Classification of Nonsimple Real AI Algebras

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Real C*-algebras

The natural definition for a real C*-algebra is that it a real Banach *-algebra that is isomorphic to a norm closed self adjoint algebra of operators on a real Hilbert space. (By *-algebra we mean that it has an involution * that is real linear and satisfies (ab)*=b*a*.) This is then analogous to the definition of complex C*-algebra. One would then like to find an abstract set of axioms, like in the complex case. It turns out that one requires one more axiom: One must assume that $x^*x + 1$ is always invertible in the unitisation. One can then form the complexification $A \otimes \mathbb{C}$ of a real C^*-algebra A and extend the norm of A to a C^*-norm on $A \otimes \mathbb{C}$. On the complexification we then get a map $\varphi : A \otimes \mathbb{C} \to A \otimes \mathbb{C}$ defined by $\varphi(x + iy) = x^* + iy^*$ (note the $+$, which makes it different from just the adjoint).

This map satisfies $\varphi(a + \lambda b) = \varphi(a) + \lambda\varphi(b)$ for all $a, b \in A \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}$, $\varphi(ab) = \varphi(b)\varphi(a)$, $\varphi(a^*) = \varphi(a)^*$, and $\varphi^2 = \mathit{identity}$. In words, it is an involutive *-antiautomorphism. We can identify A inside of $A \otimes \mathbb{C}$ as $\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\}$. Conversely, if we are given a complex C^*-algebra, and an involutive *-antiautomorphism φ on it, the subset above is a real C^*-algebra whose complexification is the given one. We thus have two ways of viewing real C^*-algebras, as real Banach algebras themselves, or via involutive *-antiautomorphisms (henceforth called real structures) on complex C^*-algebras. We shall write (A, τ) for a complex C^*-algebra with real structure τ.

Example: Group C^*-algebras

If G is a finite group, we get a real structure on $C^*(G)$ defined by $\tau(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g g^{-1}$. The real form may give additional information. For example, for the dihedral group D_8 and quaternion group Q_8 we have $C^*(D_8) \cong C^*(Q_8) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})$ but the real form for D_8 is $\mathbb{R}^4 \oplus M_2(\mathbb{R})$ and the real form for Q_8 is $\mathbb{R}^4 \oplus \mathbb{H}$.

Commutative Real C^*-algebras

If A is a commutative real C^*-algebra, then there exists a locally compact Hausdorff space X and a homeomorphism τ of X with $\tau^2 = id$ such that

$$A \cong C_0(X, \tau) = \{f \in C_0(C) \,|\, f(\tau(x)) = \overline{f(x)} \text{ for all } x \in X\}.$$

Finite Dimensional Real C^*-algebras

The most familiar non-trivial real structure on a C^*-algebra is probably the transpose operation on $M_n(\mathbb{C})$. In this case, $\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\}$ is just $M_n(\mathbb{R})$.
On the 2×2 matrices there is another real structure, usually denoted with a $\#$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\#} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In this case, $\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\}$ is $\mathbb{H}$. On $M_{2n}(\mathbb{C})$, we get an extension of $\#$ by $(x \otimes y)^{\#} = x^{tr} \otimes y^{\#}$.

Up to unitary equivalence, these are the only real structures on $M_q(\mathbb{C})$. On $M_q(\mathbb{C}) \oplus M_q(\mathbb{C})$ we also have $\varphi(x, y) = (y^{tr}, x^{tr})$. In this case,
$\{a \in A \otimes \mathbb{C} \,|\, \varphi(a) = a^*\} = \{(x, \bar{x}) \,|\, a \in M_q(\mathbb{C})\} \cong M_q(\mathbb{C})$.
Any finite dimensional real C^*-algebra is isomorphic to a finite direct sum of full matrix algebras, each of which is of the form $M_n(\mathbb{C})$, $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$.

Real AF Algebras

Real AF algebras were classified by Giordano using an invariant consisting of $K_0(A_\varphi)$, $K_2(A_\varphi)$,$K_4(A_\varphi)$, and an order structure on $K_0(A_\varphi) \oplus K_2(A_\varphi)$, and by Stacey using a diagram

$$K_0(A_\varphi) \to K_0(A) \to K_0(A_\varphi \otimes \mathbb{H}).$$

The range of invariant problem for this invariant has also been solved.

The Real Structure on the CAR Algebra

It was shown by Blackadar, in his paper on symmetries on the CAR algebra, that the K-theory of any real structure on the CAR algebra is completely determined by homological considerations. Stacey has since shown that up to isomorphism there is a unique real structure on the CAR algebra, so the obvious AF one is the only one. (Very different from the case of $\mathbb{Z}_2$ actions.)

Real Structures on Factors

It was shown by Størmer, and independently by Giordano and Jones, that there is a unique real structure, up to conjugacy, on the hyperfinite II_1 factor R. There is also a unique real structure on the injective II_∞ factor. (This in spite of there being two distinct real structures on $B(H)$. Notice that $R_{\mathbb{R}} \otimes \mathbb{H} \cong R_{\mathbb{R}}$.)

Purely Infinite Real C^*-algebras

Theorem (Boersema, Ruiz, Stacey)

Two real stable Kirchberg algebras A and B are isomorphic if, and only if, $K^{CRT}(A) \cong K^{CRT}(B)$. Two real unital Kirchberg algebras A and B are isomorphic if, and only if, $(K^{CRT}(A), [1_A]) \cong (K^{CRT}(B), [1_B])$.

