

DECOMPOSITION RANK OF APPROXIMATELY SUBHOMOGENEOUS C^* -ALGEBRAS

GEORGE A. ELLIOTT, ZHUANG NIU, LUIS SANTIAGO,
AND AARON TIKUISIS

ABSTRACT. It is shown that every Jiang-Su stable approximately subhomogeneous C^* -algebra has finite decomposition rank. This settles a key direction of the Toms–Winter conjecture for simple approximately subhomogeneous C^* -algebras. A key step in the proof is that subhomogeneous C^* -algebras are locally approximated by a certain class of more tractable subhomogeneous algebras, namely, a non-commutative generalization of the class of cell complexes. The result is applied to show finite decomposition rank for crossed product C^* -algebras associated to minimal homeomorphisms with mean dimension zero.

1. INTRODUCTION

Kirchberg and Winter defined the decomposition rank of a C^* -algebra, marrying the topological concept of Lebesgue covering dimension with the functional-analytic ideas of nuclearity and quasidiagonality. This notion subsequently gained currency, particularly in the programme of classifying C^* -algebras, where finite decomposition rank has been shown to imply strong structural conditions on a C^* -algebra [3, 28, 42, 46, 47, 48].

The concept of decomposition rank is now enshrined in the so-called Toms-Winter conjecture about simple, separable, unital, nuclear, finite C^* -algebras, which says, in particular, that finite decomposition rank and tensorial absorption of the Jiang-Su algebra (\mathcal{Z}) should be equivalent for such C^* -algebras. At the time that this conjecture was made ([42, Remark 3.5]; see also [10]), the only evidence in support of the component “ \mathcal{Z} -absorption implies finite decomposition rank” came very indirectly from classification. For example, Lin’s classification [20] implies that simple, separable, unital, nuclear, \mathcal{Z} -stable C^* -algebras which satisfy the UCT and rationally have tracial rank at most one have decomposition rank at most two. A more general classification

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result, due to Gong, Lin, and one of us (Z.N.), continues to imply finite decomposition rank for the larger class of C^* -algebras in which rational tracial rank one is weakened to rationally tracially approximated by point–line algebras [14].

The problem of showing directly that \mathcal{Z} -absorption implies finite decomposition rank (and in particular, avoiding appealing to simplicity or the UCT) has proven a pivotal problem, which has spurred the introduction of novel techniques in two very different streams of attack.

On the one hand, for non-simple C^* -algebras of a very particular form—namely, algebras $C(X) \otimes \mathcal{Z}$ (and by permanence properties, also limits of C^* -algebras Morita equivalent to these)—a direct computation, giving decomposition rank at most two, was obtained by one of us (A.T.) and Winter [40], using quasidiagonality of the cone over \mathcal{O}_2 , together with a one-dimensional approximation result inside $C(Y) \otimes \mathcal{O}_2$ due to Kirchberg and Rørdam [18]. (The present paper uses this result.) On the other hand, for simple, quasidiagonal C^* -algebras, techniques inspired by—and building on—Connes’s proof that injective von Neumann algebras are hyperfinite have been developed by Matui and Sato in the unique trace case [26], and subsequently by Bosa, Brown, Sato, Tikuisis, White, and Winter in the case of compact extreme trace space (where the hypothesis of quasidiagonality needs to be strengthened slightly) [1] (see also [38]). For the class considered in [1], the optimal decomposition rank estimate of one was proven.

The present article builds on the \mathcal{Z} -stabilized commutative case dealt with in [40], showing how to replace the algebra $C(X)$ by an arbitrary subhomogeneous algebra, i.e., a C^* -algebra for which there is a finite bound on the (Hilbert space) dimension of its irreducible representations. The main result is as follows.

Theorem A. *Let A be a locally subhomogeneous, \mathcal{Z} -stable C^* -algebra. Then $\text{dr}A \leq 2$.*

This result is novel even in the case of simple, unital, separable, locally subhomogeneous, \mathcal{Z} -stable C^* -algebras. With this result, it is now possible to prove that such C^* -algebras are classifiable; more on this below. The class of simple locally subhomogeneous C^* -algebras is very broad: it is known to include many natural examples (see the introduction to [41]), it exhausts the range of the Elliott invariant [6]. There are no simple, separable, nuclear, stably finite C^* -algebras which are known to lie outside the class of locally subhomogeneous C^* -algebras.

While there is some overlap between the class of C^* -algebras in this theorem and in the result of [1] (namely, all simple, locally subhomogeneous algebras with compact extreme trace space), the two results mostly complement each other. The present theorem allows non-simple C^* -algebras, and even in the simple case, allows C^* -algebras with non-compact extreme trace space. On the other hand, the result of [1],

where it applies, gives an optimal bound of one instead of two for the decomposition rank. (Also, the result in [1] holds under (potentially) much weaker hypotheses than local subhomogeneity—not even the UCT is required.)

In a striking application of our main result, we show in Section 4 that it applies to the UHF-stabilization of a minimal \mathbb{Z} -crossed product, by using an argument involving Berg’s technique. Using this result, Lin has shown that \mathcal{Z} -stable minimal \mathbb{Z} -crossed products are classifiable [21]; this includes all crossed products given by minimal homeomorphisms with mean dimension zero by [9].

Our proof of Theorem A consists of two major steps. The first step consists in proving the following result, which is perhaps of independent interest.

Theorem B. *Let A be a unital subhomogeneous C^* -algebra. Then A is locally approximated by non-commutative cell complexes. Moreover, when A is separable, the topological dimension of the approximating non-commutative cell complexes is equal to the topological dimension of A .*

The non-commutative cell complexes in this result, defined formally in Section 2.1, are special subhomogeneous algebras, akin to the non-commutative CW complexes of Eilers-Loring-Pedersen, though slightly more general for technical reasons. As the name suggests, they arise by gluing together matrix algebras over spaces D^n along the boundaries S^{n-1} . What is most important here is that the boundary S^{n-1} is a neighbourhood retract in D^n .

The topological dimension of a separable subhomogeneous C^* -algebra measures a sort of dimension of its primitive ideal space, and agrees with the decomposition rank of the C^* -algebra; see Definition 1.3.

Theorem B is a non-commutative analogue of the result that n -dimensional compact metrizable spaces are inverse limits of finite n -dimensional CW complexes. Just as CW complexes form a class of spaces that are amenable to deep analysis, so too are the non-commutative cell complexes of Theorem B. Their tractable structure allows them to be used to prove Theorem A. The author A.T. has also made use of Theorem B, in an argument that shows that $C(X, \mathcal{Q})$ is not locally approximated by subhomogeneous C^* -algebras of topological dimension less than the dimension of X , where \mathcal{Q} is the universal UHF algebra [39].

Generalizing the result that cell complexes are absolute neighbourhood retracts, we demonstrate that our non-commutative cell complexes satisfy a restricted semiprojectivity condition (Theorem 2.12); this condition then plays a key role in the proof of Theorem B. From the proof that a subhomogeneous C^* -algebra is locally approximated by non-commutative cell complexes, ideas are borrowed to also prove

that any \mathcal{W} -stabilized, locally subhomogeneous C^* -algebra is an inductive limit of point–line algebras (Theorem 2.20), where \mathcal{W} is the stably projectionless C^* -algebra studied in [17]; this result builds on the case of $C(X) \otimes \mathcal{W}$, which was handled by one of us (L.S.) in [37].

The second step in proving Theorem A is to show that the \mathcal{Z} -stabilization of a non-commutative cell complex has decomposition rank at most 2. In fact, in this part, we are able to directly reduce to the problem of the decomposition rank of \mathcal{Z} -stabilized commutative C^* -algebras (which was solved in [40]), by turning coloured c.p.c. approximations (as in the definition of decomposition rank) of certain commutative C^* -algebras into coloured c.p.c. approximations of non-commutative cell complexes (Theorem 3.1). The commutative C^* -algebras arise in a sort of mapping cone construction involving the eigenvalue patterns of the gluing maps. The cell structure is absolutely essential, as it allows us to relate these non-commutative C^* -algebras to such commutative algebras.

The latest results concerning particular values of the decomposition rank suggest that the only possible values in the simple case are 0, 1, and ∞ (to be precise, this is what is shown in [1] in the case of compact extreme tracial boundary). It is unclear whether the upper bound of two, proven here, is optimal for non-simple, \mathcal{Z} -stable, locally subhomogeneous C^* -algebras. Our result shows, however, that this depends purely on whether a better bound can be found for $C(X) \otimes \mathcal{Z}$. That is, if the bound of two found in [40] is improved to one (which would be optimal, since decomposition rank zero implies AF), then the same estimate applies to \mathcal{Z} -stable, locally subhomogeneous C^* -algebras.

Building on the results here, the authors G.A.E. and Z.N., in collaboration with Guihua Gong and Huaxin Lin, have gone on to prove that simple, separable unital, \mathcal{Z} -stable, locally subhomogeneous C^* -algebras are rationally tracially approximated by point–line algebras [8], and are thereby classified by the results of [14]. Their result depends on both Theorems A and B. It follows that every simple, separable, unital, \mathcal{Z} -stable, locally subhomogeneous C^* -algebra is an inductive limit of subhomogeneous C^* -algebras of topological dimension at most two (this does not constitute an alternative proof of Theorem A since it uses said theorem).

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1.2. Preliminaries and notation. Let A be a C^* -algebra. Denote the positive cone of A by A_+ . For $a, b \in A$ and $\epsilon > 0$, write $a \approx_\epsilon b$ to mean $\|a - b\| < \epsilon$. If $\mathcal{F}, B \subset A$ with \mathcal{F} finite, and $\epsilon > 0$, write $\mathcal{F} \subset_\epsilon B$ to mean that for every $a \in \mathcal{F}$,

$$(1.1) \quad \text{dist}(a, B) < \epsilon.$$

For a unital C^* -algebra A , let $U(A)$ denote the unitary group. Write M_m for $M_m(\mathbb{C})$ and U_m for $U(M_m(\mathbb{C}))$.

If A, B are unital C^* -algebras, let us consider $\text{Hom}(A, B)$, the space of *unital* $*$ -homomorphisms from A to B , with the point-norm topology. Provided that A is separable, this topology on $\text{Hom}(A, B)$ is metrizable.

For a compact Hausdorff space X , there is a one-to-one correspondence between $\text{Hom}(A, C(X, B))$ and $C(X, \text{Hom}(A, B))$, defined as follows. For $\phi \in \text{Hom}(A, C(X, B))$, define $\hat{\phi} \in C(X, \text{Hom}(A, B))$ by

$$(1.2) \quad \hat{\phi}(x)(a) := \phi(a)(x), \quad x \in X, a \in A.$$

Thus, if Y is a closed subspace of X and $r_Y: C(X, B) \rightarrow C(Y, B)$ denotes the restriction map, then for $\phi \in \text{Hom}(A, C(X, B))$,

$$(1.3) \quad (r_Y \circ \phi)^\wedge = \hat{\phi}|_Y.$$

Definition 1.1 ([19]). *Let A be a C^* -algebra and let $n \in \mathbb{N}$. One says that the **decomposition rank** of A is at most n (in abbreviated form, $\text{dr}A \leq n$) if, for every finite subset \mathcal{F} of A and every $\epsilon > 0$, there exist finite dimensional C^* -algebras F_0, \dots, F_n and c.p.c. maps*

$$(1.4) \quad A \xrightarrow{\psi} F_0 \oplus \dots \oplus F_n \xrightarrow{\phi} A$$

such that $\phi|_{F_i}$ is orthogonality preserving (also called order zero), and $\phi(\psi(a)) \approx_\epsilon a$ for every $a \in \mathcal{F}$.

Definition 1.2. *A C^* -algebra A is **subhomogeneous** if there is a finite upper bound on the dimension of the irreducible representations of A .*

Let A be a subhomogeneous C^* -algebra and denote by $\text{Prim}(A)$ the space of primitive ideals of A with the hull-kernel topology (see [29, Chapter 3]). Then the subset

$$(1.5) \quad \text{Prim}_{\leq k}(A) := \{\ker \pi \mid \pi \in \text{Hom}(A, M_{k'}) \text{ is irreducible, } k' \leq k\}$$

is a closed subspace of $\text{Prim}(A)$. Moreover,

$$(1.6) \quad \text{Prim}_k(A) = \text{Prim}_{\leq k}(A) \setminus \text{Prim}_{\leq k-1}(A)$$

is a Hausdorff subspace of $\text{Prim}(A)$, and the subquotient

$$(1.7) \quad A|_{\text{Prim}_k(A)}$$

is a k -homogeneous C^* -algebra (i.e., every irreducible representation has dimension exactly k). (To define $A|_{\text{Prim}_k(A)}$, recall that closed sets in $\text{Prim}(A)$ correspond to ideals, in a containment-reversing fashion; thus, $A|_{\text{Prim}_k(A)}$ is the quotient of the ideal corresponding to $\text{Prim}_{\leq k-1}(A)$ by the ideal corresponding to $\text{Prim}_{\leq k}(A)$.)

Definition 1.3. *If A is a separable subhomogeneous C^* -algebra then the **topological dimension** of A refers to the value*

$$(1.8) \quad \max_k \dim \text{Prim}_k(A).$$

We stick to the separable case in the above definition, due to the following subtlety in the non-separable case: If a locally compact Hausdorff space X is not second countable, the nuclear dimension of $C_0(X)$ is not equal to the covering dimension of X . If the definition of Lebesgue covering dimension is modified by replacing “an open set of X ” by “a set of the form $f^{-1}(U)$ where $f \in C_0(X, \mathbb{R})$ and $U \subseteq \mathbb{R}$ is open,” then the resulting notion of “dimension” of X agrees with the nuclear dimension of $C_0(X)$. This notion of dimension would also be the correct one to use in Definition 1.3, in order for, say Theorem 1.4 to generalize to non-separable algebras.

Rob Archbold pointed out to us the following connection to the primitive ideal space (as a topological space without additional structure). Brown and Pedersen gave another definition of topological dimension in [2, 2.2 (v)], valid for any C^* -algebra, which agrees with the definition just given in the subhomogeneous case, by [2, Proposition 2.4]. Incidentally, this shows that the topological dimension is a function of the primitive ideal space—although it is not the covering dimension of the primitive ideal space.

Theorem 1.4 ([45]). *If A is a separable subhomogeneous C^* -algebra then the topological dimension, nuclear dimension, and decomposition rank of A coincide.*

As pointed out above, separability is not needed, provided that topological dimension is defined appropriately in the non-separable case. In Corollary 3.21, we give an alternative proof of this theorem.

Definition 1.5. *Let \mathcal{C} be a class of C^* -algebras and let A be a C^* -algebra. Let us say that A is **locally approximated** by (algebras in) \mathcal{C} (or simply, locally \mathcal{C}) if, for every finite subset \mathcal{F} of A and every $\epsilon > 0$, there exists a subalgebra $C \subseteq A$ such that $C \in \mathcal{C}$ and $\mathcal{F} \subset_\epsilon C$.*

In particular, **locally subhomogeneous** means locally approximated by the class of subhomogeneous C^* -algebras.

Here is a well-known result which is clear from the definition of decomposition rank.

Proposition 1.6. *If A is locally approximated by a class \mathcal{C} then*

$$(1.9) \quad \text{dr}(A) \leq \sup_{C \in \mathcal{C}} \text{dr}(C).$$

Pull-backs exist in the category of C^* -algebras, and in many places we shall use an explicit realization of them, which we describe now. Let A, B, C be C^* -algebras and let $\alpha: A \rightarrow C, \beta: B \rightarrow C$ be $*$ -homomorphisms. Set

$$(1.10) \quad D := \{(a, b) \in A \oplus B \mid \alpha(a) = \beta(b)\},$$

and let $\pi_A: D \rightarrow A$ and $\pi_B: D \rightarrow B$ denote the first and second coordinate projections. Then

$$(1.11) \quad \begin{array}{ccc} D & \xrightarrow{\pi_A} & A \\ \pi_B \downarrow & & \downarrow \alpha \\ B & \xrightarrow{\beta} & C \end{array}$$

is a pull-back diagram.

2. NON-COMMUTATIVE CELL COMPLEXES

Non-commutative cell complexes are introduced in this section, and it is shown that all subhomogeneous algebras can be locally approximated by these. This result can be viewed as a (non-commutatively) generalized—though weakened—version of the result that n -dimensional compact metrizable spaces are inverse limits of n -dimensional simplicial complexes. In particular, as in this topological result, our result has the feature that the topological dimension of the approximating subalgebras is controlled by the dimension of the given subhomogeneous algebra.

With regard to how it is weakened, the (non-commutative) result is more like saying that n -dimensional compact metrizable spaces are inverse limits of n -dimensional cell complexes. As with the commutative case, the result provides building blocks for (approximately) subhomogeneous algebras that are much more tractable and amenable to further analysis (e.g., Theorem A, [8, 34, 39]), compared to general subhomogeneous algebras—or even to Phillips’s recursive subhomogeneous algebras.

In proving this approximation theorem, a key result is that non-commutative cell complexes are semiprojective with respect to C^* -algebras of the form $C(X, M_m)$, for any fixed $m \in \mathbb{N}$. This result

is a generalization of the classical fact that finite cell complexes are absolute neighbourhood retracts; recalling the fact that a compact Hausdorff space X is an absolute neighbourhood retract if and only if $C(X)$ is semiprojective with respect to the class of unital commutative C^* -algebras, we see that this classical result is the special case that the non-commutative cell complex is commutative and $m = 1$. In fact, the proof of the new semiprojectivity result is inspired by the proof in the classical case (see [15, Appendix A], particularly Corollary A.10, which deals with CW complexes, although the arguments work for finite cell complexes). The classical proof entails showing (i) that cell complexes embed into Euclidean space (equivalently, they have finite topological dimension) and (ii) that cell complexes are locally contractible. Then [15, Theorem A.7] implies that they embed into Euclidean space as neighbourhood retracts, which implies that they are absolute neighbourhood retracts.

Here, using the correspondence between $*$ -homomorphisms $A \rightarrow C(X, M_m)$ and continuous maps $X \rightarrow \text{Hom}(A, M_m)$, the semiprojectivity result is equivalent to showing that $\text{Hom}(A, M_m)$ is an absolute neighbourhood retract, and as in the classical case, we appeal to [15, Theorem A.7], requiring us to show that $\text{Hom}(A, M_m)$ has finite topological dimension and is locally contractible. Showing that $\text{Hom}(A, M_m)$ is finite dimensional is fairly straightforward. Showing that it is locally contractible is somewhat more involved, although the idea behind the commutative case underpins even that argument.

2.1. Definition of non-commutative cell complexes. Non-commutative cell complexes are defined as recursive subhomogeneous algebras (as in [31, Definition 1.1]) for which the gluing pairs are always of the form $S^{n-1} \subset D^n$. Here is a formal definition.

Definition 2.1. *The class of (unital) **non-commutative (NC) cell complexes** is the smallest class \mathcal{C} of C^* -algebras such that:*

- (i) every finite dimensional algebra is in \mathcal{C} ; and
- (ii) if $B \in \mathcal{C}$, $k, n \in \mathbb{N}$, $\phi: B \rightarrow C(S^{n-1}, M_k)$ is a unital $*$ -homomorphism, and A is given by the pull-back diagram

$$(2.1) \quad \begin{array}{ccc} A & \longrightarrow & C(D^n, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{S^{n-1}} \\ B & \xrightarrow{\phi} & C(S^{n-1}, M_k), \end{array}$$

then $A \in \mathcal{C}$.

Remark 2.2. (i) Of course, every NC cell complex can be constructed by finitely many iterated pull-backs as in (2.1). The topological dimension of an NC cell complex is precisely the largest value of n such that D^n appears in one of these pull-backs. (Note that one need not take the

minimum over all possible decompositions, since if at one stage, D^n appears, then the algebra at this stage is a quotient of the final algebra, and its topological dimension is at least n .)

(ii) A commutative C^* -algebra is an NC cell complex if and only if it is isomorphic to $C(X)$ for some finite cell complex X .

(iii) The definition of NC cell complexes is closely related to the definition, due to Eilers-Loring-Pedersen, of NCCW complexes [5, Section 2.4]: all NCCW complexes are NC cell complexes, although the converse does not hold (see example below).

It is not hard to see that all NC cell complexes of topological dimension at most one are, in fact, one-dimensional NCCW complexes (these have elsewhere been called point–line algebras and occur prominently in the classification results of [14] and [34]). This fails, even in the commutative case, in dimension two.

For example, let $\alpha: S^1 \rightarrow D^2$ be a space-filling curve, and then define X by the push-forward diagram

$$(2.2) \quad \begin{array}{ccc} X & \longleftarrow & D^2 \\ \uparrow & & \uparrow \\ D^2 & \xleftarrow{\alpha} & S^1, \end{array}$$

so that X is a (finite) cell complex (i.e., $C(X)$ is an NC cell complex) but not a CW complex (i.e., $C(X)$ is not an NCCW complex—since every commutative NCCW complex is equal to $C(X)$ for a CW complex X).

