Finite-dimensionality in the non-commutative Choquet boundary

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Based on joint work with Ian Thompson.

Residual finite-dimensionality

Definition

A C*-algebra $\mathfrak{A}$ is residually finite-dimensional (RFD) if there is an isometric $*$-homomorphism $\pi : \mathfrak{A} \to \prod_\lambda \mathbb{M}_{n_\lambda}$.

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- $\mathfrak{A}$ *is RFD.*
- *The finite-dimensional irreducible* $*$*-representations are dense in the spectrum of* $\mathfrak{A}$.
- *Every* $*$*-representation of* $\mathfrak{A}$ *can be approximated pointwise in the SOT by finite-dimensional* $*$*-representations.* (Exel–Loring 1992)

The non self-adjoint world

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Let $\mathcal{A}$ be a finite-dimensional operator algebra. Then, $C^*_{max}(\mathcal{A})$ is RFD (C.–Ramsey 2018), but it can happen that $C^*_{min}(\mathcal{A})$ has simply no finite-dimensional $*$-representations.

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The choice of representation of $\mathcal{A}$ matters!

The minimal representation: the C*-envelope

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This is the smallest C*-algebra that a copy of $\mathcal{A}$ can generate (Arveson, Hamana, Muhly–Solel, Dritschel–McCullough, Davidson–Kennedy).

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Consequence: if lots of boundary representations for $\mathcal{A}$ are finite-dimensional, then $C_e^*(\mathcal{A})$ should be RFD.

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Main question

Let $\mathcal{A}$ be a unital operator algebra such that $C^*_e(\mathcal{A})$ is RFD. Does there exist a finite-dimensional boundary representation for $\mathcal{A}$?

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Inside of the spectrum $\widehat{\mathrm{C}^*_e(\mathcal{A})}$, the following two sets are dense:

$$\mathcal{B} = \{[\pi] : \pi \text{ is a boundary representation for } \mathcal{A}\}$$

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Our approach is to try to identify isolated points in $\widehat{\mathrm{C}^*_e(\mathcal{A})}$, which would then lie in $\mathcal{B} \cap \mathcal{F}$.

Non-commutative peaking behaviour

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$$\|\pi^{(n)}(T)\| > \sup_{\sigma \not\cong \pi} \|\sigma^{(n)}(T)\|.$$

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Theorem (C.–Thompson 2021)

Assume that $\mathrm{C}^*_e(\mathcal{A})$ is RFD. Then, strongly peaking representations are necessarily finite-dimensional boundary representations for $\mathcal{A}$.

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Theorem (C.–Thompson 2021)

Assume that $C_e^*(\mathcal{A})$ is RFD. Then, strongly peaking representations are necessarily finite-dimensional boundary representations for $\mathcal{A}$.

π is a locally peaking representation for $\mathcal{A}$ if there is $T \in \mathbb{M}_n(\mathcal{A})$ such that

$$\|\pi^{(n)}(T)\| > \|P_{F^{(n)}} \sigma^{(n)}(T)|_{F^{(n)}}\|$$

for every $\sigma \ncong \pi$ and every finite-dimensional subspace F.

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Theorem (C.–Thompson 2021)

Locally peaking representations for $\mathcal{A}$ are boundary representations for $\mathcal{A}$.

Detecting locally peaking representations

Theorem (Glicksberg 1962)

Let X be a compact metric space, and let $\mathcal{A} \subset C(X)$ be a unital norm-closed subalgebra which separates the points. Let $E \subset X$ be a closed subset. The following statements are equivalent.

- *There is a function $f \in \mathcal{A}$ such that $f = 1$ on E and $|f(x)| < 1$ for every $x \in X \setminus E$.*
- *Viewed as an element in $C(X)^{**}$, we have $\chi_E \in \mathcal{A}^{\perp\perp}$.*

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Theorem (Hay 2007, Read 2011, C.–Thompson 2021)

Let $\mathcal{A} \subset B(\mathcal{H})$ be a separable and norm-closed unital subalgebra. Let π be an irreducible finite-dimensional $*$-representation of $\mathrm{C}^*_e(\mathcal{A})$. If $\mathfrak{s}_\pi \in \mathcal{A}^{\perp\perp}$, then π is a locally peaking representation for $\mathcal{A}$.

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The converse fails, even for strongly peaking representations for $\mathcal{A}$.

The extremal case: C*-liminality

Recall: a unital C*-algebra is said to be liminal (or CCR) if all its irreducible $*$-representations are finite-dimensional.

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Theorem (C.–Thompson 2021)

Let $\mathcal{A}$ be a unital operator algebra. Consider the following statements.

1. *The algebra $\mathcal{A}$ is C*-liminal.*
2. *Every matrix state of $\mathcal{A}$ is locally finite-dimensional.*
3. *The algebra $\mathrm{C}^*_e(\mathcal{A})$ is RFD.*

Then, we have $(1) \Rightarrow (2) \Rightarrow (3)$. *Moreover,* $(3) \not\Rightarrow (2)$.

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A standard example of a non-liminal RFD C*-algebra is $C^*(\mathbb{F}_2)$ (Choi 1980). For $\mathcal{A} = \mathrm{Alg}(I, u, v)$, all irreducible $*$-representations are boundary representations, so that $\mathcal{A}$ is not C*-liminal.

Thank you!