Affine Operator Systems

Martin Argerami

University of Regina

June 08, 2021

People involved:

- M.A.
- Matt Kennedy
- Chadi Hamzo

A crash course in Operator Systems

- **Operator System:** $\mathcal{S} \subset B(H)$, subspace, $1 \in \mathcal{S}$, $\mathcal{S}^* = \mathcal{S}$
- $\mathcal{OS}(T_1, \ldots, T_m) = \mathrm{span}\{I, T_1, T_1^*, \ldots, T_m, T_m^*\}$
- **Morphisms:** unital completely positive maps
- **Isomorphisms:** complete isometries
- **C*-Envelope:** $C_{\mathrm{e}}^*(\mathcal{S})$ is the smallest C*-algebra that (a copy of) $\mathcal{S}$ generates

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\psi"] \arrow[dr, "\iota"'] & C^*(\psi(\mathcal{S})) \arrow[d, dashed, "\pi \text{ epimorphism}"] \\ & C_{\mathrm{e}}^*(\mathcal{S}) \end{tikzcd}$$

- $\mathcal{S}$ **is reduced** when $\mathcal{S} \subset C_{\mathrm{e}}^*(\mathcal{S})$.

Affine Maps

Let V, W be complex vector spaces, $E \subset V$ be convex. A map $T : E \to W$ is *affine* if it preserves convex combinations.
$A(E, W)$: all continuous affine maps $E \to W$. $A(E) = A(E, \mathbb{C})$.

Proposition

Let V, W be complex vector spaces, $E \subset V$ convex, and $T : E \to W$. TFSAE:

1. $T \in A(E, W)$;
2. *there exists $e_0 \in E$ and $L : V \to W$ real-linear with $Te = Te_0 + L(e - e_0)$ for all $e \in E$;*
3. *there exists a real-linear map $L : V \to W$ such that $L(e_1 - e_2) = Te_1 - Te_2$ for any $e_1, e_2 \in E$;*
4. *there exists $w \in W$ and a real-linear map $L : V \to W$ such that $T = w + L$.*

Affine and Operator Duality

$E \simeq_A F$ *affinely homeomorphic* if there exists a bijection $T \in A(E, F)$. If V is locally convex and E is compact: $A(E) \subset C(E)$ is an operator system.

Proposition (Affine Duality)

Let V, W be locally convex spaces, and $K_1 \subset V$, $K_2 \subset W$ be compact and convex. The following statements are equivalent:

1. $K_1 \simeq_A K_2$;
2. $A(K_1) \simeq_O A(K_2)$.

Analog to $X \simeq Y \iff C(X) \simeq C(Y)$.

Affine and Operator Duality (continued)

Proposition (Kadison's Representation Theorem)

Let $E \subset V$ be compact convex, with V locally compact. Then

$$E \simeq_A \mathrm{St}(A(E))$$

(homeomorphism with the weak*-topology)

Let K be compact Hausdorff and $\mathcal{S}_0 \subset C(K)$ an operator system. Then

$$A(\mathrm{St}(\mathcal{S}_0)) \simeq_O \overline{\mathcal{S}_0}$$

Commuting tuples of normal operators

$T_1, \ldots, T_m \in B(H)$ normal, $T_k T_j = T_j T_k$
Gelfand Transform: $\mathcal{A} = C^*(I, T_1, \ldots, T_m) \simeq C(\Sigma(\mathcal{A}))$

Joint Spectrum:
$\sigma(T_1, \ldots, T_m) = \{(\gamma(T_1), \ldots, \gamma(T_m)) : \gamma \in \Sigma(\mathcal{A})\} \subset \prod_{j=1}^m \sigma(T_j) \subset \mathbb{C}^m$

$$C^*(I, T_1, \ldots, T_m) \simeq C(\sigma(T_1, \ldots, T_m)).$$

Proposition (AHK)

Let $T_1, \ldots, T_m \in B(H)$ and $S_1, \ldots, S_m \in B(K)$ be tuples of mutually commuting normal operators such that

$$\sigma(S_1, \ldots, S_m) = \overline{\operatorname{Ext}\operatorname{conv}\sigma(T_1, \ldots, T_m)}.$$

Then

$$\mathcal{OS}(T_1, \ldots, T_m) \simeq_O \mathcal{OS}(S_1, \ldots, S_m).$$

Normal Operators (continued)

Theorem (AHK)

Let $T_1, \ldots, T_m \in B(H)$ be commuting normal operators such that

$$\sigma(T_1, \ldots, T_m) = \overline{\mathrm{Ext\,conv}\,\sigma(T_1, \ldots, T_m)}.$$

Then $\mathcal{OS}(T_1, \ldots, T_m)$ is reduced.