2.2. Basic facts about $\text{Hom}(A, M_m)$. In preparation for the proof of our semiprojectivity result for NC cell complexes, we will prove a number of results about $\text{Hom}(A, M_m)$, particularly applicable when A is subhomogeneous.

A finite dimensional representation of a C^* -algebra decomposes as a direct sum of irreducible representations; we introduce notation to keep combinatorial track of such decompositions.

Fix $m \in \mathbb{N}$. Define

$$(2.3) \quad T_m := \{ \alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p) \in \mathbb{N}_{>0}^{2p} \mid p \in \mathbb{N} \text{ and } k_1 \alpha_1 + \dots + k_p \alpha_p = m \}.$$

Let A be a unital C^* -algebra. Then for $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p) \in T_m$, $u \in U_m$, and $\pi_i \in \text{Hom}(A, M_{k_i})$ for $i = 1, \dots, p$, define

$$(2.4) \quad \begin{aligned} \sigma_{\alpha, u, (\pi_1, \dots, \pi_p)} &:= \text{Ad}(u) \circ \text{diag}(\pi_1 \otimes 1_{\alpha_1}, \dots, \pi_p \otimes 1_{\alpha_p}) \\ &\in \text{Hom}(A, M_m). \end{aligned}$$

For $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p) \in T_m$, define

$$(2.5) \quad H_\alpha := \{ \sigma_{\alpha, u, (\pi_1, \dots, \pi_p)} \mid u \in U_m \text{ and } \pi_i \in \text{Hom}(A, M_{k_i}) \text{ for } i = 1, \dots, p \},$$

and let H_α^{pure} denote the subset of H_α consisting of those $\sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}$ for which π_1, \dots, π_p are mutually inequivalent, irreducible representations. For a C^* -algebra A which is not assumed to be unital, use $\text{Hom}_1(A, M_m)$ to denote the set of $*$ -homomorphisms whose image contains the unit (this coincides with $\text{Hom}(A, M_m)$ if A is unital), and define $\sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}$, H_α , and H_α^{pure} as in the unital case, with $\text{Hom}_1(A, M_m)$ in place of $\text{Hom}(A, M_m)$.

Proposition 2.3. *Fix $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p) \in T_m$ and a unital C^* -algebra A . Then the map $U_m \times \text{Hom}(A, M_{k_1}) \times \dots \times \text{Hom}(A, M_{k_p}) \rightarrow \text{Hom}(A, M_m)$ given by*

$$(2.6) \quad (u, \pi_1, \dots, \pi_p) \mapsto \sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}$$

is continuous.

Proof. Obvious. □

Lemma 2.4. *Let A be a C^* -algebra. Fix $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p) \in T_m$, and fix pairwise inequivalent irreducible representations $\pi_i \in \text{Hom}_1(A, M_{k_i})$ for $i = 1, \dots, p$. Then for $u, v \in U_m$,*

$$(2.7) \quad \sigma_{\alpha,u,(\pi_1,\dots,\pi_p)} \approx \sigma_{\alpha,v,(\pi_1,\dots,\pi_p)}$$

if and only if uv^* is approximately contained in

$$(2.8) \quad U(1_{k_1} \otimes M_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes M_{\alpha_p}) = 1_{k_1} \otimes U_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes U_{\alpha_p}.$$

To be more precise:

(i) Given $\mathcal{F} \subset A$ finite and $\epsilon > 0$, there exists $\delta > 0$ such that if $u, v \in U_m$ and

$$(2.9) \quad d(uv^*, 1_{k_1} \otimes U_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes U_{\alpha_p}) < \delta$$

then

$$(2.10) \quad \sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}(a) \approx_\epsilon \sigma_{\alpha,v,(\pi_1,\dots,\pi_p)}(a)$$

for all $a \in \mathcal{F}$; and

(ii) Given $\epsilon > 0$, there exists $\mathcal{F} \subset A$ finite and $\delta > 0$, such that if $u, v \in U_m$ and

$$(2.11) \quad \sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}(a) \approx_\delta \sigma_{\alpha,v,(\pi_1,\dots,\pi_p)}(a)$$

for all $a \in \mathcal{F}$, then

$$(2.12) \quad d(uv^*, 1_{k_1} \otimes U_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes U_{\alpha_p}) < \epsilon.$$

Proof. Note that, by (2.4),

$$(2.13) \quad \sigma_{\alpha,u,(\pi_1,\dots,\pi_p)}(a) \approx_\epsilon \sigma_{\alpha,v,(\pi_1,\dots,\pi_p)}(a)$$

is equivalent to saying that

$$(2.14) \quad \|[uv^*, \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p})]\| < \epsilon.$$

(i): Given \mathcal{F} finite and $\epsilon > 0$, set $M := \max_{a \in \mathcal{F}} \|a\|$ and then

$$(2.15) \quad \delta := \frac{\epsilon}{2M}.$$

Then if $u, v \in U_m$ and

$$(2.16) \quad uv^* \approx_\delta w \in 1_{k_1} \otimes U_{\alpha_1} \oplus \cdots \oplus 1_{k_p} \otimes U_{\alpha_p},$$

then for $a \in A$,

$$(2.17) \quad \begin{aligned} & uv^* \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p}) \\ & \approx_{\epsilon/2} w \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p}) \\ & = \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p}) w \\ & \approx_{\epsilon/2} \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p}) uv^*, \end{aligned}$$

as required.

(ii): On the other hand, suppose we are given $\epsilon > 0$. Since being a unitary is a stable relation, it suffices to find \mathcal{F} and $\delta > 0$ such that, instead of (2.12), we have

$$(2.18) \quad d(uv^*, 1_{k_1} \otimes M_{\alpha_1} \oplus \cdots \oplus 1_{k_p} \otimes M_{\alpha_p}) < \epsilon.$$

By our hypothesis on π_1, \dots, π_p , we may choose $\mathcal{F} \subset A$ finite such that the set

$$(2.19) \quad \{\text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p}) \mid a \in \mathcal{F}\}$$

generates

$$(2.20) \quad D := M_{k_1} \otimes 1_{\alpha_1} \oplus \cdots \oplus M_{k_p} \otimes 1_{\alpha_p}$$

as a C^* -algebra. Therefore (and using the fact that D is finite dimensional), we may pick $\delta > 0$ so that if w is unitary and

$$(2.21) \quad \|[w, \text{diag}(\pi_1(a) \otimes 1_{\alpha_1}, \dots, \pi_p(a) \otimes 1_{\alpha_p})]\| < \delta$$

for every $a \in \mathcal{F}$ then

$$(2.22) \quad \|[w, x]\| < \epsilon$$

for every contraction $x \in D$. By applying the conditional expectation onto

$$(2.23) \quad M_m \cap D' = 1_{k_1} \otimes M_{\alpha_1} \oplus \cdots \oplus 1_{k_p} \otimes M_{\alpha_p},$$

we see that this implies that

$$(2.24) \quad d(w, M_m \cap D') < \epsilon,$$

as required. \square

Proposition 2.5. *Let $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p), \beta = (l_1, \beta_1, \dots, l_q, \beta_q) \in T_m$. The following statements are equivalent:*

(i) $H_\alpha \subseteq H_\beta$;

(ii) There exists $S \in M_{p \times q}(\mathbb{N})$ such that

$$(2.25) \quad \begin{pmatrix} k_1 \\ \vdots \\ k_p \end{pmatrix} = S \begin{pmatrix} l_1 \\ \vdots \\ l_q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_q \end{pmatrix} = S^{tr} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix}.$$

If H_α^{pure} is non-empty, then these are also equivalent to

(iii) $H_\alpha^{pure} \cap H_\beta \neq \emptyset$.

Also, $H_\alpha = H_\beta$ if and only if $p = q$ and there exists a permutation ρ on $\{1, \dots, p\}$ such that

$$(2.26) \quad k_i = l_{\rho(i)} \quad \text{and} \quad \alpha_i = \beta_{\rho(i)}$$

for all $i = 1, \dots, p$.

Proof. Straightforward. \square

Proposition 2.6. *Let A be a C^* -algebra. Then $\text{Hom}_1(A, M_m) = \bigcup_{\alpha \in T_m} H_\alpha^{pure}$.*

Proof. Obvious. \square

2.3. Dimension of $\text{Hom}(A, M_m)$.

Lemma 2.7. *Let A be a unital separable subhomogeneous C^* -algebra such that $A|_{\text{Prim}_k(A)}$ is locally trivial for each k . Let $m \in \mathbb{N}$. Then there exist q and a finite increasing sequence of sets,*

$$(2.27) \quad \emptyset = Z_0 \subset Z_1 \subset \dots \subset Z_q = \text{Hom}(A, M_m)$$

such that:

(i) Z_i is closed for each i ; and

(ii) For each $\sigma \in Z_i \setminus Z_{i-1}$, there exists a neighbourhood W of σ in $Z_i \setminus Z_{i-1}$ (that is, W is relatively open in Z_i) which is homeomorphic to

$$(2.28) \quad V \times W_1 \times \dots \times W_p$$

where

(a) $V = U_m/G$ where G is a Lie subgroup of U_m ;

(b) $p \leq m$; and

(c) For each $j = 1, \dots, m'$, W_j is a relatively open subset of $\text{Prim}_k(A)$ for some k .

Proof. Consider the equivalence relation \sim on T_m given by $\alpha \sim \beta$ if $H_\alpha = H_\beta$. Enumerate one representative for each equivalence class as $\alpha^1, \dots, \alpha^q \in T_m$, in such a way that for each i, j , if $H_{\alpha^i} \subseteq H_{\alpha^j}$ then $i \leq j$. Then, define

$$(2.29) \quad Z_i := \bigcup_{j \leq i} H_{\alpha_j}.$$

Since U_m and $\text{Hom}(A, M_k)$ are compact spaces, Proposition 2.3 implies that H_α is closed for each α . Consequently, Z_i is closed.

By Propositions 2.5 and 2.6,

$$(2.30) \quad Z_i \setminus Z_{i-1} = H_{\alpha_i}^{\text{pure}}.$$

So, in order to establish (ii), we must show that every element of H_{α}^{pure} has a neighbourhood in H_{α}^{pure} of the form described in (ii).

For this, fix $\alpha = (k_1, \alpha_1, \dots, k_p, \alpha_p)$ and let $\sigma_{\alpha, u, (\pi_1, \dots, \pi_p)} \in H_{\alpha}^{\text{pure}}$. For each i , let W_i be a neighbourhood of $[\pi_i]$ in $\text{Prim}_{k_i}(A)$ such that $A|_{W_i}$ is trivial, and such that $W_i \cap W_j = \emptyset$ whenever $i \neq j$. Since $A|_{W_i}$ is trivial, let $\dot{W}_i \subset \text{Hom}(A, M_{k_i})$ be such that $\pi_i \in \dot{W}_i$ and $\pi \mapsto [\pi]$ is a homeomorphism from \dot{W}_i onto W_i .

W_i being open in $\text{Prim}_{k_i}(A)$ means that $W_i \cup \text{Prim}_{>k_i}(A)$ is open in $\text{Prim}(A)$. Using the fact that A is separable, we may therefore find $a_i \in A$ such that, for each irreducible representation π of A , $\pi(a_i) = 0$ if and only if $[\pi] \in \text{Prim}_{\leq k_i}(A) \setminus W_i$. Set

$$(2.31) \quad V := U_m / (1_{k_1} \otimes U_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes U_{\alpha_p}).$$

Define

$$(2.32) \quad W := \{\sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)} \mid [v] \in V, \pi'_i \in \dot{W}_i \text{ for all } i = 1, \dots, p\}.$$

Let us first show that W is a neighbourhood of $\sigma_{\alpha, u, (\pi_1, \dots, \pi_p)}$ in H_{α}^{pure} . Let $\sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)} \in H_{\alpha}^{\text{pure}}$ be close to $\sigma_{\alpha, u, (\pi_1, \dots, \pi_p)}$. If it is close enough, then

$$(2.33) \quad \sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)}(a_i) \neq 0,$$

and therefore, for some j , $[\pi'_j] \in W_i$. By the pigeonhole principle, permuting indices, we may assume that $[\pi'_i] \in W_i$ for each i . Replacing π'_i by a unitarily equivalent homomorphism (altering v in the process), we may then assume that $\pi'_i \in \dot{W}_i$ for each i .

This concludes the proof that W is a neighbourhood of $\sigma_{\alpha, u, (\pi_1, \dots, \pi_p)}$.

Now define $\Phi: V \times \dot{W}_1 \times \dots \times \dot{W}_p \rightarrow W$ in the obvious way, by

$$(2.34) \quad \Phi([v], \pi'_1, \dots, \pi'_p) = \sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)}.$$

This is continuous by Proposition 2.3, and it is clearly surjective. Since the sets W_1, \dots, W_p are disjoint, and by Lemma 2.4, Φ is also injective.

Let us show that Φ is open. Suppose that $\sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)} \approx \sigma_{\alpha, w, (\pi''_1, \dots, \pi''_p)}$. Then the argument for why W is a neighbourhood of $\sigma_{\alpha, u, (\pi_1, \dots, \pi_p)}$ shows that, up to a permutation, we have $\pi'_i \approx \pi''_i$ for all i . Thus

$$(2.35) \quad \sigma_{\alpha, v, (\pi'_1, \dots, \pi'_p)} \approx \sigma_{\alpha, w, (\pi''_1, \dots, \pi''_p)} \approx \sigma_{\alpha, w, (\pi'_1, \dots, \pi'_p)},$$

so by Lemma 2.4, vw^* is approximately contained in $1_{k_1} \otimes U_{\alpha_1} \oplus \dots \oplus 1_{k_p} \otimes U_{\alpha_p}$, i.e., $[v] \approx [w]$ in V . This shows that Φ is open.

Since W_i is homeomorphic to \dot{W}_i , we conclude that W is homeomorphic to $V \times W_1 \times \dots \times W_p$, as required. \square

Corollary 2.8. *Let A be a unital separable subhomogeneous C^* -algebra of finite topological dimension, and let $m \in \mathbb{N}$. Then $\text{Hom}(A, M_m)$ has finite dimension.*

Proof. Suppose that A has topological dimension at most n , so that $\text{Prim}_k(A)$ has dimension at most n for each k , and hence by [31, Theorem 2.16], it satisfies the hypotheses of Lemma 2.7. Note also that any quotient of U_m by a Lie subgroup has dimension at most m^2 . From Proposition 2.7 and standard permanence properties for dimension, it follows directly that

$$(2.36) \quad \dim \text{Hom}(A, M_m) \leq m^2 n^m.$$

□

2.4. Local contractibility of $\text{Hom}(A, M_m)$. To outline the argument here, consider the purely commutative case: $\text{Hom}(A, \mathbb{C})$ where A is commutative (note that $\text{Hom}(C(X), \mathbb{C}) \cong X$). In order to show that a cell complex is locally contractible, one may use induction, showing that any point on the boundary of a newly-attached cell has small neighbourhoods, each of which can be retracted to a contractible neighbourhood of the old complex (see [15, Proposition A.4] and its proof). This is the idea behind showing that $\text{Hom}(A, M_m)$ is locally contractible when A is a non-commutative cell complex, although even in the case that A is commutative, the argument is more complicated: if $A = C(Y)$, then homomorphisms from A to M_m are parametrized by an m -tuple of points in Y together with a unitary $u \in M_m$ (each homomorphism is the direct sum of point-evaluations, conjugated by a unitary), so part of the argument is that the space of m -multisets of points in Y is locally contractible. For more general non-commutative cell complexes, we have a similar picture, but with “points in Y ” replaced by irreducible representations. These need to be handled carefully, chiefly because of the non-Hausdorff nature of the space of irreducible representations.

The result, in the case that A is commutative (or a matrix algebra over such), turns out to be a necessary stepping stone to the overall result, and we prove this special case now. Recall that $\text{Hom}_1(A, M_m)$ denotes the space of $*$ -homomorphisms whose image contains the unit. We recall from topology that a **deformation retraction** of a space X onto a subspace Y is a continuous function

$$(2.37) \quad r = (r_t)_{t \in [0,1]}: [0, 1] \times X \rightarrow X$$

such that $r_0 = \text{id}_X$, $r_1(X) = Y$, and $r_1|_Y = \text{id}_Y$. Thus, a space X is contractible to a point x_0 exactly when there exists a deformation retraction from X to $\{x_0\}$.

Lemma 2.9. *Let X be a locally compact Hausdorff space. If X is locally contractible then so also is $\text{Hom}_1(C_0(X, M_k), M_{sk})$ for any $k, s \in \mathbb{N}$.*

Proof. Using $\alpha := (1, 1, \dots, 1, 1) \in \mathbb{N}^{2s}$, note that $\text{Hom}_1(C_0(X, M_k), M_{sk}) = H_\alpha$. To keep notation more concise, write

$$(2.38) \quad \check{\sigma}_{u, (x_1, \dots, x_s)} := \sigma_{\alpha, u, (ev_{x_1}, \dots, ev_{x_s})} \in \text{Hom}_1(C_0(X, M_k), M_{ks})$$

for $x_1, \dots, x_s \in X$ and $u \in U_{sk}$.

Consider a point $\sigma = \check{\sigma}_{u, (x_1, \dots, x_s)} \in \text{Hom}_1(C_0(X, M_k), M_{sk})$. Choose, for each $x \in \{x_1, \dots, x_s\}$, an arbitrarily small contractible neighbourhood V_x of x , such that $V_x \cap V_y = \emptyset$ whenever $x \neq y$. Fix a deformation retraction $\alpha_x: [0, 1] \times V_x \rightarrow V_x$ of V_x onto $\{x\}$, (Note that, by our choice of notation, if $x_i = x_j$ then $V_{x_i} = V_{x_j}$ and $\alpha_{x_i} = \alpha_{x_j}$.) Let $\epsilon > 0$ be small (to be determined).

Define $\tilde{U} \subset U_{sk} \times X^s$ to consist of those points $(v, (y_1, \dots, y_s))$ for which there exist a unitary $u' \in U_{sk}$ such that

- (i) $\sigma = \check{\sigma}_{u', (x_1, \dots, x_s)}$;
- (ii) $\|v - u'\| < \epsilon$; and
- (iii) $y_i \in V_{x_i}$ for each i .

The set

$$(2.39)$$

$$U := \{\check{\sigma}_{v, (y_1, \dots, y_s)} \in \text{Hom}_1(C_0(X, M_k), M_{sk}) \mid (v, (y_1, \dots, y_s)) \in \tilde{U}\}$$

is an open neighbourhood of a , and by making V_1, \dots, V_s and ϵ small enough, we make this neighbourhood arbitrarily small.

Define $\tilde{W} \subset U_{sk} \times X^s$ to consist of those points $(v, (x_1, \dots, x_s))$ for which there exist a unitary $u' \in U_{sk}$ such that

- (i) $\sigma = \check{\sigma}_{u', (x_1, \dots, x_s)}$; and
- (ii) $\|v - u'\| < \epsilon$,

and set

$$(2.40)$$

$$W := \{\check{\sigma}_{v, (x_1, \dots, x_s)} \in \text{Hom}_1(C_0(X, M_k), M_{sk}) \mid (v, (x_1, \dots, x_s)) \in \tilde{W}\}.$$

Define $\tilde{\beta} = (\tilde{\beta}_t): [0, 1] \times \tilde{U} \rightarrow \tilde{U}$ as follows. Given $(v, (y_1, \dots, y_s)) \in \tilde{U}$, define

$$(2.41) \quad \tilde{\beta}_t(v, (y_1, \dots, y_s)) = (v, (\alpha_{x_1}(t, y_1), \dots, \alpha_{x_s}(t, y_s))),$$

where we recall that α_x is a deformation retract of V_x onto $\{x\}$. It is not hard to see that $\tilde{\beta}$ is a deformation retraction of \tilde{U} onto \tilde{W} . It is also not hard to see that, via the surjection $\tilde{U} \rightarrow U$, $\tilde{\beta}$ induces a continuous map $\beta: [0, 1] \times U \rightarrow U$, which is therefore a deformation retraction of U onto W .

Next, define $\dot{W} \subset U_{sk} \times X^s$ to consist of those points $(v, (x_1, \dots, x_s))$ such that $\|v - u\| < \epsilon$. Note that $\check{\sigma}$ induces a homeomorphism of \dot{W} onto W .

For unitaries u, v and $t \in [0, 1]$, define

$$(2.42) \quad \lambda_t(u, v) := tu + (1 - t)v.$$

If u and v are sufficiently close then $\lambda_t(u, v)$ is invertible, and we may define

$$(2.43) \quad \theta_t(u, v) := \lambda_t(u, v)(\lambda_t(u, v)^* \lambda_t(u, v))^{-1/2},$$

which is unitary. Thus $(\theta_t(u, v))_{t \in [0, 1]}$ is a path of unitaries from u to v .