Real Structures on the Jiang-Su Algebra

Theorem (P. J. Stacey)

There is a real structure ρ on the Jiang-Su algebra Z such that $K^{CRT}(Z_\rho) \cong K^{CRT}(\mathbb{R})$, and $Z_\rho \otimes Z_\rho \cong Z_\rho$.

It is not known if the real structure with these properties is unique.

Real Interval Algebras

There are the following five basic real forms for interval algebras:

$$A(n, \mathbb{R}) = \{f \in C([0,1], M_n(\mathbb{C})) \mid f(1) \in M_n(\mathbb{R})\}$$

$$A(n, \mathbb{H}) = \{f \in C([0,1], M_{2n}(\mathbb{C})) \mid f(1) \in M_n(\mathbb{H})\}$$

$$M_n(C_{\mathbb{F}}[0,1]) = M_n(\{f : [0,1] \to \mathbb{F} \mid f \text{ is continuous}\})$$

for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Simple Real AI algebras

Theorem (P. J. Stacey)

*Let A and B be two unital real C^*-algebras each arising as an inductive limit of finite direct sums of real interval algebras. Suppose there exist isomorphisms $\phi_T : T(B \otimes_{\mathbb{R}} \mathbb{C}) \to T(A \otimes_{\mathbb{R}} \mathbb{C})$ and $(\phi_K^1, \phi_K^2, \phi_K^3)$ of $(K_0(A), [1]) \to (K_0(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (K_0(A \otimes_{\mathbb{R}} \mathbb{H}), [1])$ with $(K_0(B), [1]) \to (K_0(B \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (K_0(B \otimes_{\mathbb{R}} \mathbb{H}), [1])$ such that ϕ_T is compatible with ϕ_K^2 in the usual way. Then there exists a *-isomorphism $\varphi : A \to B$ giving rise to these maps on the invariant.*

Cuntz Equivalence

Definition

Let A be a C^*-algebra, either real or complex, and let a, b be positive elements of A. We say that a is Cuntz sub-equivalent to b, and write $a \precsim b$ if there exists a sequence $d_n \in A$ such that $d_n b d_n^* \to a$. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Then $\sim$ is an equivalence relation on the set of positive elements of A, called Cuntz equivalence.

The Cuntz Semigroup

Definition

Let A be a separable C^*-algebra, either real or complex. Let $Cu(A)$ denote the set of Cuntz equivalence classes of positive elements of $A \otimes_{\mathbb{R}} K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ is the real C^*-algebra of compact operators on a separable real Hilbert space. Fix an isomorphism of $K_{\mathbb{R}}$ with $M_2(K_{\mathbb{R}})$, and define addition on $Cu(A)$ by $[a] + [b] = [\begin{pmatrix} a & 0 \\ o & b \end{pmatrix}]$ (this does not depend on the choice of isomorphism). Define a partial order on $Cu(A)$ by $[a] \leq [b]$ if, and only if, $a \preccurlyeq b$ (this does not depend on choice of representatives). With these definitions, $Cu(A)$ becomes a partially ordered abelian semigroup with neutral element.

An Invariant for Nonsimple Real AI algebras

Given a unital real C^*-algebra A, our invariant, denoted $Inv(A)$, consists of the triple $(Cu(A), [1]) \to (Cu(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) \to (Cu(A \otimes_{\mathbb{R}} \mathbb{H}), [1])$ of Cuntz semigroups with distinguished elements, where the connecting maps are induced by the inclusions. A morphism of invariants $\eta : Inv(A) \to Inv(B)$ consists of a triple (η_r, η_c, η_h) of unital homomorphisms of ordered abelian partial semigroups preserving suprema of increasing sequences, zero elements, and compact containment such that the following diagram commutes:

$$\begin{array}{ccccc} (Cu(A), [1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{C}), [1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{H}), [1]) \\ \downarrow \eta_r & & \downarrow \eta_c & & \downarrow \eta_h \\ (Cu(B), [1]) & \longrightarrow & (Cu(B \otimes_{\mathbb{R}} \mathbb{C}), [1]) & \longrightarrow & (Cu(B \otimes_{\mathbb{R}} \mathbb{H}), [1]). \end{array}$$

Existence for Interval Algebras

Theorem (A.D. and L.S.)

*Let A be a real interval algebra and let B be a unital real AI algebra. Then if η is a morphism of invariants from $Inv(A)$ to $Inv(B)$, there exists a unital *-homomorphism $\varphi : A \to B$ such that $\eta = Inv(\varphi)$.*

Uniqueness for Real Interval Algebras

Theorem (A.D. and L.S.)

Let A be a real interval algebra and let B be a real AI algebra. If $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv(\varphi) = Inv(\psi)$, then φ and ψ are approximately unitarily equivalent (via unitaries in the real C^*-algebra B).

Classification of Real AI Algebras

Theorem (A.D. and L.S.)

Let A and B be unital real AI algebras. Then if $(\eta_r, \eta_c, \eta_h) : Inv(A) \to Inv(B)$ is a morphism of invariants, there exists a unital $*$-homomorphism $\varphi : A \to B$ such that $Cu(\varphi) = \eta_r$, $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = \eta_c$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = \eta_h$. Moreover, if $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv(\varphi) = Inv(\psi)$, then φ and ψ are approximately unitarily equivalent.

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