Classification of embeddings, II

Aaron Tikuisis

University of Ottawa

Joint with José Carrión, Jamie Gabe, Chris Schafhauser, and Stuart White.

Classification of embeddings

Theorem (CGSTW)

Let:

- A be a unital, sep., nuclear C*-algebra satisfying the UCT,
- B be a unital, sep., simple, nuclear, finite $\mathcal{Z}$-stable C*-algebra,
- D be either B or $B_\infty := l^\infty(\mathbb{N}, B)/c_0(\mathbb{N}, B)$.

Then for any faithful morphism $\alpha : \underline{K}T_u(A) \to \underline{K}T_u(D)$, $\exists$ a unital *-hom. $\phi : A \to D$ inducing α. Moreover, ϕ is unique up to approximate unitary equivalence.

Note. When $D = B_\infty$, this could be called "approximate classification" (or "approx. existence" and "approx. uniqueness").

Outline

- Intertwining.
- The trace-kernel extension.
- Classification into B^∞.
- Cuntz pairs.

Intertwining

Two-sided:

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[d] & \cdots & A \arrow[d, dashed, "\exists \cong"] \\
B \arrow[r, "\mathrm{id}"'] \arrow[ur] & B \arrow[r, "\mathrm{id}"'] \arrow[ur] & B & & B
\end{tikzcd}
$$

Theorem

If there are *-homomorphisms $\phi : A \to B$, $\psi : B \to A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are approximately inner, then $A \cong B$.

Consequence

Classifying embeddings $\implies$ classifying algebras.

Intertwining

One-sided:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, rightsquigarrow] \arrow[dr, phantom, "\circlearrowright\approx"] & A \arrow[r, "\mathrm{id}"] \arrow[d, rightsquigarrow] \arrow[dr, phantom, "\circlearrowright\approx"] & A \arrow[d, rightsquigarrow] & \cdots & A \arrow[d, dashed, "\exists"] \\
B \arrow[r, "\mathrm{id}"'] & B \arrow[r, "\mathrm{id}"'] & B & & B
\end{tikzcd}$$

Theorem

If there is a sequence of approx. multiplicative *-linear maps $A \to B$ that are "approx. unitarily equivalent", then they induce a *-homomorphism $A \to B$.

Consequence

Classifying embeddings to $B_\infty \Longrightarrow$ classifying embeddings to B.

The trace-kernel extension

Let B be a unital C*-algebra with nonempty set of traces $T(B)$.

Definition

$$B_\infty := l^\infty(\mathbb{N}, B)/c_0(\mathbb{N}, B), \qquad \text{the "sequence algebra"},$$

$$J := \{(x_n) \in B_\infty : \lim_{n\to\infty} \sup_{\tau\in T(B)} \tau(b_n^* b_n) = 0\}, \text{ the "trace-kernel"},$$

$$0 \to J_B \to B_\infty \to B^\infty \to 0, \qquad \text{the "trace-kernel extension"}.$$

If B has unique trace and we use $\lim_{n\to\omega}$ (an ultrafilter) instead, then we get $B^\omega \cong [\pi_\tau(B)'']^\omega$, a von Neumann algebra.

If B is $\mathcal{Z}$-stable then B_∞ is morally (or more technically, "separably") $\mathcal{Z}$-stable, and J_B is morally (separably) stable.

Classification into B^∞

Theorem (Classification into B^∞)

Let:

- A be a ~~unital,~~ sep., nuclear C*-algebra ~~satisfying the UCT~~,
- B be a unital, sep., ~~simple, nuclear,~~ finite $\mathcal{Z}$-stable C*-algebra with at least one trace.

Then for any continuous affine map $\alpha : T(B^\infty) \to T(A)$, there is a *-homomorphism $\phi : A \to B^\infty$ inducing α. Moreover, ϕ is unique up to unitary equivalence.

Ideas: local-to-global transfer ("CPoU"), and Connes' Theorem.

Upshot: classifying into B_∞ now becomes a problem of classifying lifts.

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[dr] & B_\infty \arrow[d, two heads] \\ & B^\infty \end{tikzcd}$$

Cuntz pairs

Going from classification into B^∞ to classification into B_∞ involves KK-theory. Why?

Definition

Let A, C be C*-algebras with C stable. An (A, C)-*Cuntz pair* is a pair of *-homomorphisms $\phi, \psi : A \to E$, where $E \rhd C$ such that

$$\phi(a) \equiv \psi(a) \mod C, \quad a \in A.$$

$KK(A, C)$ consists of homotopy classes of Cuntz pairs $A \to C$.

If $\phi, \psi : A \to B_\infty$ agree on traces, then by taking a unitary conjugation, they can be made to agree mod J_B—and thus form a Cuntz pair!

(Ignoring some technicalities around separability.)

Cuntz pairs

Theorem (KK-uniqueness) (CGSTW)

Let:

- A be a unital, sep., nuclear C*-algebra ~~satisfying the UCT~~,
- B be a unital, sep., simple, nuclear, finite $\mathcal{Z}$-stable C*-algebra.
- $\phi, \psi : A \to B_\infty$ form an (A, J_B)-Cuntz pair.

Then ϕ, ψ are unitarily equivalent iff $[\phi, \psi] = 0$ in $KK(A, J_B)$.