For $(v, (x_1, \dots, x_s)) \in \dot{W}$, define $\dot{\gamma} = (\dot{\gamma}_t): [0, 1] \times \dot{W} \rightarrow \dot{W}$ by

$$(2.44) \quad \dot{\gamma}_t(v, (x_1, \dots, x_s)) = (\theta_t(u, v), (x_1, \dots, x_s))$$

(assume ϵ is sufficiently small so that $\theta_t(u, v)$ is always defined.) This is a deformation retraction of \dot{W} onto $\{(u, (x_1, \dots, x_s))\}$, so by the homeomorphism $\dot{W} \rightarrow W$, we obtain a deformation retraction γ of W onto $\{\sigma\}$.

Combining the deformation retracts β and γ provides a deformation retraction of U onto $\{a\}$, as required. \square

The next lemma generalizes the commutative argument that, for points on the boundary of a cell attached to a cell complex, every relative neighbourhood in the old cell complex is a deformation retract of a neighbourhood in the enlarged cell complex. This lemma does not, in itself, imply that $\text{Hom}(A, M_m)$ is locally contractible when A is an NC cell complex, since a $*$ -homomorphism $A \rightarrow M_m$ can contain a mix of irreducible representations from the boundary of the new cell, the interior of the new cell, and purely from the old NC cell complex.

We continue to use $\text{Hom}_1(A, M_m)$ to denote the set of $*$ -homomorphisms $A \rightarrow M_m$ whose images contain the unit of M_m .

Lemma 2.10. *Let A be given by the pull-back diagram*

$$(2.45) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C_0(X \times [0, 1], M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X \times \{0\}} \\ B & \xrightarrow{\phi} & C(X, M_k), \end{array}$$

where B is a unital C^* -algebra and ϕ is a unital $*$ -homomorphism. Let U be an open set in $\text{Hom}(B, M_m)$. Then there exists an open set V in $\text{Hom}_1(A, M_m)$ such that

$$(2.46) \quad U \circ \lambda = V \cap (\text{Hom}(B, M_m) \circ \lambda),$$

and this set is a deformation retract of V .

Proof. To keep notation more concise, for $l \in \mathbb{N}$ such that $lk \leq m$, denote

$$(2.47) \quad \alpha_l := (1, 1, \dots, 1, 1, m - kl, 1) \in T_m$$

(where $1, 1$ appears l times), and for $v_1, \dots, v_l \in X \times [0, 1)$, $\pi_B \in \text{Hom}(B, M_{m-kl})$, and $u \in U_m$, denote

$$\begin{aligned} \check{\sigma}_{u, (v_1, \dots, v_l), \pi_B} &:= \sigma_{\alpha_l, u, (\text{ev}_{v_1} \circ \rho, \dots, \text{ev}_{v_l} \circ \rho, \pi_B \circ \lambda)} \\ &= \text{Ad}(u) \circ \text{diag}(\text{ev}_{v_1} \circ \rho, \dots, \text{ev}_{v_l} \circ \rho, \pi_B \circ \lambda) \\ (2.48) \quad &\in \text{Hom}_1(A, M_m). \end{aligned}$$

Define

$$\begin{aligned} \dot{U} &:= \{(u, (x_1, \dots, x_l), \pi_B) \mid l \in \mathbb{N}, \pi_B \in \text{Hom}(B, M_{m-kl}), \\ (2.49) \quad &x_1, \dots, x_l \in X, u \in U_m, \check{\sigma}_{u, ((x_1, 0), \dots, (x_l, 0)), \pi_B} \in U \circ \lambda\}. \end{aligned}$$

Note that

$$(2.50) \quad \check{\sigma}_{u, ((x_1, 0), \dots, (x_l, 0)), \pi_B} = \sigma_{\alpha_l, u, (\text{ev}_{x_1} \circ \phi, \dots, \text{ev}_{x_l} \circ \phi, \pi_B)} \circ \lambda,$$

where $\sigma_{\alpha_l, u, (\text{ev}_{x_1} \circ \phi, \dots, \text{ev}_{x_l} \circ \phi, \pi_B)}$ is in $\text{Hom}(B, M_m)$.

Set

$$\begin{aligned} \dot{V} &:= \{(u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B) \mid l \in \mathbb{N}, \\ (2.51) \quad &(u, (x_1, \dots, x_l), \pi_B) \in \dot{U}, s_1, \dots, s_l \in [0, 1)\}. \end{aligned}$$

Define $\dot{\theta} = (\dot{\theta}_t): [0, 1] \times \dot{V} \rightarrow \dot{V}$ by

$$\begin{aligned} \dot{\theta}_t(u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B) \\ (2.52) \quad &:= (u, ((x_1, ts_1), \dots, (x_l, ts_l)), \pi_B). \end{aligned}$$

Define

$$\begin{aligned} V &:= \{\check{\sigma}_{u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B} \mid \\ (2.53) \quad &(u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B) \in \dot{V}\}. \end{aligned}$$

We shall check two things: (i) that V is open, and (ii) that $\dot{\theta}$ induces a homotopy θ on V , by

$$(2.54) \quad \theta_t(\check{\sigma}_{u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B}) = \check{\sigma}_{\dot{\theta}_t(u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B)}, \quad t \in [0, 1].$$

It is obvious from the definition that (2.46) holds and that θ is a deformation retraction onto U .

(i): To see that V is open, let $\check{\sigma}_{u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B} \in V$ for some $(u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B) \in \dot{V}$. By possibly modifying the choice of lift in \dot{V} , we may arrange that π_B has no subrepresentation that factors through ϕ , that $s_1, \dots, s_{l_1} > 0$, and that $s_{l_1+1} = \dots = s_l = 0$. Suppose that σ is a representation near to $\check{\sigma}_{u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B}$. Let us write σ as

$$(2.55) \quad \check{\sigma}_{v, ((y_1, t_1), \dots, (y_p, t_p)), \pi'_B}.$$

Assuming that σ is near enough, taking into account how far each subrepresentation of π_B is from a representation that factors through ϕ , and by rearranging, we may assume that $l_1 \leq p \leq l$ and

$$(2.56) \quad (y_i, t_i) \approx (x_i, s_i), \quad i \leq p.$$

It follows that, for all i, j , if $(x_i, s_i) \neq (x_j, s_j)$ then $(y_i, t_i) \neq (y_j, t_j)$, and that

$$(2.57) \quad \check{\sigma}_{v,((x_1, s_1), \dots, (x_p, s_p)), \pi'_B} \approx \check{\sigma}_{u,((x_1, s_1), \dots, (x_l, s_l)), \pi_B}.$$

By the same argument as used to prove Lemma 2.4, there exists a unitary $u' \in U_m$ such that

$$(2.58) \quad \begin{aligned} \check{\sigma}_{u,((x_1, s_1), \dots, (x_l, s_l)), \pi_B} &= \check{\sigma}_{u',((x_1, s_1), \dots, (x_l, s_l)), \pi_B} \quad \text{and} \\ (u')^* v &\approx \begin{pmatrix} 1_{kp} & 0 \\ 0 & w_2 \end{pmatrix} =: w \end{aligned}$$

for some $w_2 \in U_{m-kp}$. Thus,

$$(2.59) \quad \begin{aligned} \check{\sigma}_{u',((x_1, s_1), \dots, (x_l, s_l)), \pi_B} &\stackrel{(2.57)}{\approx} \check{\sigma}_{v,((x_1, s_1), \dots, (x_p, s_p)), \pi'_B} \\ &\stackrel{(2.58)}{\approx} \check{\sigma}_{u'w,((x_1, s_1), \dots, (x_p, s_p)), \pi'_B}, \end{aligned}$$

so conjugating by $(u')^*$, and looking at $(m - kl) \times (m - kl)$ block on the bottom-right, one obtains

$$(2.60) \quad \text{diag}(\text{ev}_{x_{p+1}} \circ \phi, \dots, \text{ev}_{x_l} \circ \phi, \pi_B) \approx \text{Ad}(w_2) \circ \pi'_B.$$

It follows that

$$(2.61) \quad \begin{aligned} \sigma_{\alpha_p, v, (\text{ev}_{y_1} \circ \phi, \dots, \text{ev}_{y_p} \circ \phi), \pi'_B} &\stackrel{(2.58)}{\approx} \sigma_{\alpha_p, u', (\text{ev}_{y_1} \circ \phi, \dots, \text{ev}_{y_p} \circ \phi), \text{Ad}(w_2) \circ \pi'_B} \\ &\stackrel{(2.56)}{\approx} \sigma_{\alpha_p, u', (\text{ev}_{x_1} \circ \phi, \dots, \text{ev}_{x_p} \circ \phi), \text{Ad}(w_2) \circ \pi'_B} \\ &\stackrel{(2.60)}{\approx} \sigma_{\alpha_l, u', (\text{ev}_{x_1} \circ \phi, \dots, \text{ev}_{x_l} \circ \phi), \pi_B} \in U. \end{aligned}$$

These approximations can be made arbitrarily close by asking at the outset that σ is sufficiently close to $\check{\sigma}_{u,((x_1, s_1), \dots, (x_l, s_l)), \pi_B}$. It then follows that $\sigma_{\alpha_p, v, (\text{ev}_{y_1} \circ \phi, \dots, \text{ev}_{y_p} \circ \phi), \pi'_B} \in U$, i.e., $(v, (y_1, \dots, y_p), \pi'_B) \in \dot{U}$. Hence,

$$(2.62) \quad (v, ((y_1, t_1), \dots, (y_p, t_p)), \pi'_B) \in \dot{V}, \quad \text{i.e.,} \quad \sigma \in V.$$

This establishes that V is open.

(ii): To check that $\check{\theta}$ induces a homotopy $\theta = (\theta_t)_{t \in [0,1]}$ on V , suppose that

$$(2.63) \quad (u, ((x_1, s_1), \dots, (x_l, s_l)), \pi_B), (v, ((y_1, t_1), \dots, (y_p, t_p)), \pi'_B) \in \dot{V}$$

give rise to the same element of V , i.e., that

$$(2.64) \quad \check{\sigma}_{u,((x_1, s_1), \dots, (x_l, s_l)), \pi_B} = \check{\sigma}_{v,((y_1, t_1), \dots, (y_p, t_p)), \pi'_B}.$$

It is not hard to see that we may assume that $s_i, t_j > 0$ for all i, j . Then it follows that

$$(2.65) \quad \{(x_1, s_1), \dots, (x_l, s_l)\} = \{(y_1, t_1), \dots, (y_p, t_p)\}$$

as multisets, and so $l = p$. Replacing a unitary by its composition with a permutation, if necessary, we may assume that $(x_i, s_i) = (y_i, t_i)$ for all i . When this is true, it is clear from the formula for $\dot{\theta}_t$ that

$$(2.66) \quad \dot{\theta}_t(u, (x_1, s_1), \dots, (x_l, s_l), \pi_B))$$

and

$$(2.67) \quad \theta_t(v, (y_1, t_1), \dots, (y_p, t_p), \pi_P))$$

induce the same representation of A . \square

Finally, we prove the local contractibility result.

Proposition 2.11. *Let A be an NC cell complex. Then $\text{Hom}(A, M_m)$ is locally contractible.*

Proof. Using induction, we need only prove that if $\text{Hom}(B, M_m)$ is locally contractible for every m and A is given by the pull-back diagram

$$(2.68) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C(D^n, M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{S^n \times \{0\}} \\ B & \xrightarrow{\phi} & C(S^{n-1}, M_k) \end{array}$$

then $\text{Hom}(A, M_m)$ is also locally contractible for every m .

Therefore, let us take a point $\pi \in \text{Hom}(A, M_m)$. Up to unitary equivalence, π decomposes as

$$(2.69) \quad \pi = \text{diag}(\pi_B \circ \lambda, \pi_D \circ \rho),$$

where $\pi_B \in \text{Hom}(B, M_{m_B})$ and $\pi_D \in \text{Hom}_1(C_0(D^n \setminus S^{n-1}, M_k), M_{m_D})$, where $m_B + m_D = m$. Since π_D is a direct sum of finitely many irreducible representations, it actually comes from $\text{Hom}_1(C_0(Y_D, M_k), M_{m_D})$, for some compactly-contained open subset Y_D of $D^n \setminus S^{n-1}$. Let Y_B be an open neighbourhood of S^{n-1} in $D^n \setminus \bar{Y}$, such that Y_B can be identified with $S^{n-1} \times [0, 1)$, compatible with the identification of S^{n-1} with $S^{n-1} \times \{0\}$.

By Lemma 2.9, there exists a contractible neighbourhood U_D of π_D in

$$(2.70) \quad \text{Hom}_1(C_0(Y_D, M_k), M_{m_D});$$

let $\theta^D = (\theta_t^D): [0, 1] \times U_D \rightarrow U_D$ be a deformation retraction of U_D onto $\{\pi_D\}$. By the inductive hypothesis, π_B has a neighbourhood in $\text{Hom}(B, M_{m_B})$ which is contractible to π_B . Combining the deformation retraction provided by Lemma 2.10 with this contraction, we see that there exists a neighbourhood U_B of $\pi_B \circ \lambda$ in

$$(2.71) \quad \text{Hom}_1(C_0(Y_B, M_k) \oplus_{C(S^{n-1}, M_k)} B, M_{m_B})$$

and a deformation retraction $\theta^B = (\theta_t^B): [0, 1] \times U_B \rightarrow U_B$ of U_B onto $\{\pi_B \circ \lambda\}$.

Define U to be the set of $*$ -homomorphisms $A \rightarrow M_m$ of the form

$$(2.72) \quad \text{Ad}(u) \circ \text{diag}(\pi'_B, \pi'_D \circ \rho),$$

where $\pi'_B \in U_B$, $\pi'_D \in U_D$, and $u \in U_m$ such that $\|u - v\| < \epsilon$ for some $v \in U_m$ such that

$$(2.73) \quad \text{Ad}(v) \circ \pi = \pi.$$

It is not hard to see that U is an open neighbourhood of π , and by making U_D, Y_B , and ϵ sufficiently small, we turn U into an arbitrarily small neighbourhood of σ . In particular, by making U small enough, we ensure that if

$$(2.74) \quad \text{Ad}(u) \circ \text{diag}(\pi'_B, \pi'_D \circ \rho) = \text{Ad}(v) \circ \text{diag}(\pi'_B, \pi_D \circ \rho) \in U$$

where $\pi'_B \in U_B$ and $\pi'_D \in U_D$ then

$$(2.75) \quad \text{Ad}(u) \circ \pi = \text{Ad}(v) \circ \pi.$$

To show that U is a contractible neighbourhood of π , we imitate the steps of the proof of Lemma 2.9: first, we produce a deformation retraction of U onto the set W of all $*$ -homomorphisms $A \rightarrow M_m$ of the form

$$(2.76) \quad \text{Ad}(u) \circ \text{diag}(\pi_B \circ \lambda, \sigma_D \circ \rho),$$

where $u \in U_m$ is such that $\|u - v\| < \epsilon$ for some $v \in U_m$ such that

$$(2.77) \quad \text{Ad}(v) \circ \sigma = \sigma,$$

and then we contract W onto σ . \square

2.5. Restricted semiprojectivity of non-commutative cell complexes. We are now prepared to show the following, which says that NC cell complexes are semiprojective with respect to the class of C^* -algebras of the form $C(X, M_m)$, for any fixed m . (Note, of course, that most NC cell complexes do not belong to this class.)

Theorem 2.12. *Let A be an NC cell complex. Let X be a compact Hausdorff space, and Y a closed subspace. Let $\phi: A \rightarrow C(Y, M_m)$ be a unital $*$ -homomorphism. Then there exists a closed neighbourhood Z of Y in X such that ϕ lifts to a map $\tilde{\phi}: A \rightarrow C(Z, M_m)$, where by “lifts,” it is meant that $\phi = \tilde{\phi}(\cdot)|_Y$.*

Remark. Of course, the conclusion of this theorem, in the case that X is metrizable, is the same as saying that if $B := C(X, M_m)$, and I_n is an increasing sequence of ideals of B , then any unital $*$ -homomorphism $A \rightarrow B/\bigcup I_n$ lifts to a map $A \rightarrow B/I_n$ for some n . (It is stronger in the non-metrizable case.)

Proof of Theorem 2.12. Recall the definition of $\hat{\phi} \in C(Y, \text{Hom}(A, M_m))$ from Section 1.2, and that the existence of the lift $\tilde{\phi}$ corresponds to extending $\hat{\phi}$ to a continuous map $\tilde{\phi}: Z \rightarrow \text{Hom}(A, M_m)$. To prove

the theorem, we must therefore show that $\text{Hom}(A, M_m)$ is an absolute neighbourhood extensor with respect to the class of compact Hausdorff spaces (as defined in [16, II.2]), equivalently, an absolute neighbourhood retract [16, Lemma 5.1].

Since $\text{Hom}(A, M_m)$ is finite-dimensional (Corollary 2.8) and metrizable, it embeds into \mathbb{R}^n for some n [12, Theorem 1.11.4]. It is, in addition, compact, since it can be identified with a weak*-closed, bounded subset of $(A \otimes M_m)^*$ (by identifying a *-homomorphism $\phi: A \rightarrow M_m$ with the functional that maps $a \otimes e_{i,j}$ to the (i, j) -th entry of the matrix $\phi(a)$). Also, it is locally contractible by Proposition 2.11. Therefore, by [15, Theorem A.7], it is an absolute neighbourhood retract. \square

The following corollary says that the relations defining an NC cell complex A are stable in the class of C*-algebras of the form $C(X, M_m)$ (for fixed m).

Corollary 2.13. *Given an NC cell complex A , a finite set $\mathcal{F} \subset A$, a tolerance $\epsilon > 0$ and an integer $m \in \mathbb{N}$, there exists a finite set $\mathcal{G} \subset A$ and $\delta > 0$ such that the following holds: if X is a compact Hausdorff space and $\phi: A \rightarrow C(X, M_m)$ is a *-linear map that is (\mathcal{G}, δ) -multiplicative, then there is a *-homomorphism $\bar{\phi}: A \rightarrow C(X, M_m)$ such that*

$$(2.78) \quad \phi(a) \approx_\epsilon \bar{\phi}(a)$$

for all $a \in \mathcal{F}$.

Proof. This is a standard argument, involving lifting a map

$$(2.79) \quad A \rightarrow \prod_n C(X_n, M_m) / \bigoplus_n C(X_n, M_m).$$

See [25, Theorem 4.1.4]. \square

Corollary 2.14. *Let A be an NC cell complex and let $m \in \mathbb{N}$. For any finite subset $\mathcal{F} \subset A$ and $\epsilon > 0$ there exists a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ such that, if X is a compact Hausdorff space and*

$$(2.80) \quad \phi_0, \phi_1: A \rightarrow C(X, M_m)$$

are *-homomorphisms such that

$$(2.81) \quad \phi_0(a) \approx_\delta \phi_1(a)$$

for all $a \in \mathcal{G}$, then there exists a homotopy of *-homomorphisms, $\theta = (\theta_t): A \rightarrow C([0, 1]) \otimes C(X, M_m)$, such that $\theta_i = \phi_i$ for $i = 0, 1$ and

$$(2.82) \quad \theta_t(a) \approx_\epsilon \phi_0(a)$$

for all $a \in \mathcal{F}$ and $t \in [0, 1]$.

Proof. This argument is probably well-known, although we were unable to find a reference. It is a proof by contradiction. Suppose that the result is false for \mathcal{F} and $\epsilon > 0$. Let (a_i) be a dense sequence in A .

Then for every n we may find a compact space X_n and $*$ -homomorphisms

$$(2.83) \quad \phi_0^{(n)}, \phi_1^{(n)} : A \rightarrow C(X_n, M_k)$$

such that

$$(2.84) \quad \phi_0^{(n)}(a_i) \approx_{\frac{1}{n}} \phi_1^{(n)}(a_i), \quad i = 1, \dots, n$$

yet there is no $\theta = (\theta_t) : A \rightarrow C([0, 1]) \otimes C(X, M_k)$ such that $\theta_i = \phi_i^{(n)}$ for $i = 0, 1$ and

$$(2.85) \quad \theta_t(a) \approx_\epsilon \phi_0^{(n)}(a)$$

for all $x \in \mathcal{F}$ and $t \in [0, 1]$.