Corollary

Let $T_1, \ldots, T_m \in B(H)$ be commuting normal operators. Then

$$C_{\mathrm{e}}^*(T_1, \ldots, T_m) = C(\overline{\mathrm{Ext\,conv}\,\sigma(T_1, \ldots, T_m)}).$$

Normal Operators (continued)

Theorem (AHK)

Let $T_1, \ldots, T_m \in B(H)$, $S_1, \ldots, S_n \in B(K)$ be two families of commuting normal operators. The following statements are equivalent:

1. $\mathcal{OS}(T_1, \ldots, T_m) \simeq_O \mathcal{OS}(S_1, \ldots, S_n)$;
2. $\operatorname{conv} \sigma(T_1, \ldots, T_m) \simeq_A \operatorname{conv} \sigma(S_1, \ldots, S_n)$.

Theorem (AHK)

Let $T_1, \ldots, T_m \in B(H)$ be commuting normal operators. Then

$$\mathrm{St}(\mathcal{OS}(T_1, \ldots, T_m)) \simeq_A \operatorname{conv} \sigma(T_1, \ldots, T_m)$$

and

$$\mathcal{OS}(T_1, \ldots, T_m) \simeq_O A(\operatorname{conv} \sigma(T_1, \ldots, T_m)).$$

Affine transformations of the plane

Proposition

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$. The following statements are equivalent:

1. *T is an affine transformation (i.e. $Tx = y_0 + Lx$);*
2. *under the canonical identification $\mathbb{R}^2 \simeq \mathbb{C}$, there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $T = T_{\alpha,\beta,\gamma}$, where $T_{\alpha,\beta,\gamma} : z \longmapsto \alpha + \beta z + \gamma \bar{z}$;*

Affine transformations that preserve parts of the circle

Lemma

Let $S, R \subset \mathbb{T}$. If $|S| = 3$, then $\operatorname{conv} R \simeq_A \operatorname{conv} S$ if and only if $|R| = 3$.

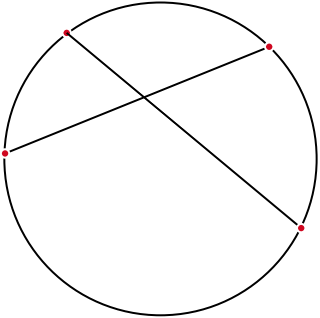
Lemma

Let $S, R \subset \mathbb{T}$. If $|S| \geq 5$, then $\operatorname{conv} R \simeq_A \operatorname{conv} S$ if and only if there exists $\xi \in [0, 2\pi)$ such that $R = e^{i\xi} S$ or $R = e^{i\xi} \overline{S}$.

Four points?

$R = \{e^{it_j}\}_{j=1}^4$, with $0 \leq t_1 < t_2 < t_3 < t_4 < 2\pi$.There exist unique $r, s \in (0, 1)$ such that

$$re^{it_1} + (1 - r)e^{it_3} = se^{it_2} + (1 - s)e^{it_4}. \tag{1}$$



Affine transformations of the circle

$$\mathrm{CI}(R) = \{r, s\}$$

(Coefficients of Intersection)

$\mathrm{CI}(R) \simeq \mathrm{CI}(S)$:

$$\big[\,\mathrm{CI}(\{r,s\})\big] = \{(r,s), (1-r,s), (r,1-s), (1-r,1-s)\}.$$

Lemma

Let $R = \{e^{ir_j}\}_{j=1}^4, S = \{e^{is_j}\}_{j=1}^4 \subset \mathbb{T}$. TFSAE:

1. $\mathrm{conv}\, R \simeq_A \mathrm{conv}\, S$;
2. $\mathrm{CI}(R) \simeq \mathrm{CI}(S)$.

A single normal operator: Unitary Operators

$U \in B(H)$. Since $\sigma(U) \subset \mathbb{T}$ and every point in $\mathbb{T}$ is extreme,

$$\sigma(U) = \operatorname{Ext}\operatorname{conv}\sigma(U).$$

So $C^*(U) = C^*_{\mathrm{e}}(\mathcal{OS}(U))$.

Theorem (AHK)

Let $U, V \in B(H)$ be unitaries. Then

1. *If $|\sigma(U)| \leq 3$, TFSAE:*
   1. $\mathcal{OS}(U) \simeq_O \mathcal{OS}(V)$;
   2. $|\sigma(V)| = |\sigma(U)|$.
2. *If $|\sigma(U)| = 4$, TFSAE:*
   1. $\mathcal{OS}(U) \simeq_O \mathcal{OS}(V)$;
   2. $|\sigma(V)| = 4$ *and* $\mathrm{CI}(\sigma(V)) = \mathrm{CI}(\sigma(U))$.
3. *If $|\sigma(U)| \geq 5$, TFSAE:*
   1. $\mathcal{OS}(U) \simeq_O \mathcal{OS}(V)$;
   2. $\exists\, \lambda \in \mathbb{T}$ *s.t.* $\sigma(V) = \lambda\,\sigma(U)$, *or* $\sigma(V) = \lambda\,\overline{\sigma(U)}$;

Examples: The class of $\mathrm{Cl}(X) = \{\frac{1}{2}, \frac{1}{2}\}$ in $\mathbb{T}$

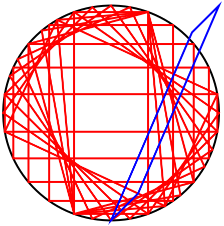
$$X \subset \mathbb{T}, \qquad \mathrm{Cl}(X) = \left\{\tfrac{1}{2}, \tfrac{1}{2}\right\} \qquad \Longleftrightarrow \qquad X = \{z, w, -z, -w\}.$$

For all choices of $z \neq w$ the four points will form a rectangle.

