Quasidiagonality and the classification of nuclear C*-algebras

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$$M_8 = \left(\begin{array}{cc|cc|cc|cc}
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\
\hline
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\
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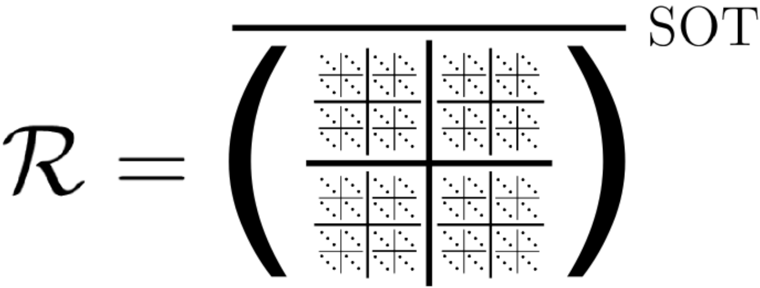

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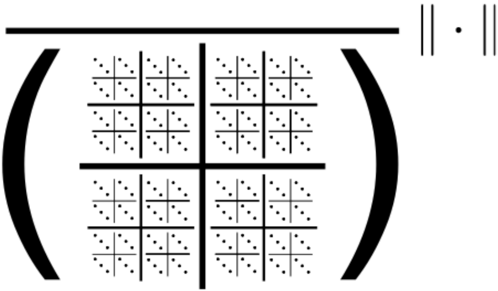

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For the von Neumann algebraic construction:

Theorem (Murray–von Neumann)

For any $p, q \in \mathbb{N} \setminus \{1\}$,

$$\overline{\bigcup M_{p^k}}^{\mathrm{SOT}} = \overline{\bigcup M_{q^k}}^{\mathrm{SOT}}.$$

For the C*-algebraic construction:

Proposition

If $p, q \in \mathbb{N} \setminus \{1\}$ are coprime then

$$M_{p^\infty} \not\cong M_{q^\infty}.$$

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The *Elliott invariant* is ordered topological K-theory paired with traces:

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)_+, [1_A]_{K_0(A)}, K_1(A), T(A), \langle \cdot, \cdot \rangle : K_0(A) \times T(A) \to \mathbb{R}).$$

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If A, B are simple, separable, nuclear, unital C*-algebras then

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Second example

Let $\theta \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ be an irrational angle. Define A_θ to be the universal C*-algebra generated by two unitaries u, v such that

$$vu = e^{2\pi i\theta} uv.$$

(An *irrational rotation algebra*.)

This is one of the most tractable (yet interesting) examples of a crossed product; namely,

$$A_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z},$$

where $\alpha : \mathbb{T} \to \mathbb{T}$ is rotation by θ.

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Elliott–Evans showed A_θ is A$\mathbb{T}$, i.e., an inductive limit of C*-algebras of the form

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Let $\alpha : X \to X$ and $\beta : Y \to Y$ be minimal homeomorphisms.
The Elliott conjecture predicts:

• $C(X) \rtimes_\alpha \mathbb{Z} \cong C(Y) \rtimes_\beta \mathbb{Z}$ if and only if the two algebras have the same Elliott invariant. (This invariant is computable from the dynamical data, using e.g., the Pimsner–Voiculescu exact sequence.)

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Every "reasonable value" of the Elliott invariant, with nonempty trace simplex, is realized by an inductive limit of subhomogeneous C*-algebras with topological dimension ≤ 2.

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Villadsen, Rørdam, Toms: There are counterexamples to the Elliott conjecture.

Giol–Kerr: There are even counterexamples of the form $C(X) \rtimes_\alpha \mathbb{Z}$.

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If $\mathcal{C}$ is a class of simple stably finite C*-algebras classified by $\mathrm{Ell}(\cdot)$, and $\mathcal{C}$ contains the algebra in the above theorem then:

- Every C*-algebra in $\mathcal{C}$ has finite nuclear dimension (a concept marrying Lebesgue covering dimension with Lance's completely positive approximation property). This is the restriction violated by the known counterexamples.

- Every C*-algebra in $\mathcal{C}$ is quasidiagonal. (Every trace on every C*-algebra in $\mathcal{C}$ is quasidiagonal.)

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If $\mathcal{C}$ is classified by $\mathrm{Ell}(\cdot)$ then:

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Question

Can the latter two restrictions be violated, by simple, separable, nuclear, unital C*-algebras?

Elliott conjecture: revised

Revised Elliott Conjecture

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if and only if

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Quasidiagonality

A separable C*-algebra A is *quasidiagonal* if there exists a c.p.c. map

$$A \to \ell_\infty(\mathbb{N}, \mathcal{Q})$$

which induces an injective *-homomorphism

$$A \to \mathcal{Q}_\omega := \ell_\infty(\mathbb{N}, \mathcal{Q}) / \{(x_n) | \lim_{n \to \omega} \|x_n\| = 0\},$$

where ω is a free ultrafilter.

In case A is nuclear, A is quasidiagonal iff it embeds into

$$\mathcal{Q}_\omega.$$

(Cf. Connes's embedding problem.)

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(i) If A is quasidiagonal and unital then it has a quasidiagonal trace.
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Rosenberg ('87): If G is a discrete group and $C^*_r(G)$ is quasidiagonal then G is amenable.

The converse ("Rosenberg's conjecture") would be a consequence of the revised Elliott conjecture, since if G is amenable then $C^*_r(G)$ embeds into a simple, separable, nuclear, unital C*-algebra of finite nuclear dimension, namely

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Classification: results

Theorem (Gong–Lin–Niu, Elliott–Gong–Lin–Niu)

Let A, B be simple, separable, nuclear, unital C*-algebras, such that:

(a) A, B have finite nuclear dimension,

(b) every trace on A and on B is quasidiagonal, and

(c) A, B satisfy the Universal Coefficient Theorem.

If $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ then $A \cong B$.

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