Define a c.p.c. map $\Phi^{(n)} : A \rightarrow C([0, 1]) \otimes C(X_n, M_k)$ by

$$(2.86) \quad \Phi^{(n)} := \text{id}_{[0,1]} \otimes \phi_1^{(n)} + (1 - \text{id}_{[0,1]}) \otimes \phi_0^{(n)}.$$

This is obviously not multiplicative, but since the $\phi_i^{(n)}$ are multiplicative and by (2.84), it induces a $*$ -homomorphism

$$(2.87) \quad \Phi : A \rightarrow \prod_n C([0, 1]) \otimes C(X_n, M_k) / \bigoplus_n C_0((0, 1)) \otimes C(X_n, M_k)$$

This map lifts to a $*$ -homomorphism

$$\tilde{\Phi} : A \rightarrow \prod_{n \geq n_0} C([0, 1]) \otimes C(X_n, M_k),$$

for some $n_0 \in \mathbb{N}$, giving a contradiction. \square

2.6. Approximating arbitrary subhomogeneous algebras. In this section, we prove the following approximation result.

Theorem 2.15 (Theorem B). *Let A be a unital, separable subhomogeneous algebra of topological dimension at most n . Then A is locally approximated by NC cell complexes of topological dimension at most n .*

Remark. Note that NC cell complexes are, by definition, unital, and so we could not expect the result above to generalize to non-unital subhomogeneous algebras. The result can, nonetheless, be applied to studying these non-unital algebras, by applying it to their unitizations.

We begin with a few technical lemmas.

Lemma 2.16. *Let A be given by the pull-back diagram*

$$(2.88) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C(X, M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C(X^{(0)}, M_k) \end{array}$$

in which B is locally approximated by subalgebras B_n . Define A_n by the pull-back diagram

$$(2.89) \quad \begin{array}{ccc} A_n & \longrightarrow & C(X, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B_n & \xrightarrow{\phi|_{B_n}} & C(X^{(0)}, M_k). \end{array}$$

Then A is locally approximated by the A_n .

Proof. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that, for each $a = (b, f) \in \mathcal{F}$, there exists $b_a \in B_n$ such that

$$(2.90) \quad b \approx_\epsilon b_a.$$

Then, since $\phi(b_a) \approx_\epsilon f|_{X^{(0)}}$, using the definition of the quotient norm, there exists $g_a \in C(X, M_k)$ such that

$$(2.91) \quad g_a|_{X^{(0)}} = \phi(b_a) \quad \text{and} \quad g_a \approx_\epsilon f.$$

Then, $(b_a, g_a) \in A_n$ and $a \approx_\epsilon (b_a, g_a)$. □

Lemma 2.17. *Let A be given by the pull-back diagram*

$$(2.92) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C(X, M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C(X^{(0)}, M_k), \end{array}$$

where B is an NC cell complex of topological dimension at most n , X is a CW complex of dimension at most n , and $X^{(0)}$ is a subcomplex. Then A is an NC cell complex of topological dimension at most n .

Proof. We prove this by induction on the dimension, m , of X .

Removing the interior of $X^{(0)}$ from X (and from $X^{(0)}$) does not change A at all, and allows $X^{(0)}$ to be replaced by its boundary (in X). Thus, we may assume that $X^{(0)}$ is an $(m-1)$ -dimensional subcomplex. Let $Y \subset X$ be the $(m-1)$ -skeleton of X , so that $X^{(0)}$ is a subcomplex of Y .

Define B' and ρ' by the following pull-back diagram:

$$(2.93) \quad \begin{array}{ccc} B' & \xrightarrow{\rho'} & C(Y, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C(X^{(0)}, M_k). \end{array}$$

By the inductive hypothesis, B' is an NC cell complex of topological dimension at most n . Moreover, it is not hard to see that the diagram

$$(2.94) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C(X, M_k) \\ \downarrow & & \downarrow f \mapsto f|_Y \\ B' & \xrightarrow{\rho'} & C(Y, M_k) \end{array}$$

commutes and is a pull-back diagram.

This reduces the problem to the case that $X^{(0)}$ is the m -skeleton of X . To deal with this case, we use induction on the number of m -cells in X . In the base case of this induction, X has no m -cells, and then $X = X^{(0)}$ and so $A = B$ and we are finished.

Otherwise, by singling out an m -cell of X , we obtain a pull-back diagram

$$(2.95) \quad \begin{array}{ccc} C(X) & \xrightarrow{\rho} & C(D^m) \\ \downarrow & & \downarrow f \mapsto f|_{S^{m-1}} \\ C(X') & \xrightarrow{\alpha} & C(S^{m-1}), \end{array}$$

where X' is a subcomplex of X given by deleting the interior of one m -cell, and α is a $*$ -homomorphism. This means that X' contains $X^{(0)}$ and has one fewer m -cell than X does.

Define B' and ρ' by the following pull-back diagram:

$$(2.96) \quad \begin{array}{ccc} B' & \xrightarrow{\rho'} & C(X', M_k) \\ \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow[\phi]{} & C(X^{(0)}, M_k). \end{array}$$

By the inductive hypothesis, B' is an NC cell complex of topological dimension at most n . Moreover, it is not hard to see that the diagram

$$(2.97) \quad \begin{array}{ccc} A & \longrightarrow & C(D^n, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{S^{n-1}} \\ B' & \xrightarrow[(\alpha \otimes \text{id}_{M_k}) \circ \rho']{} & C(S^{n-1}, M_k) \end{array}$$

is a pull-back diagram, and, therefore, A is an NC cell complex of topological dimension at most n . \square

Lemma 2.18. *Let A be given by the pull-back diagram*

$$(2.98) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C(X, M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C(X^{(0)}, M_k), \end{array}$$

where B is an NC cell complex of topological dimension at most n , and X is a simplicial complex of dimension at most n . Then A is an inductive limit of NC cell complexes of topological dimension at most n , where the maps of the inductive system are injective.

Proof. $X^{(0)}$ is a closed subset of a simplicial complex, and therefore it is the intersection of a decreasing sequence $(Y_n)_{n=1}^{\infty}$ of closed subsets, such that for each n , by changing the simplicial complex structure on X , Y_n can be viewed as a subcomplex. By Theorem 2.12, ϕ lifts to a map $\tilde{\phi}: B \rightarrow C(Y_{n_0}, M_k)$ for some n_0 .

It follows that A is an inductive limit (with injective connecting maps) of algebras A_n , defined by the pull-back diagrams

$$(2.99) \quad \begin{array}{ccc} A_n & \xrightarrow{\rho} & C(X, M_k) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{Y_n} \\ B & \xrightarrow{\tilde{\phi}(\cdot)|_{Y_n}} & C(Y_n, M_k), \end{array}$$

$n = 1, 2, \dots$. Indeed, as subalgebras of $B \oplus C(X, M_k)$, we have

$$(2.100) \quad A_1 \subseteq A_2 \subseteq \dots \quad \text{and} \quad A = \overline{\bigcup_i A_i}.$$

Moreover, by Lemma 2.17, A_n is an NC cell complex of topological dimension at most n . \square

Proof of Theorem 2.15. By [31, Theorem 2.16], A is a recursive subhomogeneous algebra. By induction on the length of the recursive subhomogeneous decomposition (see [31, Definition 1.2]), it suffices to show that if A is given by a pull-back diagram

$$(2.101) \quad \begin{array}{ccc} A & \longrightarrow & C(X, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C(X^{(0)}, M_k), \end{array}$$

where B is locally approximated by NC cell complexes of topological dimension at most n and $\dim X \leq n$, then A is locally approximated by images of NC cell complexes of topological dimension at most n . Using Lemma 2.16, we may assume that B is a single NC cell complex of topological dimension at most n .

Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Without loss of generality, since B is finitely generated, the finite set $\mathcal{G} := \{b \in B \mid (b, f) \in \mathcal{F}\}$ includes a set of generators of B . Let $\delta \in (0, \epsilon/3)$ be as provided by Corollary 2.14, with $m = k$ and $\epsilon/3$ in place of ϵ . By Corollary 2.13, there exists $\eta \in (0, \delta)$ such that, if $\alpha: B \rightarrow C$ is a *-homomorphism, D is a subalgebra of C , $\alpha(\mathcal{G}) \subset_\eta D$, and D is isomorphic to $C(W, M_k)$ for some space W , then there is a *-homomorphism $\hat{\alpha}: B \rightarrow D$ such that $\alpha(b) \approx_\delta \hat{\alpha}(b)$ for all $b \in \mathcal{G}$.

By Theorem 2.12, there exists a closed neighbourhood Y of $X^{(0)}$ such that ϕ lifts to a map $\tilde{\phi}: B \rightarrow C(Y, M_k)$. Replace Y by a smaller neighbourhood, if necessary, so that

$$(2.102) \quad \tilde{\phi}(b) \approx_\eta f|_Y, \quad a = (b, f) \in \mathcal{F}.$$

For each $a = (b, f) \in \mathcal{F}$, choose $f_a \in C(X, M_k)$ such that

$$(2.103) \quad f_a|_Y = \tilde{\phi}(b) \quad \text{and} \quad f_a \approx_\eta f.$$

Since $X \setminus Y^\circ$ has dimension at most n , by [12, Theorem 1.10.16] there exists an n -dimensional simplicial complex Γ and an injective *-homomorphism $\sigma: C(\Gamma, M_k) \rightarrow C(X \setminus Y^\circ, M_k)$ such that, for each $a = (b, f) \in \mathcal{F}$,

$$(2.104) \quad f_a|_{X \setminus Y^\circ} \approx_\eta \sigma(g_a)$$

for some $g_a \in C(\Gamma, M_k)$. Let $\bar{g}_a \in C(X, M_k)$ be such that

$$(2.105) \quad f_a \approx_\eta \bar{g}_a, \quad \bar{g}_a|_{X^{(0)}} = \phi(b), \quad \text{and} \quad \bar{g}_a|_{X \setminus Y^\circ} = \sigma(g_a).$$

Note then that for $a = (b, f) \in \mathcal{F}$,

$$(2.106) \quad \tilde{\phi}(b)|_{\partial Y} = f_a|_{\partial Y} \approx_\eta \bar{g}_a|_{\partial Y} \in \sigma(C(\Gamma, M_k))|_{\partial Y} =: D.$$

Let $\Gamma^{(0)}$ be the closed subset of Γ such that $C_0(\Gamma \setminus \Gamma)$ is the kernel of the map $\sigma(\cdot)|_{\partial Y}$. Then since D is isomorphic to $C(\Gamma^{(0)}, M_k)$, our choice of η ensures that there is a *-homomorphism $\theta_1: B \rightarrow D$ such that

$$(2.107) \quad \theta_1(b) \approx_\delta \bar{g}_a|_{\partial Y}$$

for all $a = (b, f) \in \mathcal{F}$. By Theorem 2.12, there is a closed neighbourhood W of ∂Y in Y such that $\theta_1: B \rightarrow D \subset C(\partial Y, M_k)$ extends to a map $B \rightarrow C(W, M_k)$, that we continue to denote θ_1 . Replace W by a smaller neighbourhood, if necessary, so that $\theta_1(b) \approx_\delta \tilde{\phi}(b)|_W$.

Corollary 2.14 now provides us with a homotopy $\theta = (\theta_t): B \rightarrow C([0, 1]) \otimes C(W, M_k)$, such that $\theta_0 = \tilde{\phi}(\cdot)|_W$, θ_1 agrees with the existing definition, and

$$(2.108) \quad \theta_t(b) \approx_{\epsilon/3} \tilde{\phi}(b)|_W, \quad b \in \mathcal{G}.$$

Note that ∂W (taken in X) is a disjoint union of ∂Y and the closed set $\partial W \setminus \partial Y$. Therefore, by Urysohn's Lemma, there exists a continuous function $f: W \rightarrow [0, 1]$ satisfying

$$(2.109) \quad f|_{\partial W \setminus \partial Y} \equiv 0 \quad \text{and} \quad f|_{\partial Y} \equiv 1.$$

Define $\psi: B \rightarrow C(Y, M_k)$ by

$$(2.110) \quad \psi(b)(y) = \begin{cases} \tilde{\phi}(b)(y), & \text{if } y \in Y \setminus W, \\ \theta_{f(y)}(b)(y), & \text{if } y \in W. \end{cases}$$

Since $\tilde{\phi}(b)|_{\partial W \setminus \partial Y} = \theta_0(b)|_{\partial W \setminus \partial Y}$, the function $\psi(b)$ is continuous. Also,

$$(2.111) \quad \psi(b) \approx_{\epsilon/3} \tilde{\phi}(b), \quad b \in \mathcal{G} \quad \text{and} \quad \psi(\cdot)|_{X^{(0)}} = \phi.$$

Since $\theta_1(B) \subseteq \sigma(C(\Gamma, M_k))$, $\psi(\cdot)|_{\partial Y}$ is a map from B to $\sigma(C(\Gamma, M_k))|_{\partial Y} \cong C(\Gamma^{(0)})$. Define A' by the pull-back diagram

$$(2.112) \quad \begin{array}{ccc} A' & \longrightarrow & C(\Gamma, M_k) \\ \downarrow & & \downarrow f \mapsto f|_{\Gamma^{(0)}} \\ B & \xrightarrow{\psi(\cdot)|_{\partial Y}} & C(\Gamma^{(0)}, M_k). \end{array}$$

By Lemma 2.18, A' is locally approximated by NC cell complexes of topological dimension at most n .

Let us now describe a map $\pi: A' \rightarrow A$. For $(b, g) \in A' \subseteq B \oplus C(\Gamma, M_k)$, define $f \in C(X, M_k)$ by

$$(2.113) \quad f|_{X \setminus Y^\circ} := \sigma(g)|_{X \setminus Y^\circ} \quad \text{and} \quad f|_{Y^\circ} := \psi(b).$$

The definition of B ensures that $\sigma(b)|_{\partial Y} = \psi(g)|_{\partial Y}$, so that f is indeed continuous. Note that

$$(2.114) \quad f|_{X^{(0)}} = \psi(b)|_{X^{(0)}} \stackrel{(2.111)}{=} \phi(b),$$

so $(b, f) \in A$; we define $\pi(b, g) := (b, f)$. Since σ is injective, f is only 0 if g is zero; consequently, π is injective.

Now, let us show that $\mathcal{F} \subset_{\epsilon} \pi(A')$. Let $a = (b, f) \in \mathcal{F}$. By (2.107) and (2.105),

$$(2.115) \quad g_a|_{\Gamma_0} \approx_{\epsilon/3} \theta_1(b)|_{\partial Y} = \psi(b)|_{\partial Y},$$

so let $g \in C(\Gamma, M_k)$ be such that

$$(2.116) \quad g|_{\Gamma_0} = \psi(b)|_{\partial Y} \quad \text{and} \quad g \approx_{\epsilon/3} g_a.$$

Then $(b, g) \in A'$; let us show that $\pi(b, g) \approx_{\epsilon} a$.

Let $\pi(b, g) = (b, f')$. Then

$$(2.117) \quad \begin{aligned} f'|_Y &\stackrel{(2.113)}{=} \psi(b) \\ &\stackrel{(2.111)}{\approx_{\epsilon/3}} \tilde{\phi}(b) \\ &\stackrel{(2.102)}{\approx_{\epsilon/3}} f, \end{aligned}$$

and

$$\begin{aligned}
 f'|_{X \setminus Y} &\stackrel{(2.113)}{=} \sigma(g)|_{X \setminus Z} \\
 &\stackrel{(2.116)}{\approx_{\epsilon/3}} \sigma(g_a)|_{X \setminus Z} \\
 &\stackrel{(2.104)}{\approx_{\epsilon/3}} f_a|_{X \setminus Z} \\
 (2.118) \quad &\stackrel{(2.103)}{\approx_{\epsilon/3}} f|_{X \setminus Z}.
 \end{aligned}$$

Thus, $\pi(b, g) = (b, f') \approx_{\epsilon} (b, f) = a$ as required. \square

Corollary 2.19. *Let A be a C^* -algebra. Then A is locally subhomogeneous if and only if it is locally approximated by NC cell complexes.*

Proof. Since NC cell complexes are subhomogeneous, the reverse direction is obvious. In the other direction, since a C^* -subalgebra of a subhomogeneous algebra is itself subhomogeneous, every locally subhomogeneous algebra is locally approximated by finitely generated, subhomogeneous algebras. Moreover, [27, Theorem 1.5] shows that every finitely generated, subhomogeneous algebra satisfies the hypothesis of Theorem 2.15, and is therefore itself locally approximated by NC cell complexes. \square

2.7. \mathcal{W} -stable approximately subhomogeneous algebras. We include another result here, which is proven using an adaptation of the proof of Theorem 2.15. Recall the stably projectionless, monotracial, K -theoretically trivial algebra \mathcal{W} studied in [17].

Theorem 2.20. *Let A be a separable locally subhomogeneous algebra. Then $A \otimes \mathcal{W}$ is an inductive limit of one-dimensional NC cell complexes (i.e., point-line algebras; see Remark 2.2 (iii)).*

We need some technical lemmas to prove this.

Lemma 2.21. *Suppose that*

$$(2.119) \quad 0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

is an extension of C^ -algebras, such that I and B are subhomogeneous. Then the topological dimension of A is the maximum of the topological dimension of I and of B .*

Proof. A is subhomogeneous, and for each k , $\text{Prim}_k(A)$ can be decomposed into an open set homeomorphic to $\text{Prim}_k(I)$ and a closed sets homeomorphic to $\text{Prim}_k(B)$. Therefore,

$$(2.120) \quad \dim \text{Prim}_k(A) = \max\{\dim \text{Prim}_k(I), \dim \text{Prim}_k(B)\},$$

and the result follows by the definition of topological dimension (Definition 1.3). \square

Proposition 2.22. *Let C be a $C(X)$ -algebra, let $X^{(0)} \subseteq X$ be closed and let A be given by the pull-back diagram*

$$(2.121) \quad \begin{array}{ccc} A & \xrightarrow{\rho} & C \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X^{(0)}} \\ B & \xrightarrow{\phi} & C|_{X^{(0)}}, \end{array}$$

where ϕ is a $*$ -homomorphism. If B is approximated by subhomogeneous algebras of topological dimension at most 1, and C is locally subhomogeneous, then A is locally subhomogeneous. Moreover, if C is approximated by subhomogeneous C^* -algebras of topological dimension at most $n \geq 1$, then so also is A .

Proof. This proof is along the same lines as the proof of Theorem 2.15. Let $\mathcal{F} \subset A$ be a finite subset that we wish to approximate, up to a tolerance $\epsilon > 0$. Without loss of generality, by Theorem 2.15 and Lemma 2.16, we may suppose that B is a one-dimensional NC cell complex. Choose a finite generating subset \mathcal{G} of B containing $\lambda(\mathcal{F})$.

Since B is (weakly) semiprojective [5] (see also [11, Remark 3.5]), there exists $\eta \in (0, \epsilon/3)$ such that, if $\alpha: B \rightarrow E$ is a $*$ -homomorphism and E' is a subalgebra of E such that $\alpha(\mathcal{G}) \subset_\eta E'$ then there is a $*$ -homomorphism $\hat{\alpha}: B \rightarrow E'$ such that $\alpha(b) \approx_{\epsilon/3} \hat{\alpha}(b)$ for all $b \in \mathcal{G}$. Also since B is semiprojective, there exists a closed neighbourhood Y of $X^{(0)}$ such that ϕ lifts to a map $\tilde{\phi}: B \rightarrow C|_Y$. Replace Y by a smaller neighbourhood if necessary, so that $\tilde{\phi} \circ \lambda(a) \approx_\eta \rho(a)|_Y$ for $a \in \mathcal{F}$.

For each $a = (b, f) \in \mathcal{F}$, choose $f_a \in C$ such that

$$(2.122) \quad f_a|_Y = \tilde{\phi}(b) \quad \text{and} \quad f_a \approx_\eta f.$$

Since C is locally subhomogeneous, there exists a subhomogeneous subalgebra D of C such that, for each $a \in \mathcal{G}$,

$$(2.123) \quad f_a \approx_\eta g_a$$

for some $g_a \in D$. Let Z be a closed neighbourhood of $X \setminus Y^\circ$, disjoint from $X^{(0)}$. Choose, for each $a = (b, f) \in \mathcal{F}$, an element $\bar{g}_a \in C$ such that

$$(2.124) \quad f_a \approx_\eta \bar{g}_a, \quad \bar{g}_a|_{X^{(0)}} = \phi(b)|_{X^{(0)}}, \quad \text{and} \quad \bar{g}_a|_Z = g_a|_Z \in D|_Z.$$

Consider the quotient of A by $C|_{X \setminus Y}$, which we denote by $B \oplus_{C|_{X^{(0)}}} C|_Y$. The pair $(\text{id}_B, \tilde{\phi})$ provides a $*$ -homomorphism $B \rightarrow B \oplus_{C|_{X^{(0)}}} C|_Y$, and the image of \mathcal{G} under this $*$ -homomorphism is contained, up to η , in the subalgebra

$$(2.125) \quad C^*(\{(b, \bar{g}_a|_Y) \mid a = (b, f) \in \mathcal{F}\}).$$

By the choice of η , it follows that there is a $*$ -homomorphism

$$(2.126) \quad \psi = (\psi_1, \psi_2): B \rightarrow C^*(\{(b, \bar{g}_a|_Y) \mid a = (b, f) \in \mathcal{F}\})$$

such that

$$(2.127) \quad \psi \circ \lambda(a) \approx_{\epsilon/3} (b, \bar{g}_a|_Y)$$

for all $a = (b, f) \in \mathcal{F}$.

