödinger equation is

$$H\psi = \lambda\psi. \tag{2}$$

with solutions $\psi_0, \psi_2, \ldots$

$$\psi_n(x) = c_n \cdot H_n(x)e^{-x^2}, \quad n = 0, 1, 2, \ldots.$$

The corresponding eigenvalues are $\lambda_n = (2n+1)\hbar\frac{1}{2}\sqrt{km}$.

# Dirac's idea

- method of finding the energy states $\psi_n$ without solving a differential equation. Set

$$a = \sqrt{\frac{m\omega}{2\hbar}} \cdot (x + \frac{i}{m\omega} p), I \;\; a^* = \sqrt{\frac{m\omega}{2\hbar}} \cdot (x - \frac{i}{m\omega} p)$$

Then

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^*), \;\; p = i\sqrt{\frac{\hbar m\omega}{2}}(a - a^*).$$

- $a$ appends a single quantum of energy to the oscillator and $a^*$ subtracts one
- $a$, $a^*$ act as weighted shifts on $L^2(\mathbb{R})$ with respect to the eigenbasis for $H$.

# Canonical anti-commutation relations

$H, a, a^*$ satisfy

$$H = \hbar\omega(a^* a + \frac{1}{2}), \ \ [a, a^*] = 1, \ \ [H, a] = -2a, \ [H, a^*] = 2a.$$

So $a^*$ acts on the ground state $\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$ to produce all the energy states $\psi_n$, $n = 0, 1, 2, \ldots$.

# The Dirac-Schrödinger operator

We set

$$D = \begin{bmatrix} 0 & x - \frac{d}{x} \\ x + \frac{d}{dx} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix}$$

acting on $L^2(\mathbb{R})$ so that

$$D^2 = \begin{bmatrix} H - 1 & 0 \\ 0 & H + 1 \end{bmatrix}$$

Then

- The eigenvalues of $|D|$ are $\sim \sqrt{n}$.
- $(1 + D^2)^{-1}$ is compact, indeed, is Hilbert-Schmidt.
- $[f, D]$ is bounded if $f$ is a differentiable bounded function on $\mathbb{R}$ with bounded first derivative.
- $[u_t, D]$ is bounded for any translation $U_t \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

# Spectral triples

We obtain a spectral cycle $\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), D\right)$ for the C*-algebra $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$: the *Heisenberg cycle*.
Let $\alpha$ be a smooth flow on $M$ a compact manifold. Let $p \in M$
Then $C(M)$ embeds in $C_u(\mathbb{R})$ by $f \mapsto f_p$ where

$$f_p(t) := f\left(\alpha_t(p)\right).$$

This induces a *-homomorphism

$$\pi_p\colon C(M) \rtimes \mathbb{R}_d \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B}\left(L^2(\mathbb{R})\right)$$

and pulls the Heisenberg cycle back to one for $C(M) \rtimes \mathbb{R}_d$, and to $C(M) \rtimes \Lambda$ for any subgroup $\Lambda \subset \mathbb{R}$.

# Problem

Prove that if $a \in A^\infty \subset A := C(M) \rtimes \Lambda$, with $\Lambda \subset \mathbb{R}$ finitely generated, acting through a smooth flow $\alpha$, and $A^\infty$ a suitable smooth subalgebra of $A$, then the zeta function

$$\zeta(a, s) := \mathrm{Trace}(\pi_p(a) H^{-s})$$

meromorphically extends to $\mathbb{C}$.

My contention is that this has something to do with the cohomological equation, and ergodic time averages in dynamics.

# Integral formula for the zeta function

**Theorem**

*Let $f \in C_u(\mathbb{R})$. Then*

$$\Gamma(s)\,\zeta(f,s) := \Gamma(s)\cdot \mathrm{Trace}(fH^{-s})$$
$$= \frac{1}{2\sqrt{\pi}} \cdot \int_0^1 t^{s-1}\mathrm{csch}\,t \cdot \int_{\mathbb{R}} f(x\sqrt{\coth t})e^{-x^2}\,dxdt \quad (3)$$

*up to an entire function of $s$.*

Proof follows from:

- Heat kernel is
$$k_t(x,y) = \frac{1}{\sqrt{\sinh 2t}} \cdot \exp\left(-\tanh t \cdot \frac{(x+y)^2}{4} - \coth t \cdot (\frac{(x-y)^2}{4})\right)$$
- Mellin transform and the change of variables $x \mapsto x\sqrt{\tanh t}$.

We start with the problem of computing $\lim_{s\to 1^+}(s-1)\cdot\zeta(f,s)$ for $f \in C_u(\mathbb{R})$.