So

$$\text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{-it} & 0 & 0 \\ 0 & 0 & -e^{it} & 0 \\ 0 & 0 & 0 & -e^{-it} \end{bmatrix}, \begin{bmatrix} e^{-it} & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & -e^{-it} & 0 \\ 0 & 0 & 0 & -e^{it} \end{bmatrix} \right\}$$

are all isomorphic operator systems. (others, too)



$$\boxed{T = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

Examples: operator systems from unitaries

On the other hand, the unitaries

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

generate non-isomorphic operator systems:

$$\mathrm{CI}(V) = \left\{\tfrac{1}{2}, \tfrac{1}{2}\right\} \not\simeq \left\{\tfrac{1}{\sqrt{2}}, \scriptstyle 2-\sqrt{2}\right\} = \mathrm{CI}(W).$$

Examples: Commuting Normal Operators

Example

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$C^*(T_1) = C^*(T_1, T_2) = \mathbb{C}^5.$$

$$\sigma(T_1, T_2) = \{(1,0), (2,1), (3,i), (4,-i), (5,2)\} \subset \mathbb{C}^2.$$

$$S_1 = \begin{bmatrix} 1+i & 0 & 0 & 0 & 0 \\ 0 & 2+2i & 0 & 0 & 0 \\ 0 & 0 & 5+3i & 0 & 0 \\ 0 & 0 & 0 & 2+4i & 0 \\ 0 & 0 & 0 & 0 & 5+5i \end{bmatrix}, \; S_2 = \begin{bmatrix} 5+3i & 0 & 0 & 0 & 0 \\ 0 & 10+10i & 0 & 0 & 0 \\ 0 & 0 & 14+9i & 0 & 0 \\ 0 & 0 & 0 & 21+12i & 0 \\ 0 & 0 & 0 & 0 & 25+23i \end{bmatrix},$$

then $\mathcal{OS}(S_1, S_1) \simeq_O \mathcal{OS}(T_1, T_2)$.

Commuting Normal Operators (continued)

Example (continued)

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 5 & 0 & 0 & -1 \\ 3 & 0 & 4 & 0 \end{bmatrix},$$

$$A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 + i\overline{z_1} - iz_2 + i\overline{z_2} \\ \left(\frac{5+3i}{2}\right) z_1 + \left(\frac{5+3i}{2}\right) \overline{z_1} + \frac{5i}{2}\, z_2 + \frac{3i}{2}\, \overline{z_2} \end{bmatrix}$$

$\{(1,0),(2,1),(3,i),(4,-i),(5,2)\} \xmapsto{A}$

$\{(1+i, 5+3i),(2+2i, 10+10i),(5+3i, 14+9i),(2+4i, 21+12i),(5+5i, 25+23i)\}$

Example (continued)

Example (continued)

$$\operatorname{conv} \sigma(T_1, T_2) \simeq_A \operatorname{conv} \sigma(S_1, S_2).$$

$$\mathcal{OS}(T_1, T_2) \simeq_O \mathcal{OS}(S_1, S_2).$$

Explicitly:

$$\psi^{-1}(S_1) = (1+i)T_1 + 2\operatorname{Im} T_2, \qquad \psi^{-1}(S_2) = (5+3i)T_1 + 3i\operatorname{Re} T_2 + iT_2.$$

Example (joint spectrum in the bidisk)

$f, g \in C(\overline{\mathbb{D}})$: $f(z) = z$, $g(z) = |z|$. Then

$$\sigma(f, g) = \{(z, |z|) : \ z \in \overline{\mathbb{D}}\}.$$

$$\operatorname{Ext}\operatorname{conv}\sigma(f, g) = \{(\lambda, 1) : \ \lambda \in \mathbb{T}\} \cup \{(0, 0)\}.$$

Thus

$$C^*_{\mathrm{e}}(\mathcal{OS}(f, g)) = C(\{(\lambda, 1) : \ \lambda \in \mathbb{T}\} \cup \{(0, 0)\}) \simeq C(\mathbb{T}) \oplus \mathbb{C}.$$

and

$$\mathcal{OS}(f, g) \simeq_O \mathcal{OS}((h, 0), (1, 0)) \subset C(\mathbb{T}) \oplus \mathbb{C}, \qquad h(z) = z.$$

Thank you!