Note that, for $b \in \mathcal{G}$, there exists $a \in \mathcal{F}$ such that $\lambda(a) = b$; thus $\psi_2(b)|_{Y \cap Z} \in C^*(\{\bar{g}_a|_{Y \cap Z}\}) \subseteq D|_{Y \cap Z}$. Since \mathcal{G} generates B , it follows that $\psi_2(B) \subseteq D|_{Y \cap Z}$; therefore, we may define E by the pull-back diagram

$$(2.128) \quad \begin{array}{ccc} E & \longrightarrow & D|_Z \\ \downarrow & & \downarrow f \mapsto f|_{Y \cap Z} \\ B & \xrightarrow{\psi_2(\cdot)|_{Y \cap Z}} & D|_{Y \cap Z}, \end{array}$$

which is clearly subhomogeneous since D and B are. Moreover, E can be identified with a subalgebra of A by

$$(2.129) \quad (b, d) \mapsto (\psi_1(b), c) \in B \oplus C$$

where $c|_Y := \psi_2(b)$ and $c|_Z := d$. This subalgebra approximates \mathcal{F} up to ϵ .

Applying Lemma 2.21 to E yields the final statement, about topological dimension. \square

Proof of Theorem 2.20. Since one-dimensional subhomogeneous algebras are approximated by one-dimensional NC cell complexes (Theorem 2.15), and one-dimensional NC cell complexes are semiprojective [5] (see also [11, Remark 3.5]), it suffices to show that $A \otimes \mathcal{W}$ is locally approximated by one-dimensional subhomogeneous algebras. For this, it suffices to assume that A is recursive subhomogeneous (in fact, by Theorem 2.15, we could assume it is an NC cell complex—but this isn't needed).

By induction on its decomposition, we need to show that if an algebra A is defined by a pull-back diagram as in (2.121) (with $X^{(0)}$ possibly empty), where $C := C(X, \mathcal{W})$, and B is approximated by one-dimensional subhomogeneous algebras, then A is approximated by one-dimensional subhomogeneous algebras. In this case, C is approximated by one-dimensional subhomogeneous algebras by [37, Theorem 1.4], and hence Proposition 2.22 implies that the same holds for A , as required. \square

3. DECOMPOSITION RANK

Recall the definition of the decomposition rank of a $*$ -homomorphism from [40, Definition 2.2]. If $A \subseteq B$ then we write $\text{dr}(A \subseteq B)$ to denote the decomposition rank of the inclusion map; when $A = B$, this is the same as the decomposition rank of B , whereas in the case of a first-factor embedding $A \otimes 1_D \subseteq A \otimes D$, where D is strongly self-absorbing,

this decomposition rank value relates to the decomposition rank of $A \otimes D$, see [40, Proposition 2.6].

In this section, we prove the following result:

Theorem 3.1. *Let A be an NC cell complex. Then for any unital C^* -algebra D ,*

$$(3.1) \quad \text{dr}(A \otimes 1_D \subseteq A \otimes D) \leq \sup \text{dr}(C(Z) \otimes 1_D \subseteq C(Z) \otimes D),$$

where the maximum is taken over all compact metrizable spaces Z of the same topological dimension as that of A .

In combination with the main result of [40] (and Theorem 2.15 to handle general subhomogeneous C^* -algebras), Theorem A is a consequence (see Corollary 3.20). It also provides an alternative proof of the main result of [45], see Corollary 3.21 (although both proofs are fairly technical, they are fundamentally different; A.T. and Wilhelm Winter found out the hard way that the argument in [45] probably cannot be adapted to prove Theorem 3.1).

3.1. A special case. We include the construction of c.p.c. approximations for special one-step NC cell complexes, to give an idea of how the general case works. The general case involves a significant amount of recursion and induction which can be avoided in this case.

Consider a separable C^* -algebra A which is given by a pull-back diagram

$$(3.2) \quad \begin{array}{ccc} A & \xrightarrow{\sigma} & C(X_1 \times [0, 1], M_{m_1}) \\ \lambda \downarrow & & \downarrow f \mapsto f|_{X_1 \times \{0\}} \\ C(X_0, M_{m_0}) & \xrightarrow{\theta} & C(X_1, M_{m_1}), \end{array}$$

where θ is a unital $*$ -homomorphism. This implies that $m_1 = lm_0$ for some $l \in \mathbb{N}$.

Recalling that $\hat{\theta}(x)$ means $\text{ev}_x \circ \theta$, define

$$(3.3) \quad S: \text{Hom}(C(X_0, M_{m_0}), M_{m_1}) \rightarrow \{\text{finite subsets of } X_0\}$$

by

$$(3.4) \quad S(\text{Ad}(u) \circ \text{diag}(\text{ev}_{z_1}, \dots, \text{ev}_{z_l})) := \{z_1, \dots, z_l\}$$

(note that we are taking a set, not a multiset, here). Of course, every unital $*$ -homomorphism $\alpha: C(X_0, M_{m_0}) \rightarrow M_{m_1}$ can be expressed as $\alpha = \text{Ad}(u) \circ \text{diag}(\text{ev}_{z_1}, \dots, \text{ev}_{z_l})$, and although this expression is not unique, the set $S(\alpha) = \{z_1, \dots, z_l\}$ is; in fact,

$$(3.5) \quad S(\alpha) = X_0 \setminus U$$

where $\ker(\alpha) = C_0(U, M_{m_0})$.

Define

$$(3.6) \quad Y := \{(x_1, x_0) \in X_1 \times X_0 \mid x_1 \in X_1 \text{ and } x_0 \in S(\hat{\theta}(x_1))\};$$

since S is continuous (in, say, the Hausdorff metric on finite subsets of X_0), it follows that Y is a closed, and therefore compact, subset of $X_1 \times X_0$.

Define $\bar{\xi}: C(Y, M_{m_0}) \rightarrow C(X_1, M_{m_1})$ as follows. For $x_1 \in X_1$ and $f \in C(Y, M_{m_0})$, let us express $\hat{\theta}(x_1)$ as

$$(3.7) \quad \hat{\theta}(x_1) = \text{Ad}(u) \circ \text{diag}(\text{ev}_{z_1}, \dots, \text{ev}_{z_l}),$$

and then define

$$(3.8) \quad \bar{\xi}(f)(x_1) := \text{Ad}(u)(\text{diag}(f(x_1, z_1), \dots, f(x_1, z_l))).$$

(One may check that this is well defined—it does not depend on the chosen representation of $\hat{\theta}(x)$ and $\bar{\xi}(f)$ is indeed continuous.) A critical property of $\bar{\xi}$ is that θ factors as

$$(3.9) \quad C(X_0, M_{m_0}) \rightarrow C(Y, M_{m_0}) \xrightarrow{\bar{\xi}} C(X_1, M_{m_1}),$$

where the first map is induced by the coordinate projection $Y \rightarrow X_0$.

Fix $\eta \in (0, 1)$ and define

$$(3.10) \quad Z_\eta := (X_0 \times \{0\}) \amalg (Y \times [0, \eta]) \amalg (X_1 \times [\eta, 1]) / \sim,$$

where \sim is the equivalence relation generated by the following:

(i) for $(x_1, x_0) \in Y$,

$$(3.11) \quad (x_0, 0) \sim ((x_1, x_0), 0); \text{ and}$$

(ii) for $(x_1, x_0) \in Y$,

$$(3.12) \quad ((x_1, x_0), \eta) \sim (x_1, \eta).$$

(It is easy to see that \sim is closed, and therefore Z_η is Hausdorff.)

These constructions will be used in the proof of Proposition 3.3 (a baby version of Theorem 3.1) below.

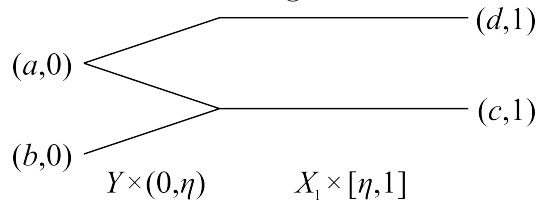
Example 3.2. This simple example illustrates the constructions. Let $X_0 := \{a, b\}$, $X_1 := \{c, d\}$, $m_0 = 1, m_1 = 2$, and define $\theta: C(X_0) \rightarrow C(X_1, M_2)$ by

$$(3.13) \quad \theta(f)(c) := \text{diag}(f(a), f(b)), \quad \theta(f)(d) := \text{diag}(f(a), f(a)).$$

In this case, we have $Y = \{(c, a), (c, b), (d, a)\}$ and $\bar{\xi}: C(Y) \rightarrow C(X_1, M_2)$ is given by

$$(3.14) \quad \bar{\xi}(f)(c) = \text{diag}(f(c, a), f(c, b)), \quad \bar{\xi}(f)(d) = \text{diag}(f(d, a), f(d, a)).$$

The space Z_η looks like the following:



Proposition 3.3. *Consider the C^* -algebra A given by the pull-back diagram (3.2) and let D be an infinite dimensional UHF algebra. Then $\text{dr}(A \otimes D) \leq 2$.*

Proof. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$; we must show that we can (\mathcal{F}, ϵ) -approximately factorize the identity map on A in a 3-colourable way. We may assume that \mathcal{F} consists of positive elements. Moreover, by approximating \mathcal{F} , we may assume that there exists $\eta > 0$ such that, for every $a = (f, g) \in \mathcal{F} \subset C(X_0, M_{m_0}) \oplus C(X_1 \times [0, 1], M_{m_1})$, $x \in X_1$, and $t \in [0, 2\eta]$,

$$(3.15) \quad g(x, t) = g(x, 0).$$

Set $Z := Z_\eta$ as defined in (3.10). Let us identify the spaces X_0 , $Y \times (0, \eta)$, and $X_1 \times [\eta, 1]$ with subspaces of Z .

Let \mathcal{U}_0 be an open cover of X_0 such that for every $(f, g) \in \mathcal{F} \subset C(X_0, M_{m_0}) \oplus C(X_1 \times [0, 1], M_{m_1})$ and every $U \in \mathcal{U}_0$, the restriction $f|_U$ is approximately constant (to within ϵ). Choose an open cover \mathcal{U}_1 of Z such that for $U \in \mathcal{U}_1$:

(a) If $U \cap X_0 \neq \emptyset$ then $U \cap (X \times [\eta, 1]) = \emptyset$, and there exists $V \in \mathcal{U}_0$ such that $U \cap X_0 \subseteq V$ and for every $((x, y), t) \in (Y \times (0, \eta)) \cap U$, it is the case that $y \in V$;

(b) If $U \cap (Y \times (0, \eta)) \neq \emptyset$ then $U \cap (X \times [2\eta, 1]) = \emptyset$, and there exist disjoint open sets $V_1, \dots, V_q \in \mathcal{U}_0$ such that for any $(x, t) \in U \cap (X \times [\eta, 2\eta])$ or $((x, y), t) \in U \cap (Y \times (0, \eta))$,

$$(3.16) \quad S(\hat{\theta}(x)) \subset V_1 \cup \dots \cup V_q; \text{ and}$$

(c) For every $(f, g) \in \mathcal{F} \subset C(X_0, M_{m_0}) \oplus C(X_1 \times [0, 1], M_{m_1})$, the restriction $g|_{U \cap (X \times [\eta, 1])}$ is approximately constant (to within ϵ).

Since the embedding $C(Z) \otimes 1_D \subset C(Z) \otimes D$ has decomposition rank at most 2 [40, Theorem 4.1], we may find a 3-colourable approximate (to within ϵ) finite partition of unity $(a_i)_{i=1}^r$ in $C(Z, D)_+$ subordinate to \mathcal{U}_1 (by [40, Proposition 3.2], where this concept is defined; see also Definition 3.17). By possibly rescaling, assume that $\sum a_i \leq 1_D$.

Let's now define our c.p.c. approximation

$$(3.17) \quad A \xrightarrow{\psi} F_1 \oplus \dots \oplus F_r \xrightarrow{\phi} D \otimes A,$$

by defining finite dimensional (in fact matrix) algebras F_i together with the parts of ψ and ϕ corresponding to each F_i . This is done in cases.

Case 1: The support of a_i intersects X_0 . Let $b_i \in C(Y \times [0, \eta], D)$ be given by composing a_i with the natural map $Y \times [0, \eta] \rightarrow Z$ (from the definition of Z). Using $\bar{\xi}: C(Y, M_{m_0}) \rightarrow C(X_1, M_{m_1})$ as defined in (3.8), identify the domain and codomain appropriately so as to view $\bar{\xi} \otimes \text{id}_{C([0, \eta]) \otimes D}$ as a map from $C(Y \times [0, \eta], D) \otimes M_{m_0}$ to $D \otimes C(X_1 \times [0, \eta], M_{m_1})$. Then, note that by condition (a) on the open cover \mathcal{U}_1 ,

$\bar{\xi}(b_i \otimes 1_{m_0})$ vanishes on $X_1 \times \{\eta\}$, i.e.,

$$(3.18) \quad (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_i \otimes 1_{m_0}) \in D \otimes C_0(X_1 \times [0, \eta], M_{m_1})$$

and

$$(3.19) \quad \begin{aligned} (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_i \otimes 1_{m_0})|_{X_1 \times \{0\}} &= (\bar{\xi} \otimes \text{id}_D)(b_i|_{Y \times \{0\}} \otimes 1_{m_0}) \\ &\stackrel{(3.9)}{=} (\theta \otimes \text{id}_D)(a_i|_{X_0} \otimes 1_{m_0}), \end{aligned}$$

where, again, we make appropriate identifications of domains and codomains of the maps involved. This shows that

$$(3.20) \quad e_i := (a_i|_{X_0} \otimes 1_{m_0}, (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_i \otimes 1_{m_0})) \in (D \otimes A)_+.$$

Set

$$(3.21) \quad F_i := M_{m_0}$$

and define the c.p.c. order zero map $\phi_i: M_{m_0} \rightarrow D \otimes A$ by

$$(3.22) \quad \phi_i(\kappa) := (a_i|_{X_0} \otimes \kappa, (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_i \otimes \kappa));$$

(the range of this map is contained in $D \otimes A$, for similar reasons as for e_i). Fix a choice of $x_i \in X_0 \cap \text{supp } a_i$, and use this to define $\psi_i: A \rightarrow M_k$ by

$$(3.23) \quad \psi_i(f, g) = f(x_i).$$

Case 2: The support of a_i intersects $Y \times (0, \eta)$, but not X_0 . Let $U \in \mathcal{U}_1$ contain the support of a_i and for this set, let $V_1, \dots, V_q \in \mathcal{U}_0$ be disjoint sets as in (b), which we relabel $V_{i,1}, \dots, V_{i,q(i)}$. We define $F_i := F_{i,1} \oplus \dots \oplus F_{i,q(i)}$ where we define $F_{i,j}$ (and the parts of ϕ, ψ relating to $F_{i,j}$) now. Let us therefore fix $j = 1, \dots, q(i)$.

Define $b_{i,j}: Y \times (0, 2\eta) \rightarrow D$ by

$$(3.24) \quad b_{i,j}((x, y), t) := \begin{cases} a_i((x, y), t), & \text{if } t \in (0, \eta) \text{ and } y \in V_{i,j}; \\ a_i(x, t), & \text{if } t \in [\eta, 2\eta) \text{ and } S(\hat{\theta})(x) \cap V_{i,j} \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

One may check (using (b)) that $b_{i,j}$ is continuous, and vanishes on $Y \times \{0, 2\eta\}$; therefore, we may view it as an element of

$$(3.25) \quad C_0(Y \times (0, 2\eta), D) \subset C_0(Y \times (0, 1], D).$$

We note, for future use, that for $((x, y), t) \in Y \times (0, 2\eta)$,

$$(3.26) \quad \sum_{j=1}^{q(i)} b_{i,j}((x, y), t) = \begin{cases} a_i((x, y), t), & \text{if } t \in (0, \eta); \\ a_i(x, t), & \text{if } t \in [\eta, 2\eta). \end{cases}$$

Define

$$(3.27) \quad e_{i,j} := (0_{X_0}, (\bar{\xi} \otimes \text{id}_{C([0,1]) \otimes D})(b_{i,j} \otimes 1_{m_0})) \in A_+.$$

Define the summand

$$(3.28) \quad F_{i,j} := M_{m_0}$$

and define $\phi_{i,j}: M_{m_0} \rightarrow D \otimes A$ by

$$(3.29) \quad \phi_{i,j}(\kappa) := (0_Y, (\bar{\xi} \otimes \text{id}_{C([0,1]) \otimes D})(b_{i,j} \otimes \kappa)).$$

Fix a choice of $y_{i,j} \in V_{i,j}$ and use this to define $\psi_{i,j}: A \rightarrow M_{m_0}$ by

$$(3.30) \quad \psi_{i,j}(f, g) = f(y_{i,j}).$$

Case 3: The support of a_i is contained in $X_1 \times (\eta, 1]$. In this case, $a_i|_{X_1 \times (\eta, 1]} \in C_0(X_1 \times (\eta, 1], D)_+ \subset C(X_1 \times [0, 1], D)_+$, so

$$(3.31) \quad e_i := (0_{X_0}, a_i|_{X_1 \times (\eta, 1]} \otimes 1_{m_1}) \in (D \otimes A)_+.$$

Define the summand

$$(3.32) \quad F_i := M_{m_1}$$

and define $\phi_i: M_{m_1} \rightarrow D \otimes A$ by

$$(3.33) \quad \phi_i(\kappa) := (0_{X_0}, a_i|_{X_1 \times (\eta, 1]} \otimes \kappa).$$

Choose $z_i \in \text{supp } a_i \subseteq X_1 \times (\eta, 1]$ and use this to define $\psi_i: A \rightarrow M_k$ by

$$(3.34) \quad \psi_i(f, g) := g(z_i).$$

Set G_1, G_2 and G_3 equal to the set of all i from Cases 1, 2 and 3 respectively. Thus, we have

$$(3.35) \quad F = \bigoplus_{i \in G_1} F_i \oplus \bigoplus_{i \in G_2} \bigoplus_{j=1}^{q(i)} F_{i,j} \oplus \bigoplus_{i \in G_3} F_i.$$

The maps ψ and ϕ are given as (direct) sums in the same way.

It is easy to see that whenever $a_i a_{i'} = 0$, we also have $e_i e_{i'} = 0$ (or $e_i e_{i',j'} = 0$, etc.), and that for $i \in G_2$, the elements $e_{i,1}, \dots, e_{i,q(i)}$ are mutually orthogonal. Therefore, ϕ is 3-colourable. Since $\sum a_i \leq 1_Z$, we also have

$$(3.36) \quad \phi(1_F) = \sum_{i \in G_1} e_i + \sum_{i \in G_2} \sum_{j=1}^{q(i)} e_{i,j} + \sum_{i \in G_3} e_i \leq 1_{D \otimes A},$$

and so ϕ is contractive.

Next, we must check that $\phi\psi(a) \approx 1_D \otimes a$ for $a = (f, g) \in \mathcal{F}$. We see that $\phi\psi(a) = (f', g')$ where

$$(3.37) \quad f' := \sum_{i \in G_1} (a_i|_{X_0} \otimes f(x_i))$$

and

$$\begin{aligned}
(3.38) \quad g' &:= \sum_{i \in G_1} (\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_i \otimes f(x_i)) + \\
&\quad \sum_{i \in G_2} \sum_{j=1}^{q(i)} (\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j})) + \\
&\quad \sum_{i \in G_3} a_i|_{X_1 \times [0,1]} \otimes g(z_i).
\end{aligned}$$

For $x \in X_0$, we have that $a_i(x) \neq 0$ only if $f(x_i) \approx_\epsilon f(x)$. Also, by the definition of G_1 , $\sum_{i \in G_1} a_i(x) = \sum_i a_i(x) \approx_\epsilon 1_D$. Therefore, it follows from (3.37) that $f'(x) \approx_{2\epsilon} 1_D \otimes f(x)$. Hence $\|f' - f\| \leq 2\epsilon$.