**Lemma**

*If $f \in C_u(\mathbb{R})$ then*

$$\lim_{s\to 1^+}(s-1)\cdot\zeta(f,s) \sim \frac{1}{2\sqrt{\pi}}\cdot\lim_{\lambda\to\infty}\int_0^1\int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt. =: \mu_u(f)$$

*in the sense that if either limit exists they both do and are equal.*

Let $\mathcal{D}$ be the closed linear subspace of $C_u(\mathbb{R})$ of all $f$ such that the limits above exist.

# Birkhoff ergodic

## Proposition

*If $\alpha$ is a smooth flow on $M$ compact and $\mu$ an $\alpha$-invariant measure and $f \in C(M)$, then $f_p \in \mathcal{D}$ and $\mu_u(f_p) = \int_M fd\mu$. for a.e $\in M$.*

Proof: $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = \int_M fd\mu$ by Birkhoff Ergodic.
Existence of this limit implies that
$\frac{1}{2\sqrt{\pi}}\int_0^1 \int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt \to \int_M fd\mu$ as well as $\lambda \to \infty$.

If $f \in C_u(\mathbb{R})$ admits $n$ successive bounded anti-derivatives, ${}^1 f, {}^2 f, \ldots, {}^n f$, then $\mathrm{Trace}(fH^{-s})$ extends analytically to $\mathrm{Re}(s) > 1 - \frac{n}{2}$.
Indeed,

$$\Gamma(s)\cdot\mathrm{Trace}(fH^{-s}) \sim \frac{1}{2\sqrt{\pi}}\cdot\int_0^1\int_{\mathbb{R}} t^{s-1}\mathrm{csch}\, t\cdot\int_{\mathbb{R}} f(x\sqrt{\coth t})\cdot e^{-x^2}\,dxdt \tag{4}$$

Let $F = {}^1 f$, then integration by parts gives

$$= \frac{1}{\sqrt{\pi}}\cdot\int_0^1\int_{\mathbb{R}} t^{s-1}\mathrm{csch}\, t\sqrt{\tanh t}\cdot F(x\sqrt{\coth t})\,xe^{-x^2}\,dxdt$$

$$= \frac{1}{\sqrt{\pi}}\int_0^1 t^{s-1}\mathrm{csch}\, t\sqrt{\tanh t}\cdot\phi(t)dt, \tag{5}$$

with $\phi(t) = \int_{\mathbb{R}} F(x\sqrt{\coth t})\,xe^{-x^2}dx$ similar version of same thing but analytic for $\mathrm{Re}(s) > 1/2$ and $f$ replaced by ${}^1 f$.

# Periodic functions

In the case of the usual flow on the circle:

**Theorem**

*If $f \in C_u(\mathbb{R})$ is periodic of period $\rho$ then $\mathrm{Trace}(fH^{-s})$ extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and residue*

$$\mathrm{Res}_{s=1}\mathrm{Trace}(fH^{-s}) = \frac{1}{\rho}\int_0^{\rho} f(t)dt$$

If $f$ is $\rho$-periodic on $\mathbb{R}$ with zero mean $\mu(f) = 0$ then $\int_0^T f(t)dt$ is also $\rho$-periodic with zero mean. So having zero mean for periodic functions is equivalent to having a periodic (*e.g.* continuous) anti-derivative. Now

$$\zeta(f, s) = \zeta(f - \mu(f), s) + \mu(f)\zeta(1, s)$$

and the first term extends analytically and $\zeta(1, s)$ is the Riemann zeta function, giving that $\zeta(f, s)$ has a single simple pole at $s = 1$ and residue there the 'obstruction' $\mu(f)$.

# The cohomological equation

If $X$ generates a smooth flow $\alpha$ and $f \in C^\infty(M)$, is $Xu = f$ solvable by $u \in C^\infty(M)$? (Analogy: is a form $\omega$ exact: $\omega = d\tau$ some $\tau$.)
Obstruction: if $\mu$ is $\alpha$-invariant then $\int_M u \circ \alpha_t d\mu$ is constant for any smooth $u$ and differentiating gives $\int_M Xu \, d\mu = 0$, whence $\int_M f d\mu = 0$ is necessary (analogy: exact implies closed $d\omega = 0$).

If $\ker(\mu) = \mathrm{ran}(X)$ acting on $C^\infty(M)$ then say the flow is $\mu$-cohomologically trivial.

