The Category Cu. Which maps are the correct ones? the *-homomorphisms or cpc order zero maps?

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14th March, 2014

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The order in $\mathrm{W}(A)$ is usually not the algebraic order.

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Properties

- $\mathrm{Cu}(A)$ *belongs to a category of semigroups called* Cu *that admits inductive limits that are not algebraic.*
- *The assignment $A \mapsto \mathrm{Cu}(A)$ is sequentially continuous.*

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Definition

Let a, b be elements in a partially ordered set S. Then, we will say that $a \ll b$ (way-below) if for any increasing sequence $\{y_n\}$ with supremum in S such that $b \leq \sup(y_n)$, there exists m such that $a \leq y_m$.

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Definition (Cu)

An object of Cu is a partially ordered semigroup with zero element S such that:

- The order, in S, is compatible with the addition, i.e., if $x_i \leq y_i$, $i \in \{1, 2\}$ then $x_1 + x_2 \leq y_1 + y_2$,
- every increasing sequence in S has a supremum,
- for all $x \in S$ there exists a sequence $\{x_n\}$ such that $x = \sup(x_n)$ where $x_n \ll x_{n+1}$,
- the relation $\ll$ and suprema are compatible with addition.

The maps of Cu are those morphisms which preserve the order, the zero, the suprema of increasing sequences and the relation $\ll$.

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In fact, $\langle (a-\varepsilon)_+ \rangle \ll \langle a \rangle$ *in* $\mathrm{Cu}(A)$ *for all* $\varepsilon > 0$ *and for all* $a \in A_+$.

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Example

- *Let* X *be a compact metric space. Then, if* $\mathcal{O}(X)$ *is the set of open sets in* X *ordered by inclusion, it follows that* $\mathcal{O}(X) \in \mathrm{Cu}$. *In this, we have that* $U \ll V$ *for* $U, V \in \mathcal{O}(X)$, *if there exists a compact subset* $K \subseteq X$ *such that* $U \subseteq K \subseteq V$.

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- *Let* X *be a finite-dimensional compact metric space, then* $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \in \mathrm{Cu}$, *where* $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

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Remark

Not all the maps between semigroups preserve $\ll$, *usually maps between two semigroups just preserve* $(+, \leq, \sup)$.

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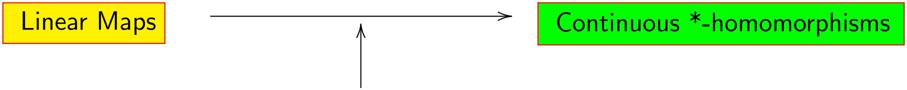

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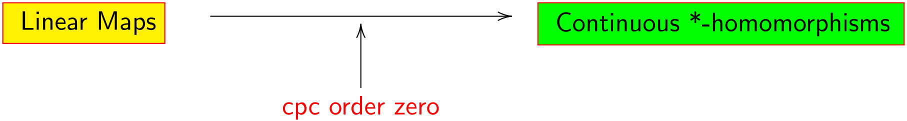

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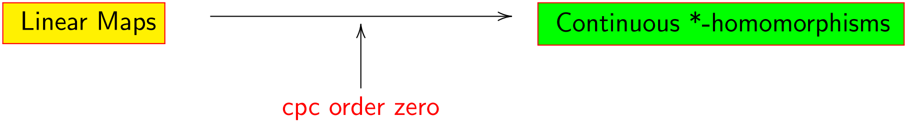

Definition

- A map $\varphi : A \to B$ is positive if $\forall a \geq 0 \implies \varphi(a) \geq 0$, and it is completely positive (c.p.) if $\varphi^n : M_n(A) \to M_n(B)$ is positive.
- A c.p. map $\varphi : A \to B$ is order zero if for $a, b \in A^+$ such that $ab = 0 \implies \varphi(a)\varphi(b) = 0$.

Theorem (Winter-Zacharias '09)

Let A, B be C-algebras and $\varphi : A \to B$ a cpc order zero map and set $C = C^*(\varphi(A)) \subseteq B$. Then, there exists*

- $h_\varphi \in \mathcal{M}(C) \cap C'$ *a positive element*
- *a *-homomorphism* $\pi_\varphi : A \to \mathcal{M}(C) \cap \{h\}'$

such that

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With the same notation:

- *(Functional calculus on cpc$_\perp$)If* $f \in C_0((0,1])$, *then* $f(\varphi) : A \to B$ *given by* $f(\varphi)(a) = f(h_\varphi)\pi_\varphi(a)$ *is a well-defined c.p. order zero map.*

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Proposition

Let A, B be C-algebras. Then every cpc order zero map $\varphi : A \to B$ naturally induces a map $\mathrm{Cu}(\varphi) : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ which preserves addition, order, the zero element and the suprema of increasing sequences, but, in general, not the way-below.*

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(Possible answer) Study a bivariant verion of Cuntz Semigroup (as done by *KK*-theory)

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It follows that it is an abelian semigroup