Strongly Peaking Representations and Compressions of Operator Systems

Kenneth R. Davidson

University of Waterloo

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joint work with Ben Passer.

Problem.

Find complete unitary invariants for an d-tuple of operators.

This is overly ambitious in general.
For commuting normal operator d-tuples: joint spectrum, spectral measure, multiplicity.

The following invariant is due to Arveson (1970) for $d = 1$.

Definition.

If $T = (T_1, \dots, T_d) \in \mathcal{B}(H)^d$, the matrix range of T is

$$\mathcal{W}(T) = \bigcup_{n \geq 1} \mathcal{W}_n(T)$$

where for $n \geq 1$,

$$\mathcal{W}_n(T) = \{\varphi(T) : \varphi : \mathcal{B}(H) \to \mathcal{M}_n \text{ is u.c.p.}\}.$$

A prototype for the type of result we want is

Theorem (Arveson, 1970).

Suppose that K and L are irreducible compact operators. Then

$$K \simeq L \iff \mathcal{W}(K) = \mathcal{W}(L).$$

Definition.

A d-tuple $T \in \mathcal{B}(H)^d$ is minimal if whenever $M \subset H$ is a proper reducing subspace for T, then $\mathcal{W}(T|_M) \neq \mathcal{W}(T)$.

Theorem (DDSS, 2017).

Let $K, L \in \mathcal{B}(H)^d$ be two minimal non-singular d-tuples of compact operators. Then

$$K \simeq L \iff \mathcal{W}(K) = \mathcal{W}(L).$$

Operator system: a unital s.a. subspace $\mathcal{S}$ of a C*-algebra.
If $T \in \mathcal{B}(H)^d$, let $\mathcal{S}_T = \text{span}\{I, T_1, \dots, T_d, T_1^*, \dots, T_d^*\}$.
If $j : \mathcal{S} \to \mathcal{B}(H)$ is u.c.p., then $C^*(j\mathcal{S})$ is a C*-cover of $\mathcal{S}$.

Theorem (Hamana, 1979).

There is a unique minimal C-cover, the C*-envelope* $C_e^*(\mathcal{S})$.

Definition.

An operator system $\mathcal{S} \subset \mathcal{B}(H)$ is fully compressed if for any proper subspace $G \subset H$, the map $\mathcal{S} \ni s \to s|_G$ is not completely isometric.

Theorem (Passer-Shalit, 2019).

If $K \in \mathcal{K}(H)^d$ is a compact d-tuple, TFAE

1. K is minimal and non-singular
2. $\mathcal{S}_K$ is fully compressed
3. K is multiplicity free and $C^*(K) = C_e^*(\mathcal{S}_K)$.

Definition (Arveson, 1969).

$\mathcal{S}$ op. system, $A = C^*(\mathcal{S})$, $\pi \in \mathrm{Irrep}(A)$ is boundary representation if $\pi|_{\mathcal{S}}$ has a unique u.c.p. extension to $C^*(A)$. (i.e. π has u.e.p.)

The set $\partial\mathcal{S}$ of all boundary reps is the Choquet boundary of $\mathcal{S}$.
$\pi \in \partial\mathcal{S}$ must factor through $C^*_e(\mathcal{S})$.

Theorem (Arveson, 2008; D-Kennedy, 2015).

$\mathcal{S}$ op. system. Then $\partial\mathcal{S}$ completely norms $\mathcal{S}$. So

$$C^*\big(\big(\bigoplus_{\partial\mathcal{S}} \pi\big)(\mathcal{S})\big) = C^*_e(\mathcal{S}).$$

Definition (Arveson, 2011).

$\mathcal{S}$ op. system, $A = C^*(\mathcal{S})$, $\pi \in \mathrm{Irrep}(A)$ is *strongly peaking* *if* $\exists S \in M_n(\mathcal{S})$, $\|\pi(S)\| > \sup_{\sigma\in\mathrm{Irrep}(A),\ \sigma\not\simeq\pi} \|\sigma(S)\|$.

Strongly peaking reps are isolated points in $\hat{A}$, and belong to $\partial\mathcal{S}$.
Strongly peaking reps are GCR, $\pi(A) \supset \mathcal{K}(H_\pi)$.
Non-GCR reps are not fully compressed.

Theorem 1.

Let $\mathcal{S}$ be a separable op. system, $A = C^*(\mathcal{S})$. TFAE

1. $\mathcal{S}$ is fully compressed.
2. $\mathrm{id}_A \simeq \bigoplus_{\pi_i\in\Omega} \pi$, $\Omega \subset \partial\mathcal{S}$, $\pi_i \not\simeq \pi_j$, $\pi|_{A\cap\mathcal{K}(H)} \neq 0$.
3. $\mathcal{S}$ is minimal, $A = C^*_e(\mathcal{S})$, and $\mathrm{id}_A \simeq \bigoplus_{\pi\in\Xi} \pi$, where $\Xi \subset \mathrm{Irrep}(A)$, $\pi|_{A\cap\mathcal{K}(H)} \neq 0$.
4. $\mathrm{id}_A \simeq \bigoplus_{\pi\in\Lambda} \pi$, Λ strongly peaking reps without multiplicity.

A C*-algebra A is GCR if every $\pi \in \mathrm{Irrep}(A)$ is GCR.
A C*-algebra A is NGCR if no $\pi \in \mathrm{Irrep}(A)$ is GCR.
$\exists! J \lhd A$ s.t. J is GCR and A/J is NGCR.

A C*-algebra A is type I if for every $\pi \in \mathrm{Rep}(A)$, $\pi(A)''$ is type I.

Theorem (Glimm, 1961).

A is GCR $\iff$ A is type I.
A is NGCR $\iff$ A has faithful type II and type III reps.
$\iff \exists$ inequivalent families in $\mathrm{Irrep}(A)$, each sums to faithful rep.

An ideal $J \lhd A$ is essential if $J \cap I \neq \{0\}$ for all $\{0\} \neq I \lhd A$.

Theorem 2.

Let $\mathcal{S} \subset \mathcal{B}(H)$ separable op. system, $A = C^*(\mathcal{S})$. TFAE

1. $\mathcal{S}$ is fully compressed.
2. $A = C_e^*(\mathcal{S})$ and $\mathrm{id}_A \simeq \bigoplus_{\pi \in \Xi} \pi$, $\Xi \subset \mathrm{Irrep}(A)$, each $[\pi_i]$ is isolated in $\widehat{A}$.
3. $\mathcal{S}$ is minimal, $A = C_e^*(\mathcal{S})$, and $A \cap K(H)$ is essential.
4. $\mathcal{S}$ is minimal, $A = C_e^*(\mathcal{S})$, the GCR ideal J is essential.

A does not have to be GCR.

Applications

Corollary.

An operator system has a fully compressed representation $\iff$ the isolated points of $\widehat{C_e^*(\mathcal{S})}$ is dense.

Corollary.

An operator system has a fully compressed representation if $\widehat{C_e^*(\mathcal{S})}$ is countable.

Careful! The topology is usually not Hausdorff.

An operator or op. system is block diagonal if $\exists$ sequence of finite rank projections $P_n \leq P_{n+1}$ increasing to I in the commutant.

Corollary.

If two operator systems are minimal, block diagonal and order isomorphic, then the order isomorphism arises from a unitary equivalence.

Corollary.

If two d-tuples S and T are minimal, block diagonal and $\mathcal{W}(S) = \mathcal{W}(T)$, then $S \simeq T$.

The end. Thanks.