Next, let $(x, t) \in X_1 \times [0, \eta]$. Express $\hat{\theta}(x)$ as

$$(3.39) \quad \hat{\theta}(x) = \text{Ad}(u) \circ \text{diag}(\text{ev}_{z_1}, \dots, \text{ev}_{z_l}).$$

Then, for $i \in G_1$,

$$\begin{aligned}
(3.40) \quad &(\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_i \otimes f(x_i))(x, t) \\
&\stackrel{(3.8)}{=} \text{Ad}(1_D \otimes u)(\text{diag}(b_i((x, z_1), t) \otimes f(x_i), \dots, b_i((x, z_l), t) \otimes f(x_i))) \\
&= \text{Ad}(1_D \otimes u)(\text{diag}(a_i((x, z_1), t) \otimes f(x_i), \dots, a_i((x, z_l), t) \otimes f(x_i))).
\end{aligned}$$

using the definition of b_i . Likewise, for $i \in G_2$ and $j = 1, \dots, q(i)$,

$$\begin{aligned}
(3.41) \quad &(\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j}))(x, t) \\
&\stackrel{(3.8)}{=} \text{Ad}(1_D \otimes u)(\text{diag}(b_{i,j}((x, z_1), t) \otimes f(y_{i,j}), \dots, \\
&\quad b_{i,j}((x, z_l), t) \otimes f(y_{i,j}))).
\end{aligned}$$

Combining these, we find that

$$\begin{aligned}
(3.42) \quad g'(x, t) &= \sum_{i \in G_1} (\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_i \otimes f(x_i))(x, t) + \\
&\quad \sum_{i \in G_2} \sum_{j=1}^{q(i)} (\bar{\xi} \otimes \text{id}_{\mathbb{C}([0,\eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j}))(x, t) \\
&= \text{Ad}(1_D \otimes u)(\text{diag}(c_1, \dots, c_l))
\end{aligned}$$

where

$$\begin{aligned}
 c_k &:= \sum_{i \in G_1} a_i(x, z_k) \otimes f(x_i) + \\
 &\quad \sum_{i \in G_2} \sum_{j=1}^{q(i)} b_{i,j}((x, z_k), t) \otimes f(y_{i,j}) \\
 &\approx_\epsilon \sum_{i \in G_1} a_i(x, z_k) \otimes f(z_k) + \\
 &\quad \sum_{i \in G_2} \sum_{j=1}^{q(i)} b_{i,j}((x, z_k), t) \otimes f(z_k) \\
 (3.43) \quad &\approx_\epsilon 1_D \otimes f(z_k),
 \end{aligned}$$

using the fact that f is approximately constant on each set in \mathcal{U}_0 for the first approximation, and using (3.26) and that $\sum_{i \in G_1 \cup G_2} a_i(x, z_k) \approx_\epsilon 1$ for the second approximation. Putting (3.43) into (3.42) we obtain

$$\begin{aligned}
 g'(x, t) &\approx_{2\epsilon} \text{Ad}(1_D \otimes u)(\text{diag}(1_D \otimes f(z_1), \dots, 1_D \otimes f(z_k))) \\
 (3.44) \quad &\stackrel{(3.9)}{=} 1_D \otimes \theta(f)(x)
 \end{aligned}$$

$$(3.45) \quad \stackrel{(3.15)}{=} 1_D \otimes g(x, t).$$

Next, let $(x, t) \in X_1 \times [\eta, 2\eta)$. Express $\hat{\theta}(x)$ once again as in (3.39). For $i \in G_2$ and $j = 1, \dots, q(i)$, the computation (3.41) remains valid, and so

$$\begin{aligned}
 (3.46) \quad &\sum_{i \in G_2} \sum_{j=1}^{q(i)} (\bar{\xi} \otimes \text{id}_{C([0, \eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j}))(x, t) = \text{Ad}(1_D \otimes u)(\text{diag}(c_1, \dots, c_l))
 \end{aligned}$$

where

$$\begin{aligned}
 c_k &:= \sum_{i \in G_2} \sum_{j=1}^{q(i)} b_{i,j}((x, z_k), t) \otimes f(y_{i,j}) \\
 &\approx_\epsilon \sum_{i \in G_2} \sum_{j=1}^{q(i)} b_{i,j}((x, z_k), t) \otimes f(z_k) \\
 (3.47) \quad &\stackrel{(3.26)}{=} \sum_{i \in G_2} a_i(x, t) \otimes f(z_k),
 \end{aligned}$$

using the fact that f is approximately constant on each set in \mathcal{U}_0 , in the approximation. Thus we have

$$\begin{aligned}
& \sum_{i \in G_2} \sum_{j=1}^{q(i)} (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j}))(x, t) \\
& \approx_{\epsilon} \sum_{i \in G_2} a_i(x, t) \otimes \text{Ad}(u)(\text{diag}(f(z_1), \dots, f(z_k))) \\
& = \sum_{i \in G_2} a_i(x, t) \otimes \theta(f)(x) \\
(3.48) \quad & \stackrel{(3.9)}{=} \sum_{i \in G_2} a_i(x, t) \otimes g(x, t).
\end{aligned}$$

Using this and the fact that $\sum_{i \in G_2 \cup G_3} a_i(x, t) \approx_{\epsilon} 1_D$, we obtain

$$\begin{aligned}
g'(x, t) &= \sum_{i \in G_2} \sum_{j=1}^{q(i)} (\bar{\xi} \otimes \text{id}_{C([0,\eta]) \otimes D})(b_{i,j} \otimes f(y_{i,j}))(x, t) \\
&\quad + \sum_{i \in G_3} a_i(x, t) \otimes g(z_i) \\
&\approx_{\epsilon} \sum_{i \in G_2 \cup G_3} a_i(x, t) \otimes g(x, t) \\
(3.49) \quad &\approx_{\epsilon} 1_D \otimes g(x, t).
\end{aligned}$$

Finally, for $(x, t) \in X_1 \times [2\eta, 1]$, we have

$$(3.50) \quad g'(x, t) = \sum_{i \in G_3} a_i(x, t) \otimes g(z_i) \approx_{2\epsilon} 1_D \otimes g(x, t)$$

This concludes the verification that $1_D \otimes (f, g) \approx_{3\epsilon} \phi\psi(f, g)$.

This shows that $\text{dr}(A \otimes 1_D \subset A \otimes D) \leq 2$. Since D is UHF, it follows that $\text{dr}(A \otimes D) \leq 2$. \square

3.2. Set-up. Let us fix an NC cell complex A ; below we shall fix a decomposition of it. In subsequent sections, we shall produce a construction based on this fixed data.

For a topological space X , let CX denote the cone over X , defined by

$$(3.51) \quad X \times [0, 1] / \sim$$

where $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. Note that this is the commutative space corresponding to the unitization of the standard C^* -algebraic cone over $C(X)$. We view CX as the (non-topological) disjoint union of $X \times [0, 1)$ and $\{1\}$.

Let A have an NC cell decomposition, in which it is the k_0 -th stage, as follows. The first stage is $A_0 = C(CX_0, M_{m_0})$ and, for $k \geq 1$, the

k -th stage is A_k which satisfies the pull-back diagram

$$(3.52) \quad \begin{array}{ccc} A_k & \xrightarrow{\sigma_k} & C(CX_k, M_{m_k}) \\ \lambda_k \downarrow & & \downarrow f \mapsto f|_{X_k \times \{0\}} \\ A_{k-1} & \xrightarrow{\theta_k} & C(X_k, M_{m_k}), \end{array}$$

where each X_k is a sphere of some dimension. Thus, $A = A_{k_0}$. Occasionally, it will be convenient to use $\sigma_0: A_0 \rightarrow C(CX_0, M_{m_0})$ to denote the identity map on $C(CX_0, M_{m_0})$.

We will frequently use the notation

$$(3.53) \quad \lambda_k^{k'} := \lambda_{k'+1} \circ \cdots \circ \lambda_k: A_k \rightarrow A_{k'}$$

for $k' \leq k$.

Presently we will initiate a construction that will lead to c.p.c. approximations for the inclusion $A_{k_0} \rightarrow A_{k_0} \otimes D$. These will involve tuples of parameters $(\eta_1, \dots, \eta_{k_0}) \in (0, 1)^{k_0}$. We will frequently abbreviate this tuple as $\bar{\eta}$, and also use $\bar{\eta}$ to denote (η_1, \dots, η_k) for values $k < k_0$ where appropriate.

Throughout this section, we will use the notation x_k (or x'_k , etc.) to denote a point in X_k , v_k to denote a point in CX_k , and similar notation for points in sets $Z_{k,\bar{\eta}}$ and $Y_{k,\bar{\eta}}$ to be introduced in Section 3.4. This is not to be confused with the k^{th} entry from a tuple or sequence of points. When we wish to refer to an element of a tuple or sequence of points in X_k , we use the notation $x_k^{(i)}$ (where the tuple or sequence is indexed over i).

Remark 3.4. In the construction to follow, it is not important that the spectrum of A_0 is a cone; nor is it important that X_k is a sphere. It only matters that at each stage (after A_0), the space is a cone where the gluing occurs at the “wide end” of the cone. (In fact, what is pertinent is that the gluing space is a neighbourhood retract.) However, no stronger theorem (than Theorem A) is obtained by replacing CX_0 and X_k by more general spaces.

3.3. Approximating subalgebras. Let us now define approximating subalgebras $A_{k,(\eta_1, \dots, \eta_k)}$ of A_k , parametrized by $\bar{\eta} = (\eta_1, \dots, \eta_k) \in (0, 1)^k$. Set

$$(3.54) \quad A_{0,() } := A_0.$$

Having defined $A_{k-1,\bar{\eta}}$, define $A_{k,\bar{\eta}}$ to consist of $a \in A_k$ such that

$$(3.55) \quad \begin{aligned} & \lambda_k(a) \in A_{k-1,\bar{\eta}}, \text{ and} \\ & \sigma_k(a)(x, t) = \sigma_k(a)(x, 0), \text{ for all } x \in X, t \in [0, \eta_k]. \end{aligned}$$

Evidently, A_k is locally approximated by the subalgebras $A_{k,\bar{\eta}}$.

3.4. Construction of $Z_{k,\bar{\eta}}$. Let us now construct Hausdorff spaces $Y_{k,(\eta_1,\dots,\eta_{k-1})}$ (for $k \geq 1$ and $\eta_1, \dots, \eta_{k-1} \in (0, 1)$) and $Z_{k,(\eta_1,\dots,\eta_k)}$ (for $k \geq 0$ and $\eta_1, \dots, \eta_{k-1} \in (0, 1)$) recursively, together with a map

$$(3.56) \quad S_{k,(\eta_1,\dots,\eta_k),m} : \text{Hom}(A_k, M_m) \rightarrow \{\text{finite subsets of } Z_{k,\bar{\eta}}\};$$

we emphasize that the values of this map are sets, not multisets. Define

$$(3.57) \quad Z_{0,() } := CX_0,$$

and for $\alpha \in \text{Hom}(A_0, M_m)$, define $S_{0,(),m}(\alpha)$ to consist of all $x_0 \in CX_0$ for which ev_{x_0} is equivalent to a subrepresentation of α .

Now, having defined $Z_{k-1,\bar{\eta}}$ and $S_{k-1,\bar{\eta},m}$, set

$$(3.58) \quad Y_{k,\bar{\eta}} := \{(x_k, z_{k-1}) \in X_k \times Z_{k-1,\bar{\eta}} \mid z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k))\},$$

where we recall that $\hat{\theta}_k : X_k \rightarrow \text{Hom}(A_{k-1}, M_{m_k})$ is the continuous map induced by $\theta_k : A_{k-1} \rightarrow C(X_k, M_{m_k})$. Next, for $\eta_k > 0$, set

$$(3.59) \quad Z_{k,\bar{\eta}} := ((Z_{k-1,\bar{\eta}} \times \{0\}) \amalg (Y_{k,\bar{\eta}} \times [0, \eta_k]) \amalg (X_k \times [\eta_k, 1])) / \sim,$$

where \sim is the equivalence relation generated by the following:

(i) For every $z_{k-1} \in Z_{k-1,\bar{\eta}}$ and $(x_k, z_{k-1}) \in Y_{k,\bar{\eta}}$,

$$(3.60) \quad (z_{k-1}, 0) \sim ((x_k, z_{k-1}), 0);$$

(ii) For every $x_k \in X_k$ and $(x_k, z_{k-1}) \in Y_{k,\bar{\eta}}$,

$$(3.61) \quad ((x_k, z_{k-1}), \eta_k) \sim (x_k, \eta_k); \quad \text{and}$$

(iii) For every $x_k, x'_k \in X_k$,

$$(3.62) \quad (x_k, 1) \sim (x'_k, 1).$$

As this equivalence relation is closed, $Z_{k,\bar{\eta}}$ is a Hausdorff space. Identify $Z_{k-1,\bar{\eta}} \times \{0\}$, $Y_{k,\bar{\eta}} \times (0, \eta_k)$, $X_k \times [\eta_k, 1)$, and $\{1\}$ in the natural way with subsets of $Z_{k,\bar{\eta}}$; note that $Z_{k,\bar{\eta}}$ is in fact the (non-topological) disjoint union of these four subsets.

Now let us define maps

$$(3.63) \quad S_{k,\bar{\eta},m} : \text{Hom}(A_k, M_m) \rightarrow \{\text{finite subsets of } Z_{k,\bar{\eta}}\}.$$

For $\alpha \in \text{Hom}(A_k, M_m)$, let $S_{k,\bar{\eta},m}(\alpha)$ be the set consisting of all $z_k \in Z_{k,\bar{\eta}}$ for which one of the following properties obtains:

(i) $z_k = 1$ and $\text{ev}_1 \circ \sigma_k$ is equivalent to a subrepresentation of α ;

(i)' $z_k = (x_k, t) \in X_k \times [\eta_k, 1)$ and $\text{ev}_{(x_k,t)} \circ \sigma_k$ is equivalent to a subrepresentation of α ;

(ii) $z_k = ((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta_k)$ and $\text{ev}_{(x_k,t)} \circ \sigma_k$ is equivalent to a subrepresentation of α ; or

(iii) $z_k = (z_{k-1}, 0) \in Z_{k-1,\bar{\eta}}$ and α contains a subrepresentation that factors as

$$(3.64) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\alpha'} M_{m'},$$

such that $z_{k-1} \in S_{k-1,\bar{\eta},m'}(\alpha')$.

This concludes the inductive definition of the spaces Y , Z , and the maps S .

We may regard $Y_{k,\bar{\eta}}$ as an “ X -fibred” space, with its fibres being finite subsets of $Z_{k-1,\bar{\eta}}$. For a subset A of $Y_{k,\bar{\eta}}$ and for $x_k \in X_k$, let us write $A(x_k)$ to denote the set

$$(3.65) \quad \{z_{k-1} \in Z_{k-1,\bar{\eta}} \mid (x_k, z_{k-1}) \in A\},$$

and for $B \subseteq X_k$, let us also consider the subset

$$(3.66) \quad A|_B := \{(z_{k-1}, x_k) \mid x_k \in B \text{ and } z_{k-1} \in A(x_k)\} \subseteq A.$$

(Note that, while $A(x_k)$ can be canonically identified with $A|_{\{x_k\}}$, they are technically different.) Let us also regard $Z_{k,\bar{\eta}}$ and CX_k as $[0, 1]$ -fibred spaces, and use similar notation: for $A \subseteq Z_{k,\bar{\eta}}$,

$$(3.67) \quad \begin{aligned} A(0) &:= \{z_{k-1} \mid (z_{k-1}, 0) \in A\} \subseteq Z_{k-1,\bar{\eta}}, \\ A(t) &:= \{(x_k, z_{k-1}) \mid ((x_k, z_{k-1}), t) \in A\} \subseteq Y_{k,\bar{\eta}}, \quad t \in (0, \eta), \\ A(t) &:= \{x_k \mid (x_k, t) \in A\} \subseteq X_k, \quad t \in [\eta, 1], \text{ and} \\ A(1) &:= \{*\}; \end{aligned}$$

and for $B \subseteq [0, 1]$,

$$(3.68) \quad A|_B := \{(w, t) \mid t \in [0, 1] \text{ and } w \in A(t)\} \subseteq A.$$

It is clear how to define similar notation for subsets of CX_k .

Remark 3.5. A more explicit description of the points in $Z_{k,\bar{\eta}}$ is as follows. The points of $Z_{k,\bar{\eta}}$ are tuples $(v_{k_1}, \dots, v_{k_p})$ where

- (i) $v_{k_i} \in CX_{k_i}$ with $0 \leq k_1 < \dots < k_p \leq k$;
- (ii) $v_{k_1} \in CX_{k_1}|_{[\eta_{k_1}, 1]}$ (or CX_0 in case $k_1 = 0$);
- (iii) $v_{k_i} = (x_{k_i}, t_i) \in CX_{k_i}|_{(0, \eta_{k_i})}$ for $i = 1, \dots, p$; and
- (iv) $\text{ev}_{v_{k_i}} \circ \sigma_{k_i} \circ \lambda_{k_{i+1}}^{k_i} : A_{k_{i+1}-1} \rightarrow M_{k_i}$ is a subrepresentation of $\theta_{k_{i+1}}(x_{k_{i+1}})$ for $i = 0, \dots, p-1$.

We will refer to this description at times, although we do not use it exclusively because the topology is more accessible using the other description.

Under this description, for $\alpha \in \text{Hom}(A_k, M_m)$, the set $S_{k,\bar{\eta},m}(\alpha)$ consists of $z_k = (v_{k_1}, \dots, v_{k_p})$ for which $\text{ev}_{v_{k_p}} \circ \sigma_{k_p} \circ \lambda_k^{k_p}$ is equivalent to a subrepresentation of α .

Lemma 3.6. *Each map*

$$(3.69) \quad S_{k,\bar{\eta},m} : \text{Hom}(A_k, M_m) \rightarrow \{\text{finite subsets of } Z_{k,\bar{\eta}}\}$$

is continuous, where we use the Hausdorff metric on the set of finite subsets of $Z_{k,\bar{\eta}}$.

Proof. Implicitly, we are fixing a metric d on $Z_{k,\bar{\eta}}$, giving rise to a Hausdorff metric. (By compactness, the topology does not depend on this choice of metric).

Step 1. Let $\alpha \in \text{Hom}(A_k, M_m)$ and let $\epsilon > 0$; in this step, we show that for each $z_k \in S_{k,\bar{\eta},m}(\alpha)$, there is a neighbourhood U of α such that, for all $\alpha' \in U$, there exists $z'_k \in S_{k,\bar{\eta},m}(\alpha')$ with $d(z_k, z'_k) < \epsilon$. We prove this inductively on k , thereby allowing us to assume it is true for $k-1$. (The base case, where $k=0$, is straightforward.)

Consider cases:

(i) If $z_k = 1$ or $z_k = (x_k, t)$ for $t \in (\eta_k, 1)$, then this means that $\text{ev}_{z_k} \circ \sigma_k$ is equivalent to a subrepresentation of α . We may then choose U to be a neighbourhood of α such that every $\alpha' \in U$ contains some subrepresentation $\text{ev}_{z'_k} \circ \sigma_k$, where $z'_k \in CX_k|_{(\eta_k, 1]}$ and $d(z_k, z'_k) < \epsilon$ (inside $Z_{k,\bar{\eta}}$). Thus, for such α' , $z'_k \in S_{k,\bar{\eta},m}(\alpha')$.

(ii) If $z_k = (x_k, \eta_k)$, then this means that $\text{ev}_{z_k} \circ \sigma_k$ is equivalent to a subrepresentation of α . This is essentially the same as Case (i), but we need to be a bit more careful. Choose U_x and U_t to be neighbourhoods of x_k and t inside X_k and $(0, 1)$ respectively, such that the set

$$(3.70) \quad (Y_{k,\bar{\eta}}|_{U_x} \times (U_t \cap (0, \eta_k))) \cup (U_x \times (U_t \cap [\eta_k, 1]))$$

is contained in the ball of radius ϵ about z_k . We may then choose U to be a neighbourhood of α such that every $\alpha' \in U$ contains some subrepresentation $\text{ev}_{(x'_k, t')} \circ \sigma_k$, where $x'_k \in U_x$ and $t' \in U_t$. Thus, for such α' , if $t' \geq \eta_k$ then $z'_k := (x'_k, t') \in S_{k,\bar{\eta},m}(\alpha')$ and $d(z_k, z'_k) < \epsilon$. Otherwise, $t' < \eta_k$, and for any $z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k))$, we have

$$(3.71) \quad z'_k := ((x'_k, z_{k-1}), t') \in S_{k,\bar{\eta},m}(\alpha')$$

and $d(z_k, z'_k) < \epsilon$.

(iii) If $z_k = ((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta_k)$, then this means that $\text{ev}_{(x_k, t)} \circ \sigma_k$ is equivalent to a subrepresentation of α and that $z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k))$. Let U_x, U_z , and U_t be neighbourhoods of x_k, z_{k-1} , and t inside $X_k, Z_{k-1,\bar{\eta}}$, and $(0, \eta_k)$ respectively, such that

$$(3.72) \quad ((U_x \times U_z) \cap Y_{k,\bar{\eta}}) \times U_t$$

is contained in the ball of radius ϵ about z_k .

