# Classification of embeddings, II

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# Theorem (CGSTW)

Let:

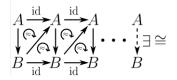
- *A* be a unital, sep., nuclear C\*-algebra satisfying the UCT,
- *B* be a unital, sep., simple, nuclear, finite *Z*-stable C\*-algebra,
- *D* be either *B* or  $B_{\infty} := l^{\infty}(\mathbb{N}, B)/c_0(\mathbb{N}, B)$ .

Then for any faithful morphism  $\alpha : \underline{K}T_u(A) \to \underline{K}T_u(D), \exists$  a unital \*-hom.  $\phi : A \to D$  inducing  $\alpha$ . Moreover,  $\phi$  is unique up to approximate unitary equivalence.

Note. When  $D = B_{\infty}$ , this could be called "approximate classification" (or "approx. existence" and "approx. uniqueness").

- Intertwining.
- The trace-kernel extension.
- Classification into  $B^{\infty}$ .
- Cuntz pairs.

Two-sided:



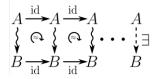
#### Theorem

If there are \*-homomorphisms  $\phi : A \to B$ ,  $\psi : B \to A$  such that  $\phi \circ \psi$  and  $\psi \circ \phi$  are approximately inner, then  $A \cong B$ .

#### Consequence

Classifying embeddings  $\implies$  classifying algebras.

One-sided:



#### Theorem

If there is a sequence of approx. multiplicative \*-linear maps  $A \rightarrow B$  that are "approx. unitarily equivalent", then they induce a \*-homomorphism  $A \rightarrow B$ .

#### Consequence

Classifying embeddings to  $B_{\infty} \Longrightarrow$  classifying embeddings to B.

Let *B* be a unital C\*-algebra with nonempty set of traces T(B).

### Definition

$$\begin{split} B_{\infty} &:= l^{\infty}(\mathbb{N}, B)/c_0(\mathbb{N}, B), & \text{the "sequence algebra",} \\ J &:= \{(x_n) \in B_{\infty} : \lim_{n \to \infty} \sup_{\tau \in T(B)} \tau(b_n^* b_n) = 0\}, \text{ the "trace-kernel",} \\ 0 \to J_B \to B_{\infty} \to B^{\infty} \to 0, & \text{the "trace-kernel extension".} \end{split}$$

If *B* has unique trace and we use  $\lim_{n\to\omega}$  (an ultrafilter) instead, then we get  $B^{\omega} \cong [\pi_{\tau}(B)'']^{\omega}$ , a von Neumann algebra.

If *B* is  $\mathbb{Z}$ -stable then  $B_{\infty}$  is morally (or more technically, "separably")  $\mathbb{Z}$ -stable, and  $J_B$  is morally (separably) stable.

## Theorem (Classification into $B^{\infty}$ )

Let:

- *A* be a unital, sep., nuclear C\*-algebra satisfying the UCT,
- *B* be a unital, sep., simple, nuclear, finite *Z*-stable C\*-algebra with at least one trace.

Then for any continuous affine map  $\alpha : T(B^{\infty}) \to T(A)$ , there is a \*-homomorphism  $\phi : A \to B^{\infty}$  inducing  $\alpha$ . Moreover,  $\phi$  is unique up to unitary equivalence.

Ideas: local-to-global transfer ("CPoU"), and Connes' Theorem.

Upshot: classifying into  $B_{\infty}$  now becomes a problem of classifying lifts.



# Cuntz pairs

Going from classification into  $B^{\infty}$  to classification into  $B_{\infty}$  involves *KK*-theory. Why?

#### Definition

Let *A*, *C* be C\*-algebras with *C* stable. An (*A*, *C*)-*Cuntz pair* is a pair of \*-homomorphisms  $\phi, \psi : A \to E$ , where  $E \triangleright C$  such that

 $\phi(a) \equiv \psi(a) \mod C, \quad a \in A.$ 

KK(A, C) consists of homotopy classes of Cuntz pairs  $A \rightarrow C$ .

If  $\phi$ ,  $\psi$  :  $A \rightarrow B_{\infty}$  agree on traces, then by taking a unitary conjugation, they can be made to agree mod  $J_B$ —and thus form a Cuntz pair!

(Ignoring some technicalities around separability.)

### Theorem (KK-uniqueness) (CGSTW)

Let:

- *A* be a unital, sep., nuclear C\*-algebra satisfying the UCT,
- *B* be a unital, sep., simple, nuclear, finite *Z*-stable C\*-algebra.
- $\phi, \psi : A \to B_{\infty}$  form an  $(A, J_B)$ -Cuntz pair.

Then  $\phi$ ,  $\psi$  are unitarily equivalent iff  $[\phi, \psi] = 0$  in  $KK(A, J_B)$ .