C*-algebras, classification, and regularity

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C*-algebras

The canonical C*-algebra is

$$\mathcal{B}(\mathcal{H}) := \{\text{continuous linear operators } \mathcal{H} \to \mathcal{H}\},$$

where $\mathcal{H}$ is a (complex) Hilbert space.

$\mathcal{B}(\mathcal{H})$ is a (complex) algebra: multiplication = composition of operators.

Operators are continuous if and only if they are bounded $\rightsquigarrow$ (operator) norm on $\mathcal{B}(\mathcal{H})$.

Every operator T has an adjoint T^* $\rightsquigarrow$ involution on $\mathcal{B}(\mathcal{H})$.

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C*-algebras: examples

Example

$M_n(\mathbb{C})$ = $n \times n$ matrices with complex entries.

This is $\mathcal{B}(\mathcal{H})$ where $\mathcal{H} = \mathbb{C}^n$.

Multiplication = matrix multiplication.

Adjoint = conjugate transpose.

$\|A\| =$ operator norm = (largest eigenvalue of $A^*A)^{1/2}$.

Every finite dimensional C*-algebra is a direct sum of matrix algebras.

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This can be identified with "diagonal" operators on $\ell^2(X)$.

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Every commutative unital C*-algebra is $C(X, \mathbb{C})$ for a unique compact Hausdorff space X.

In fact, $X \mapsto C(X, \mathbb{C})$ is an equivalence of categories.

C*-algebras are considered noncommutative topological spaces.

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C*-algebras: examples

Interesting C*-algebras can be constructed from groups, dynamical systems, directed graphs, rings, coarse metric spaces,...

Question

What do properties of a C*-algebra tell us about the object from which it is constructed?

C*-properties: amenability of a group; exactness of a group,...

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UHF algebras

Consider the following C*-algebra:

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$$M_2(\mathbb{C})$$

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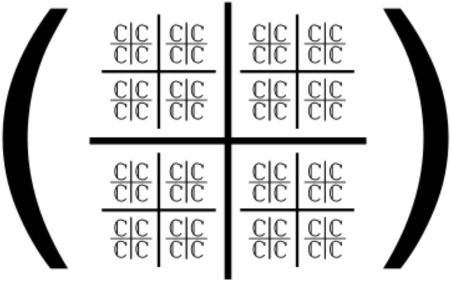

$M_8(\mathbb{C})$

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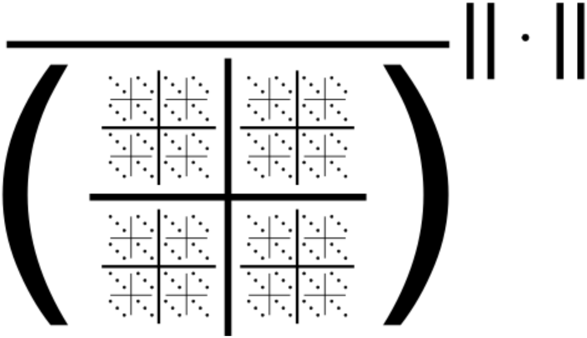

$$M_{2^\infty} = \overline{M_2(\mathbb{C})^{\otimes\infty}},$$

a **uniformly hyperfinite** algebra.

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A noncommutative Cantor set construction.

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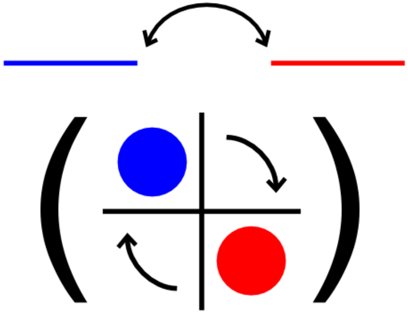

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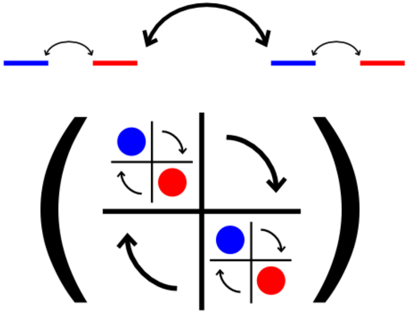

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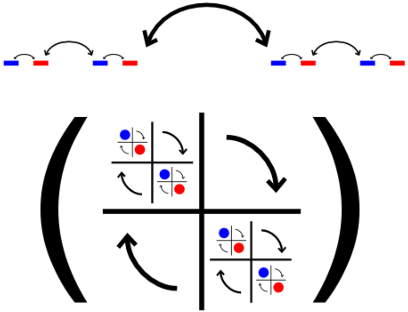

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A noncommutative Cantor set construction.