By induction and by continuity of $\hat{\theta}_k$, there is a neighbourhood U'_x of x_k inside X_k such that, for $x'_k \in U'_x$, there exists $z'_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k))$ such that $z'_{k-1} \in U_z$. We may suppose that $U'_x \subseteq U_x$. We may then choose U to be a neighbourhood of α such that every $\alpha' \in U$ contains some subrepresentation $\text{ev}_{(x'_k, t')} \circ \sigma_k$, where $x'_k \in U'_x$ and $t' \in U_t$. For such α' , it follows that there exists $z'_k := ((x'_k, z'_{k-1}), t') \in S_{k,\bar{\eta},m}(\alpha)$ with $d(z'_k, z_k) < \epsilon$.

(iv) If $z_k = (z_{k-1}, 0) \in Z_{k-1,\bar{\eta}} \times \{0\}$, then this means that α contains a subrepresentation that factors as

$$(3.73) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\beta} M_n,$$

in such a way that $z_{k-1} \in S_{k-1,\bar{\eta},n}(\beta)$.

By induction, we may pick a neighbourhood V of β in $\text{Hom}(A_{k-1}, M_n)$ such that for every $\beta' \in V$, there exists $z'_{k-1} \in S_{k-1, \bar{\eta}, n}(\beta')$ with

$$(3.74) \quad d((z_{k-1}, 0), (z'_{k-1}, 0)) < \epsilon/2$$

(inside $Z_{k, \bar{\eta}}$).

If β is not contained as a subrepresentation of $\hat{\theta}_k(x_k)$ for any x_k , then we may choose U to be a neighbourhood of α such that every $\alpha' \in U$ contains a subrepresentation that factors as

$$(3.75) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\beta'} M_n,$$

with $\beta' \in V$.

Otherwise, β is contained as a subrepresentation of $\hat{\theta}_k(x_k)$ for some x_k (possibly more than one), and we therefore have a gluing

$$(3.76) \quad (z_{k-1}, 0) \sim ((x_k, z_{k-1}), 0)$$

for any such x_k . We may then choose U to be a neighbourhood of α such that for every $\alpha' \in U$, one of the following holds:

(a) α' contains a subrepresentation that factors as

$$(3.77) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\beta'} M_n,$$

with $\beta' \in V$. In this case, there exists $z'_{k-1} \in S_{k-1, \bar{\eta}, n}(\beta')$ with

$$(3.78) \quad d((z_{k-1}, 0), (z'_{k-1}, 0)) < \epsilon/2.$$

By construction, $(z'_{k-1}, 0) \in S_{k, \bar{\eta}, m}(\alpha')$.

(b) α' contains a subrepresentation of the form $\text{ev}_{(x'_k, t)} \circ \sigma_k$ where $t \in (0, \eta_k)$ and $\hat{\theta}_k(x'_k)$ contains a subrepresentation that factors as

$$(3.78) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\beta'} M_n,$$

with $\beta' \in V$, and therefore there exists $z'_{k-1} \in S_{k-1, \bar{\eta}, n}(\beta')$ with

$$(3.78) \quad d((z_{k-1}, 0), (z'_{k-1}, 0)) < \epsilon/2.$$

We may also ask (by making U small enough) that in this case, $d(((x'_k, z'_{k-1}), t), (z'_{k-1}, 0)) < \epsilon/2$. By construction, $z'_k := ((x'_k, z'_{k-1}), t) \in S_{k, \bar{\eta}, m}(\alpha')$ and

$$(3.78) \quad d(z'_k, z_k) \leq d(z'_k, (z'_{k-1}, 0)) + d((z'_{k-1}, 0), z_k) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This concludes Step 1.

Step 2. Let $\alpha \in \text{Hom}(A_k, M_m)$. To show continuity at α , we must show that for $\epsilon > 0$, there is a neighbourhood U of α such that, for all $\alpha' \in U$, the Hausdorff distance from $S_{k, \bar{\eta}, m}(\alpha)$ to $S_{k, \bar{\eta}, m}(\alpha')$ is at most ϵ . This can be done using Step 1 and a pigeonhole principle argument.

Namely, we may associate a non-zero dimension to each $z_k \in S_{k, \bar{\eta}, m}(\alpha)$ such that the sum of these dimensions (counted with multiplicity) is exactly m . To be precise, if $z_k = (v_{k_1}, \dots, v_{k_p})$ as in Remark 3.5, then the ‘‘dimension’’ of z_k is $D(z_k) := m_{k_1}$. Then, with a bit of care, in Step 1, we achieve, for each $z_k \in S_{k, \bar{\eta}, m}(\alpha)$, a neighbourhood U_{z_k} of α such that for $\alpha' \in U_{z_k}$, there exist $z_k^{(1)}, \dots, z_k^{(q)} \in S_{k, \bar{\eta}, m}(\alpha')$ for some

q (counting with multiplicity) with $d(z_k, z_k^{(i)}) < \epsilon$ for each i , and such that $D(z_k^{(1)}) + \cdots + D(z_k^{(q)}) = D(z_k)$. Set

$$(3.78) \quad U := \bigcap_{z_k \in S_{k, \bar{\eta}, m}(\alpha)} U_{z_k}.$$

Then the pigeonhole principle implies that for $\alpha' \in U$, every $z'_k \in S_{k, \bar{\eta}, m}(\alpha')$ is of the form $z_k^{(i)}$ as above for some $z_k \in S_{k, \bar{\eta}, m}(\alpha)$, and so $d(z_k, z'_k) < \epsilon$. \square

3.5. Pointwise finite dimensional approximations. For $z_k = (v_{k_1}, \dots, v_{k_p}) \in Z_{k, \bar{\eta}}$ (using the notation of Remark 3.5), define $F_{k, \bar{\eta}, z_k} := M_{m_{k_1}}$ and define the *-homomorphism $\nu_{k, \bar{\eta}, z_k} : A_k \rightarrow F_{k, \bar{\eta}, z_k}$ by

$$(3.79) \quad \nu_{k, \bar{\eta}, z_k}(a) := (\sigma_{k_1} \circ \lambda_{k_1}^{k_1}(a))(v_{k_1}).$$

Next, we construct, for each $\alpha \in \text{Hom}(A_k, M_m)$ and $z_k \in S_{k, \bar{\eta}, m}(\alpha)$, a *-homomorphism $\mu_{k, \bar{\eta}, \alpha, z_k} : F_{k, \bar{\eta}, z_k} \rightarrow M_m$. This gives us two *-homomorphisms

$$(3.80) \quad A_k \xrightarrow{\nu_{k, \bar{\eta}, z_k}} F_{k, \bar{\eta}, z_k} \xrightarrow{\mu_{k, \bar{\eta}, \alpha, z_k}} M_m;$$

ultimately, our c.p.c. approximations used to prove Theorem 3.1 will be built out of these two maps; in fact, the c.p.c. approximations will look like

$$(3.81) \quad \begin{array}{ccc} A_{k_0} & \xrightarrow{\text{id}_{A_{k_0}} \otimes 1_D} & A_{k_0} \otimes D, \\ & \searrow \psi & \nearrow \phi \\ & & F \end{array}$$

where F will be a direct sum of certain $F_{k_0, \bar{\eta}, z_{k_0}}$ and ψ will be a corresponding direct sum of the $\nu_{k_0, \bar{\eta}, z_{k_0}}$. The components of the map ϕ will be built out of σ_k and certain $\mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}$.

The construction of $\mu_{k, \bar{\eta}, \alpha, z_k}$ is recursive in k .

For $k = 0$, $A_0 = C(CX_0, M_{m_0})$ and if $z_0 \in S_{0, (\cdot), m}(\alpha)$ then this means that $z_0 \in CX_0$ and ev_{z_0} is equivalent to a subrepresentation of α . We have $F_{0, (\cdot), z_0} = M_{m_0}$ and $\nu_{0, (\cdot), z_0} = \text{ev}_{z_0}$. Denote by α' the largest subrepresentation of α that is equivalent to a multiple of ev_{z_0} , which means that there exists a unique $\mu_{0, (\cdot), \alpha, z_0}$ such that (3.80) is a factorization of α' .

For $k > 0$, the definition breaks into cases.

(i) $z_k \in Z_{k, \bar{\eta}}|_{[\eta_k, 1]} = CX_k|_{[\eta_k, 1]}$: This means that $\text{ev}_{z_k} \circ \sigma_k$ is equivalent to a subrepresentation of α . We handle this in exactly the same way as for $k = 0$. We have $F_{k, \bar{\eta}, z_k} = M_{m_k}$ and $\nu_{k, \bar{\eta}, z_k} = \text{ev}_{z_k} \circ \sigma_k$. Let α' denote the largest subrepresentation of α that is equivalent to a multiple of $\text{ev}_{z_k} \circ \sigma_k$, and we define $\mu_{k, \bar{\eta}, \alpha, z_k}$ to be the unique *-homomorphism for

which (3.80) is a factorization of α' ; thus we have a commuting triangle

$$(3.82) \quad \begin{array}{ccc} A_k & \xrightarrow{\alpha'} & M_m. \\ & \searrow \nu_{k,\bar{\eta},z_k} & \nearrow \mu_{k,\bar{\eta},\alpha,z_k} \\ & & F_{k,\bar{\eta},z_k} \end{array}$$

(ii) $z_k = ((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta_k)$: In this case, $F_{k,\bar{\eta},z_k} = F_{k-1,\bar{\eta},z_{k-1}}$. Also, $z_{k-1} \in S(\hat{\theta}_k(x_k))$, so by recursion, we have a map $\mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}} : F_{k,\bar{\eta},z_k} \rightarrow M_{m_k}$. We also know that $\text{ev}_{(x_k,t)}$ is equivalent to a subrepresentation α . Let α' denote the largest subrepresentation that is equivalent to a multiple of $\text{ev}_{(x_k,t)}$, and factorize it as

$$(3.83) \quad A_k \xrightarrow{\text{ev}_{(x_k,t)}} M_{m_k} \xrightarrow{\mu'} M_m.$$

Now, set

$$(3.84) \quad \mu_{k,\bar{\eta},\alpha,z_k} := \mu' \circ \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}} : F_{k,\bar{\eta},z_k} \rightarrow M_m.$$

Diagrammatically (and leaving out some subscripts), we have

$$(3.85) \quad \begin{array}{ccccc} A_k & \xrightarrow{\nu_k} & F_k = F_{k-1} & \xrightarrow{\mu_k} & M_m \\ \lambda_k \downarrow & \nearrow \nu_{k-1} & & \searrow \mu_{k-1} & \uparrow \mu' \\ A_{k-1} & \xrightarrow{\alpha'} & & & M_{m_k}, \end{array}$$

where the bottom triangle approximately commutes (it exactly commutes when restricted to $A_{k-1,\bar{\eta}}$, as defined in Section 3.3).

(iii) $z_k = (z_{k-1}, 0) \in Z_{k,\bar{\eta}}$. In this case, $F_{k,\bar{\eta},z_k} = F_{k-1,\bar{\eta},z_{k-1}}$ and $\nu_{k,\bar{\eta},z_k} = \nu_{k-1,\bar{\eta},z_{k-1}} \circ \lambda_k$. Then, α contains a subrepresentation that factors as

$$(3.86) \quad A_k \xrightarrow{\lambda_k} A_{k-1} \xrightarrow{\alpha'} M_{m'} \xrightarrow{\mu'} M_m,$$

such that α' is unital, μ' is injective (usually non-unital), in such a way that $z_{k-1} \in S_{k-1,\bar{\eta},m'}(\alpha')$. Take the largest subrepresentation of this form, i.e., let m' be maximal satisfying these conditions (note that this uniquely determines α' , up to unitary equivalence).

By recursion we have a map $\mu_{k-1,\bar{\eta},\alpha',z_{k-1}} : F_{k,\bar{\eta},z_k} \rightarrow M_{m'}$. Set

$$(3.87) \quad \mu_{k,\bar{\eta},\alpha,z_k} := \mu' \circ \mu_{k-1,\bar{\eta},\alpha',z_{k-1}} : F_{k,\bar{\eta},z_k} \rightarrow M_m.$$

Diagrammatically, we have the commuting diagram (3.85) (with m' in place of m_k), except that in this case, all triangles commute exactly.

Lemma 3.7. *For $\alpha \in \text{Hom}(A_k, M_m)$, and $a \in A_{k,\bar{\eta}}$ (as defined in Section 3.3),*

$$(3.88) \quad \alpha(a) = \sum_{z_k \in S_{k,\bar{\eta},m}(\alpha)} \mu_{k,\bar{\eta},\alpha,z_k} \circ \nu_{k,\bar{\eta},z_k}(a).$$

In particular,

$$(3.89) \quad \mu_{k,\bar{\eta},\alpha,z_k}(1_{F_{k,\bar{\eta},z_k}}) \cdot \alpha(a) = \mu_{k,\bar{\eta},\alpha,z_k} \circ \nu_{k,\bar{\eta},z_k}(a)$$

and

$$(3.90) \quad (\mu_{k,\bar{\eta},\alpha,z_k}(1_{F_{k,\bar{\eta},z_k}}))_{z_k \in S_{k,\bar{\eta},m}(\alpha)}$$

is a partition of unity in M_m (consisting of orthogonal projections).

Proof. This is proven by induction on k . In the case $k = 0$, it is trivial.

Suppose that the statement holds for $k - 1$. Let $\alpha \in \text{Hom}(A_k, M_m)$. The map α decomposes as follows:

$$(3.91) \quad \alpha = \text{Ad}(u) \circ \text{diag}(\text{ev}_{v_1} \circ \sigma_k, \text{ev}_{v_p} \circ \sigma_k, \alpha' \circ \lambda_k),$$

where $v_1, \dots, v_p \in CX_k|_{(0,1]}$ and $\alpha' \in \text{Hom}(A_{k-1}, M_{m'})$ for some m' .

By possibly reordering, arrange that

$$(3.92) \quad v_1, \dots, v_{p'} \in CX_k|_{[\eta_k,1]} \quad \text{and} \quad v_{p'+1}, \dots, v_p \in CX_k|_{(0,\eta_k)}.$$

Let us assume that v_1, \dots, v_p are distinct; this is only to simplify notation—if the v_i are not distinct, the following argument still works provided we keep track of multiplicities.

For $i = 1, \dots, p'$, we have $v_i \in S_{k,\bar{\eta},m}(\alpha)$ and

$$(3.93) \quad \begin{aligned} F_{k,\bar{\eta},v_i} &= M_{m_k}, \\ \nu_{k,\bar{\eta},v_i} &= \text{ev}_{v_i} \circ \sigma_k, \\ \mu_{k,\bar{\eta},\alpha,v_i} &= \text{Ad}(u) \circ \\ &\text{diag}(0_{m_k}, \dots, 0_{m_k}, \text{id}_{M_{m_k}}, 0_{m_k}, \dots, 0_{m_k}, 0_{m'}), \end{aligned}$$

by (3.82). Thus, for $a \in A_k$,

$$(3.94) \quad \mu_{k,\bar{\eta},\alpha,v_i} \circ \nu_{k,\bar{\eta},v_i}(a) = e_i \alpha(a),$$

where

$$(3.95) \quad e_i := \text{Ad}(u)(\text{diag}(0_{m_k}, \dots, 0_{m_k}, 1_{m_k}, 0_{m_k}, \dots, 0_{m_k}, 0_{m'})),$$

with the 1_{m_k} appearing in the i th position.

For $i = p' + 1, \dots, p$, write $v_i = (x_k^{(i)}, t_i)$. Then for every $z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k^{(i)}))$,

$$(3.96) \quad ((x_k^{(i)}, z_{k-1}), t_i) \in S_{k,\bar{\eta},m}(\alpha).$$

By induction, for $a \in A_{k,\bar{\eta}}$, since $\lambda_k(a) \in A_{k-1,\bar{\eta}}$,

$$(3.97) \quad \begin{aligned} &\hat{\theta}_k(x_k^{(i)})(\lambda_k(a)) \\ &= \sum_{z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k^{(i)}))} \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k^{(i)},z_{k-1})} \circ \nu_{k-1,\bar{\eta},z_{k-1}}(\lambda_k(a)). \end{aligned}$$

By (3.84), for $z_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x_k^{(i)}))$,

$$(3.98) \quad \begin{aligned} \nu_{k, \bar{\eta}, ((x_k^{(i)}, z_{k-1}), t_i)} &= \nu_{k-1, z_{k-1}} \circ \lambda_k, \text{ and} \\ \mu_{k, \bar{\eta}, \alpha, ((x_k^{(i)}, z_{k-1}), t_i)} &= \text{Ad}(u) \circ \text{diag}(0_{m_k}, \dots, 0_{m_k}, \mu_{k-1, \hat{\theta}_k(x_k^{(i)}, z_{k-1})}, \\ &\quad 0_{m_k}, \dots, 0_{m_k}, 0_{m'}). \end{aligned}$$

Thus, defining e_i again as in (3.95), for $a \in A_{k, \bar{\eta}}$, we have

$$(3.99) \quad \begin{aligned} e_i \alpha(a) &= \text{Ad}(u) \circ \text{diag}(0, \dots, 0, \text{ev}_{(x_k^{(i)}, t)} \circ \sigma_k(a), 0, \dots, 0, 0) \\ &\stackrel{(3.55)}{=} \text{Ad}(u) \circ \text{diag}(0, \dots, 0, \text{ev}_{x_k^{(i)}} \circ \theta_k \circ \lambda_k(a), 0, \dots, 0, 0) \\ &\stackrel{(3.97)}{=} \text{Ad}(u) \circ \text{diag}(0, \dots, 0, \\ &\quad \sum_{z_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x_k^{(i)}))} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k^{(i)}, z_{k-1})} \circ \nu_{k-1, \bar{\eta}, z_{k-1}}(\lambda_k(a)), \\ &\quad 0, \dots, 0, 0) \\ &\stackrel{(3.98)}{=} \sum_{z_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x_k))} \mu_{k, \bar{\eta}, \alpha, ((x_k, z_{k-1}), t)} \circ \nu_{k, \bar{\eta}, ((x_k, z_{k-1}), t)}(a). \end{aligned}$$

Next let us consider α' . $S_{k, \bar{\eta}, m}(\alpha)$ contains $S_{k-1, \bar{\eta}, m'}(\alpha') \times \{0\}$, and by induction, for $a \in A_{k, \bar{\eta}}$, since $\lambda_k(a) \in A_{k-1, \bar{\eta}}$,

$$(3.100) \quad \alpha'(\lambda_k(a)) = \sum_{z_{k-1} \in S_{k-1, \bar{\eta}, m'}(\alpha')} \mu_{k-1, \bar{\eta}, \alpha', z_{k-1}} \circ \nu_{k-1, \bar{\eta}, z_{k-1}} \circ \lambda_k(a).$$

By (3.87), for $z_{k-1} \in S_{k-1, \bar{\eta}, m'}(\alpha')$,

$$(3.101) \quad \begin{aligned} \nu_{k, \bar{\eta}, (z_{k-1}, 0)} &= \nu_{k-1, \bar{\eta}, z_{k-1}} \circ \lambda_k, \text{ and} \\ \mu_{k, \bar{\eta}, \alpha, (z_{k-1}, 0)} &= \text{Ad}(u) \circ \text{diag}(0_{m_k}, \dots, 0_{m_k}, \mu_{k-1, \bar{\eta}, \alpha', z_{k-1}}). \end{aligned}$$

Define

$$(3.102) \quad e' := \text{Ad}(u) \circ \text{diag}(0_{m_k}, \dots, 0_{m_k}, 1_{m'}).$$

Then, for $a \in A_{k, \bar{\eta}}$,

$$(3.103) \quad \begin{aligned} e \alpha(a) &= \text{Ad}(u) \circ \text{diag}(0, \dots, 0, \alpha' \circ \lambda_k(a)) \\ &\stackrel{(3.100)}{=} \text{Ad}(u) \circ \text{diag}(0, \dots, 0, \\ &\quad \sum_{z_{k-1} \in S_{k-1, \bar{\eta}, m'}(\alpha')} \mu_{k-1, \bar{\eta}, \alpha', z_{k-1}} \circ \nu_{k-1, \bar{\eta}, z_{k-1}} \circ \lambda_k(a)) \\ &\stackrel{(3.101)}{=} \sum_{z_{k-1} \in S_{k-1, \bar{\eta}, m'}(\alpha')} \mu_{k, \bar{\eta}, \alpha, (z_{k-1}, 0)} \circ \nu_{k, \bar{\eta}, (z_{k-1}, 0)}(a). \end{aligned}$$

Putting these together, for $a \in A_{k,\bar{\eta}}$ we have

$$\begin{aligned}
& \sum_{z_k \in S_{k,\bar{\eta},m}(\alpha)} \mu_{k,\bar{\eta},\alpha,z_k} \circ \nu_{k,\bar{\eta},z_k}(a) \\
&= \sum_{i=1}^{p'} \mu_{k,\bar{\eta},\alpha,v_i} \circ \nu_{k,\bar{\eta},v_i}(a) + \sum_{i=p'+1}^p \\
&\quad \sum_{z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k^{(i)}))} \mu_{k,\bar{\eta},\alpha,((x_k^{(i)}, z_{k-1}), t_i)} \circ \nu_{k,\bar{\eta},((x_k^{(i)}, z_{k-1}), t_i)}(a) \\
&\quad + \sum_{z_{k-1} \in S_{k-1,\bar{\eta},m'}(\alpha')} \mu_{k,\bar{\eta},\alpha,(z_{k-1},0)} \circ \nu_{k,\bar{\eta},(z_{k-1},0)}(a) \\
&\stackrel{(3.94),(3.99)}{=} \sum_{i=1}^{p'} e_i \alpha(a) + \sum_{i=p'+1}^p e_i \alpha(a) + e' \alpha(a) \\
&\stackrel{(3.95)}{=} \sum_{i=1}^{p'} e_i \alpha(a) + \sum_{i=p'+1}^p e_i \alpha(a) + e' \alpha(a) \\
&\stackrel{(3.102)}{=} \alpha(a). \tag{3.104}
\end{aligned}$$

Since $\mu_{k,\bar{\eta},\alpha,z_k}$ is a $*$ -homomorphism, (3.90) is a family of projections. \square

3.6. A partition of $Z_{k,\bar{\eta}}$. We now partition $Z_{k,\bar{\eta}}$ into sets

$$(3.105) \quad Z_{k,\bar{\eta},0}^*, \dots, Z_{k,\bar{\eta},k}^*,$$

where we define $Z_{k,\bar{\eta},m,k'}^*$ by fixing k' and using recursion in k , from k' to k_0 .