## Theorem

*Let $\alpha$ be a smooth ergodic flow on $M$ with generating vector field $X$, which is $\mu$-cohomologically trivial for some $\mu$.*
*Then $\mathrm{Trace}(f_p H^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and $\mathrm{Res}_{s=1}\mathrm{Trace}(f_p H^{-s}) = \int_M f d\mu$ for all $f \in C^\infty(M)$.*

## Theorem

*Let $\alpha$ be a linear ergodic flow on $\mathbb{T}^n$ with Diophantine periods.*
*Then if $f \in C^\infty(\mathbb{T}^n)$, $p \in \mathbb{T}^n$, and $f_p(t) := f(\alpha_t(p))$, then $\mathrm{Trace}(f_p H^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and*

$$\mathrm{Res}_{s=1}\mathrm{Trace}(f_p H^{-s}) = \int_{\mathbb{T}^n} f d\mu.$$

# Theorem

If $\Lambda \subset \mathbb{R}$ is finitely generated, say by $S$, we say the embedding $\Lambda \to \mathbb{R}$ *satisfies a Diophantine condition* if $|\alpha| \geq |\alpha|_S^{-\gamma}$ for some $\gamma > 1$ and all $\alpha \in \Lambda$, where $|\alpha|_S$ is the word length.
Let $\alpha$ be a smooth ergodic flow on $\mathbb{T}^n$ with Diophantine periodic and $\Lambda \subset \mathbb{R}$ a finitely generated subgroup satisfying a Diophantine property. Set $A = C(M) \rtimes \Lambda$ and $A^\infty = \cap_{n,m} \mathrm{dom}(\delta_1^n \delta_2^m)$. where

$$\delta_1(\sum f_\alpha \, \alpha) = \sum f'_\alpha \, \alpha, \quad \delta_2(\sum f_\alpha \alpha) = \sum \alpha \cdot f_\alpha \alpha.$$

Then with the representation $\pi_p$ of $A^\infty$ on $L^2(\mathbb{R})$ obtained by restricting to an orbit, the triple $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi_p \oplus \pi_p, D)$ is a spectral triple over $A^\infty$ with the meromorphic extension property.
We have: ...

a) For $a \in A^\infty$, $\mathrm{Trace}(\pi_p(a)H^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and

$$\mathrm{Res}_{s=1}\mathrm{Trace}(\pi_p(a)H^{-s}) = \tau_\mu(a),$$

where $\tau_\mu$ is the trace on $A$ determined by the unique $\alpha$-invariant measure.

b) The functional

$$\tau_2(a^0, a^1, a^2) = \tau_\mu \left(a^0(\delta_1(a^1)\delta_2(a^2)\delta_2(a^1)\delta_1(a^2)\right)$$

is a cyclic 2-cocycle on $A^\infty$ and $\mathrm{Ch}(D_{\alpha,\Lambda}) = \tau_\mu - \tau_2$.

## Theorem

*Let $\alpha$ be Kronecker flow on $\mathbb{T}^2$ along lines of slope $\hbar$. Set*

$$\Lambda = \{n + m\hbar \mid n, m \in \mathbb{Z}\} \subset \mathbb{R}.$$

*The $C(\mathbb{T}^2) \rtimes \Lambda \cong A_\hbar \otimes A_{1/\hbar}$ and the class $\Delta_\hbar \in \mathrm{KK}_0(A_\hbar \otimes A_{1/\hbar}, \mathbb{C})$ of the corresponding Heisenberg cycle for $C(\mathbb{T}^2) \rtimes \Lambda$ is represented by a spectral triple over a smooth subalgebra $B^\infty \subset A_\hbar \otimes A_{1/h}$ with the meromorphic continuation property.*
*The class $\Delta_\hbar$ induces a KK-duality between $A_\hbar$ and $A_{1/\hbar}$ and the matrix of the induced form on $\mathrm{K}_0$ is given by*

$$\begin{bmatrix} 1 & -\lfloor 1/\hbar \rfloor \\ -\lfloor \hbar \rfloor & 1 \end{bmatrix}$$

*with respect to the bases consisting of the unit and the respective Rieffel projections.*

# Example: irrational rotation

$A_\hbar = C(\mathbb{R}/\mathbb{Z}) \rtimes \hbar\mathbb{Z}$ carries a natural $\mathbb{Z}$-parameterized family of representations on $L^2(\mathbb{R})$ with the generator of $\mathbb{Z}$ acting by $\hbar + n$, $n = 0, \pm 1, \pm 2, \ldots$.

The previous construction gives a family $[D_\hbar]$ of K-homology classes for $A_\hbar$, as $\hbar$ runs over the coset $\hbar + \mathbb{Z}$. Each is represented by a spectral triple over $A_\hbar^\infty$ with the meromorphic continuation property over the *C*-algebra* $A_\hbar$.

# Conjecture

For general smooth flows of parabolic type, results of Forni and others imply that there are typically infinitely many distributional obstructions to solving the cohomological equation. These are organized by certain asymptotic expansions (of the function $\int_0^T f(t)dt$.) My hope is that these dynamical asymptotic expansions can be used to produce meromorphic extensions of the zeta function $\zeta(f, s)$ with many other poles, with residues the distributions. This could allow us to do Noncommutative Geometry with flows on higher genus surfaces.