Can likewise define M_{p^∞} for any $p \in \mathbb{N}$.

Classification

Question

Is $M_{2^\infty} \cong M_{3^\infty}$?

If we suspect that two C*-algebras are not isomorphic, how do we go about proving it?

When do we stop trying to prove they are non-isomorphic?

Conjecture (Elliott, $\sim$1990)

If A, B are separable, simple, amenable C*-algebras then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, where $\mathrm{Ell}(A)$ is K-theory paired with traces (the **Elliott invariant**).

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Separable: $\exists$ dense sequence.

Simple: no nontrivial closed, two-sided ideals.

Amenable: many equivalent definitions, including a finite dimensional approximation property, akin to noncommutative partitions of unity.

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K-theory: arose from topological K-theory, a cohomology theory founded in vector bundles.

"Computable" (exact sequences, Künneth formula, ...)

Eg. $K_0(M_{p^\infty}) = \mathbb{Z}[\frac{1}{p}]$

Traces: a trace on A is a positive linear functional $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all a, b.

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Theorem (Villadsen, Rørdam, Toms, ∼2000)

There exist separable, simple, nuclear, unital C*-algebras A, B such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ but $A \not\cong B$.

Why? A seems to have “high topological dimension” and B has “low topological dimension”.

In fact, $B = A \otimes M_{2^\infty}$.

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If A, B are separable, simple, amenable C*-algebras then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, where $\mathrm{Ell}(A)$ is K-theory paired with traces (the **Elliott invariant**).

Theorem (Villadsen, Rørdam, Toms, $\sim$2000)

There exist separable, simple, nuclear, unital C*-algebras A, B such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ but $A \ncong B$.

Why? A seems to have "high topological dimension" and B has "low topological dimension".

In fact, $B = A \otimes M_{2^\infty}$.

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Theorem (Villadsen, Rørdam, Toms, $\sim$2000)

There exist separable, simple, nuclear, unital C*-algebras A, B such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ but $A \not\cong B$.

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The current state of classification

Can we make the dichotomy between "high topological dimension" and "low topological dimension" precise?

Conjecture (Toms-Winter, $\sim$2008)

Three diverse properties coincide for the class of separable, simple, amenable C*-algebras.

Can we prove that separable, simple, nuclear C*-algebras with "low topological dimension" are classified (as in the Elliott conjecture)?

Problem: Establish what the Toms-Winter properties mean for C*-algebra constructions, eg. group C*-algebras.

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Eg. The nuclear dimension of $C(X, \mathbb{C})$ is $\dim X$.

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Property 2: **Jiang-Su stability**

Given a C*-algebra A, the C*-algebra $A \otimes M_{2^\infty}$ has much more uniformity; is “low dimensional” in a sense.

A is M_{2^∞}**-stable** if $A \cong A \otimes M_{2^\infty}$.

Observe $M_{2^\infty} \cong M_{4^\infty} \cong M_{2^\infty} \otimes M_{2^\infty}$.

Unfortunately, many C*-algebras (such as M_{3^∞}) are not M_{2^∞}-stable.

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If A is a separable simple C*-algebra with finite nuclear dimension then A is $\mathcal{Z}$-stable (unless $A = M_n(\mathbb{C})$ or $\mathcal{K}(\mathcal{H})$).

Theorem (Bosa-Brown-Sato-T-White-Winter 2015)

If A is a separable simple amenable C*-algebra which is $\mathcal{Z}$-stable, then it has nuclear dimension ≤ 1 (provided it is unital and the set of extreme points of $T(A)$ is weak*-closed).

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