For $k = k' = 0$, set

$$(3.106) \quad Z_{0,(),0}^* := Z_{0,()}.$$

For $k = k' > 0$, set

$$(3.107) \quad Z_{k,\bar{\eta},k}^* := Z_{k,\bar{\eta}}|_{[\eta_k,1]}.$$

For $k > k'$, define $Z_{k,\bar{\eta},k'}^*$ to consist of:

(i) $((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta_k)$ such that

$$(3.108) \quad z_{k-1} \in Z_{k-1,\bar{\eta},k'}^*; \text{ and}$$

(ii) $(z_{k-1}, 0) \in Z_{k-1,\bar{\eta}} \times \{0\}$ such that

$$(3.109) \quad z_{k-1} \in Z_{k-1,\bar{\eta},k'}^*.$$

Likewise, let us define

$$\begin{aligned}
 (3.110) \quad Z_{0,(),0}^{**} &:= Z_{0,()} \\
 Z_{k,\bar{\eta},k}^{**} &:= Z_{k,\bar{\eta}}|_{(\eta_k,1]}, \quad k > 0 \quad (\text{compare, } Z_{k,\bar{\eta},k}^* = Z_{k,\bar{\eta}}|_{[\eta_k,1]}), \\
 Z_{k,\bar{\eta},k'}^{**} &:= \{((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta_k) \mid z_{k-1} \in Z_{k-1,\bar{\eta},k'}^{**}\} \\
 &\cup \{(z_{k-1}, 0) \in Z_{k-1,\bar{\eta}} \times \{0\} \mid z_{k-1} \in Z_{k-1,\bar{\eta},k'}^{**}\}.
 \end{aligned}$$

Observe that $Z_{k,\bar{\eta},k'}^{**} \subseteq Z_{k,\bar{\eta},k'}^*$ and $Z_{k,\bar{\eta},k'}^{**}$ is open in $Z_{k,\bar{\eta}}$.

Lemma 3.8. (i) *In the notation of Remark 3.5, $Z_{k,\bar{\eta},k'}^*$ consists of points $z_k = (v_{k_1}, \dots, v_{k_p}) \in Z_{k,\bar{\eta}}$ for which $k_1 = k'$; for $k' > 0$, the points in $Z_{k,\bar{\eta},k'}^{**}$ are the ones that additionally satisfy $v_{k_1} \in CX_{k_1}|_{(\eta_{k_1},1]}$.*

(ii)

$$(3.111) \quad Z_{k,\bar{\eta}} = \prod_{k'=0}^k Z_{k,\bar{\eta},k'}^*$$

Proof. (i) is easily proven by induction on k , and (ii) follows immediately. \square

The map $(\alpha, z_k) \mapsto \mu_{k,\bar{\eta},\alpha,z_k}$ has a certain continuity property, which is expressed in the following lemma.

Lemma 3.9. *For $\alpha \in \text{Hom}(A_k, M_m)$ and*

$$(3.112) \quad z_k \in S_{k,\bar{\eta},m}(\alpha) \cap Z_{k,\bar{\eta},k'}^{**},$$

*and for $\epsilon > 0$, there exist neighbourhoods U_{z_k} (respectively U_α) of z_k (α) in $Z_{k,\bar{\eta},k'}^{**}$ ($\text{Hom}(A_k, M_m)$) such that the following approximation holds: For every $\alpha' \in U_\alpha$ and every contraction $\kappa \in M_{m_{k'}}$,*

$$(3.113) \quad \mu_{k,\bar{\eta},\alpha,z_k}(\kappa) \approx_\epsilon \sum_{z'_k \in S_{k,\bar{\eta},m}(\alpha') \cap U_{z_k}} \mu_{k,\bar{\eta},\alpha',z'_k}(\kappa).$$

Proof. Decompose α :

$$(3.114) \quad \alpha = \text{Ad}(u) \circ \text{diag}(\text{ev}_{v_1} \circ \sigma_k, \dots, \text{ev}_{v_p} \circ \sigma_k, \beta \circ \lambda_k),$$

where $v_1, \dots, v_p \in CX_k|_{(0,1]}$, and $\beta \in \text{Hom}(A_{k-1}, M_{m'})$ for $m' = m - pm_k$.

Let us prove the statement inductively on $k \geq k'$. For $k = k' = 0$, it is quite easy—and uses the same idea as the case $k = k' > 0$ which is done in detail next.

For $k = k' > 0$, assume that the v_i are ordered so that $v_1 = \dots = v_{p'} = z_k$ and $v_i \neq z_k$ for $i = p' + 1, \dots, p$ (thus $p' \geq 1$).

Let U_{z_k} be a neighbourhood of z_k in $CX_k|_{(\eta_k,1]}$ such that $v_i \notin \bar{V}$ for $i = p' + 1, \dots, p$ (we identify this with a subset of $Z_{k,\bar{\eta},k}^{**}$). Then define U_α to consist of all $\alpha' \in \text{Hom}(A_k, M_m)$ that are of the form

$$(3.115) \quad \alpha' = \text{Ad}(u') \circ \text{diag}(\text{ev}_{v'_1} \circ \sigma_k, \dots, \text{ev}_{v'_{p'}} \circ \sigma_k, \beta)$$

where u' is a unitary near to u , $v'_1, \dots, v'_{p'} \in U_{z_k}$, and $\beta \in \text{Hom}(A_k, M_{m-p'm_k})$ does not contain $\text{ev}_v \circ \sigma_k$ as a subrepresentation for any $v \in \bar{V}$. Then U_α is a neighbourhood of α .

For $\alpha' \in U_\alpha$, let us write α' as in (3.115), so

$$(3.116) \quad S_{k, \bar{\eta}, m}(\alpha') \cap U_{z_k} = \{v'_1, \dots, v'_{p'}\}.$$

Thus,

$$(3.117) \quad \begin{aligned} & \sum_{z'_k \in S_{k, \bar{\eta}, m}(\alpha') \cap U_{z_k}} \mu_{k, \bar{\eta}, \alpha', z'_k}(\kappa) \\ &= \sum_{i=1}^{p'} \mu_{k, \bar{\eta}, \alpha', v'_i}(\kappa) \\ &= \text{Ad}(u')(\text{diag}(\kappa, \dots, \kappa, 0_{m-p'm_k})) \\ &\approx_\epsilon \text{Ad}(u)(\text{diag}(\kappa, \dots, \kappa, 0_{m-p'm_k})) \\ &= \mu_{k, \bar{\eta}, z_k}(\kappa), \end{aligned}$$

thus establishing (3.113).

For $k > k'$, let us consider two cases.

(i) $z_k = ((x_k, z_{k-1}), t) \in Y_{k, \bar{\eta}} \times (0, \eta_k)$: In this case, $z_{k-1} \in Z_{k-1, \bar{\eta}, k'}^{**}$. Referring to the decomposition in (3.114), we may assume that the v_i are ordered so that $v_1 = \dots = v_{p'} = (x_k, t)$ and $v_i \neq (x_k, t)$ for $i = p' + 1, \dots, p$.

By induction and by continuity of $\hat{\theta}_k$, there exist neighbourhoods $U_{z_{k-1}}, U_{x_k}$ of z_{k-1}, x_k respectively (the neighbourhood U_{x_k} arising from a neighbourhood of $\hat{\theta}_k(x_k)$), such that, for every $x'_k \in U_{x_k}$ and every contraction $\kappa \in M_{m_{k'}}$,

$$(3.118) \quad \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}(\kappa) \approx_{\epsilon/p} \sum_{z'_{k-1} \in S_{k, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k)) \cap U_{z_{k-1}}} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x'_k), z'_{k-1}}(\kappa).$$

Choose a neighbourhood $V \subset U_{x_k} \times (0, \eta_k)$ of (x_k, t) such that $v_i \notin \bar{V}$ for $i = p' + 1, \dots, p$. Define

$$(3.119) \quad U_{z_k} := \{((x'_k, z'_{k-1}), t') \in Y_{k, \bar{\eta}} \times (0, \eta_k) \mid (x'_k, t') \in V, z'_{k-1} \in U_{z_{k-1}}\}.$$

Then define U_α to consist of all $\alpha' \in \text{Hom}(A_k, M_m)$ with a decomposition

$$(3.120) \quad \alpha' = \text{Ad}(u') \circ \text{diag}(\text{ev}_{v'_1} \circ \sigma_k, \dots, v'_{p'} \circ \sigma_k, \beta)$$

where u' is a unitary near to u , $v'_1, \dots, v'_{p'} \in V$, and $\beta \in \text{Hom}(A, M_{m-p'm_k})$ does not contain $\text{ev}_v \circ \sigma_k$ as a subrepresentation for any $v \in \bar{V}$. Our set-up ensures that U_α is a neighbourhood of α .

To see (3.113), let $\alpha' \in U_\alpha$ be as in (3.120) and let $v'_i = (x_k^{(i)}, t_i)$ for each i , so that

$$(3.121) \quad S_{k,\bar{\eta},m}(\alpha') \cap U_{z_k} = \bigcup_{i=1}^n \{(x_k^{(i)}, z'_{k-1}), t_i \mid z'_{k-1} \in R_i\},$$

where $R_i := \{z'_{k-1} \in U_{z_{k-1}} \mid (x_k^{(i)}, z'_{k-1}) \in Y_{k,\bar{\eta}}\}$. For $z'_{k-1} \in R_i$, by (3.84) we have

$$(3.122) \quad \begin{aligned} & \mu_{k,\bar{\eta},\alpha',((x_k^{(i)}, z'_{k-1}), t_i)}(\kappa) = \\ & \text{Ad}(u') \circ \text{diag}(0, \dots, 0, \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k^{(i)}, z'_{k-1})}(\kappa), 0, \dots, 0, 0_{m-p'm_k}), \end{aligned}$$

where the non-zero entry is in the i^{th} position. By (3.118), it follows that

$$(3.123) \quad \begin{aligned} & \sum_{z'_{k-1} \in R_i} \mu_{k,\bar{\eta},\alpha',((x_k^{(i)}, z'_{k-1}), t_i)}(\kappa) \\ & \approx_\epsilon \text{Ad}(u')(\text{diag}(0, \dots, 0, \mu_{k-1,\bar{\eta},\theta_k(x_k^{(i)}, z_{k-1})}(\kappa), 0, \dots, 0, 0_{m-p'm_k})). \end{aligned}$$

Thus,

$$(3.124) \quad \begin{aligned} & \sum_{z'_k \in S_{k,\bar{\eta},m}(\alpha') \cap U_{z_k}} \mu_{k,\bar{\eta},\alpha',z'_k}(\kappa) \\ & = \sum_{i=1}^{p'} \sum_{z'_{k-1} \in R_i} \mu_{k,\bar{\eta},\alpha',((x_k^{(i)}, z'_{k-1}), t_i)}(\kappa) \\ & \approx_\epsilon \text{Ad}(u')(\text{diag}(1_{p'} \otimes \mu_{k-1,\bar{\eta},\theta_k(x_k^{(i)}, z_{k-1})}(\kappa), 0_{m-p'm_k})) \\ & \stackrel{(3.84)}{=} \mu_{k,\bar{\eta},\alpha}(\kappa) \end{aligned}$$

(the approximation is within ϵ because the errors, each of norm at most ϵ , are orthogonal), which establishes (3.113).

(ii) $z_k = (z_{k-1}, 0) \in Y_{k,\bar{\eta}} \times (0, \eta_k)$: Then $z_{k-1} \in Z_{k-1,\bar{\eta},k'}^{**} \cap S_{k-1,\bar{\eta},m'}(\beta)$ (referring to the decomposition (3.114)). By induction, there exist neighbourhoods $U_{z_{k-1}}$ of z_{k-1} in $Z_{k-1,\bar{\eta},k'}^{**}$ and U_β of β in $\text{Hom}(A_{k-1}, M_{m'})$ such that for every $\beta' \in U_\beta$ and every contraction $\kappa \in M_{m_{k'}}$,

$$(3.125) \quad \begin{aligned} & \mu_{k-1,\bar{\eta},\beta',z_{k-1}}(\kappa) \\ & \approx_{\epsilon/3} \sum_{z'_{k-1} \in S_{k,\bar{\eta},m'}(\beta') \cap U_{z_{k-1}}} \mu_{k-1,\bar{\eta},\beta',z'_{k-1}}(\kappa). \end{aligned}$$

Here we need to be particularly careful about our choice of neighbourhoods, to take into account the possibility that $\beta \circ \lambda_k$ can be

decomposed differently, involving point evaluations on $CX_k|_{\{0\}}$. For each $q \in \mathbb{N}$, consider the set

$$(3.126) \quad \begin{aligned} V_q := & \{(v, x_k^{(1)}, \dots, x_k^{(q)}, \gamma) \in U(M_{m'}) \times X_k^q \times \text{Hom}(A_{k-1}, M_{m'-qm_k}) \mid \\ & \text{Ad}(v) \circ \text{diag}(\text{ev}_{(x_k^{(1)}, 0)} \circ \sigma_k, \dots, \text{ev}_{(x_k^{(q)}, 0)} \circ \sigma_k, \gamma \circ \lambda_k) \in U_\beta \circ \lambda_k\}. \end{aligned}$$

Since

$$(3.127) \quad \begin{aligned} (v, x_k^{(1)}, \dots, x_k^{(q)}, \gamma) \mapsto & \text{Ad}(v) \circ \text{diag}(\text{ev}_{(x_k^{(1)}, 0)} \circ \sigma_k, \dots, \\ & \text{ev}_{(x_k^{(q)}, 0)} \circ \sigma_k, \gamma \circ \lambda_k) \end{aligned}$$

is a continuous map into $\text{Hom}(A_{k-1}, M_{m'}) \circ \lambda_k$, the set V_q is open. Let $\eta' < \eta_k$ be such that $v_1, \dots, v_p \in CX_k|_{(\eta', 1]}$. Define

$$(3.128) \quad \begin{aligned} W := & \bigcup_{q \in \mathbb{N}} \{\text{Ad}(v) \circ \text{diag}(\text{ev}_{(x_k^{(1)}, t_1)} \circ \sigma_k, \dots, \text{ev}_{(x_k^{(q)}, t_q)} \circ \sigma_k, \gamma \circ \lambda_k) \mid \\ & (v, x_k^{(1)}, \dots, x_k^{(q)}, \gamma) \in V_q \text{ and } t_1, \dots, t_q \in [0, \eta')\}, \end{aligned}$$

so that W is a neighbourhood of $\beta \circ \lambda_k$ in $\text{Hom}(A_k, M_{m'})$.

Set

$$(3.129) \quad \begin{aligned} U_\alpha := & \{\text{Ad}(v) \circ \text{diag}(\text{ev}_{w_1} \circ \sigma_k, \dots, \text{ev}_{w_p} \circ \sigma_k, \check{\beta}) \mid \\ & w_1, \dots, w_p \in CX_k|_{(\eta', 1]}, \check{\beta} \in W, v \in U(M_m), v \approx_{\epsilon/3} u\}, \end{aligned}$$

which is a neighbourhood of α , and set

$$(3.130) \quad \begin{aligned} U_{z_k} := & U_{z_{k-1}} \times \{0\} \cup \\ & \{(x_k, z'_{k-1}), t) \in Y_{k, \bar{\eta}} \times (0, \eta') \mid z'_{k-1} \in U_{z_{k-1}}\}, \end{aligned}$$

which is a neighbourhood of z_k .

Let $\alpha' \in U_\alpha$ so that

$$(3.131) \quad \alpha' = \text{Ad}(v) \circ \text{diag}(\text{ev}_{w_1} \circ \sigma_k, \dots, \text{ev}_{w_p} \circ \sigma_k, \check{\beta}),$$

for some $w_1, \dots, w_p \in CX_k|_{(\eta', 1]}$, some $\check{\beta} \in W$, and some $v \in U(M_m)$ with

$$(3.132) \quad v \approx_{\epsilon/3} u$$

Thus,

$$(3.133) \quad \check{\beta} = \text{Ad}(w) \circ \text{diag}(\text{ev}_{(x_k^{(1)}, t_1)} \circ \sigma_k, \dots, \text{ev}_{(x_k^{(q)}, t_q)} \circ \sigma_k, \gamma \circ \lambda_k)$$

where $q \in \mathbb{N}$, $(x_k^{(i)}, t_i) \in CX_k|_{[0, \eta')}$, $\gamma \in \text{Hom}(A_{k-1}, M_{m'-qm_k})$, and these satisfy

$$(3.134) \quad \check{\beta}' := \text{Ad}(w) \circ \text{diag}(\text{ev}_{(x_k^{(1)}, 0)} \circ \sigma_k, \dots, \text{ev}_{(x_k^{(q)}, 0)} \circ \sigma_k, \gamma \circ \lambda_k) \in U_\beta \circ \lambda_k.$$

Set

$$(3.135) \quad w' := \text{diag}(1_{pm_k}, w) \in U(M_m),$$

so that $\check{\beta}$ corresponds to the subrepresentation

$$(3.136) \quad \text{Ad}(vw') \circ \text{diag}(0_{pm_k}, \text{ev}_{(x_k^{(1)}, 0)} \circ \sigma_k, \dots, \text{ev}_{(x_k^{(q)}, 0)} \circ \sigma_k, \gamma \circ \lambda_k)$$

of α . Note that

$$(3.137) \quad \check{\beta}' = \text{Ad}(w) \circ \text{diag}(\hat{\theta}_k(x_k^{(1)}), \dots, \hat{\theta}_k(x_k^{(q)}), \gamma) \circ \lambda_k,$$

and since λ_k is surjective, it follows from (3.134) that

$$(3.138) \quad \beta' := \text{Ad}(w) \circ \text{diag}(\hat{\theta}_k(x_k^{(1)}), \dots, \hat{\theta}_k(x_k^{(q)}), \gamma) \in U_\beta.$$

We compute

$$S_{k, \bar{\eta}, m}(\alpha') \cap U_{z_k} = \bigcup_{i=1}^q R_i \cup R_\gamma$$

where for $i = 1, \dots, q$,

$$(3.139) \quad R_i := \begin{cases} \{((x_k^{(i)}, z'_{k-1}), t_i) \mid z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x_k^{(i)})) \cap U_{z_{k-1}}\}, & \text{if } t_i > 0; \\ \{(z'_{k-1}, 0) \mid z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x_k^{(i)})) \cap U_{z_{k-1}}\}, & \text{if } t_i = 0, \end{cases}$$

and

$$(3.140) \quad R_\gamma := \{(z'_{k-1}, 0) \mid z'_{k-1} \in S_{k-1, \bar{\eta}, m' - qm_k}(\gamma) \cap U_{z_{k-1}}\}.$$

For $z'_k \in R_i$, let $z'_k = (x_k, z'_{k-1}, t_i)$ if $t_i > 0$ or $z'_k = (z'_{k-1}, 0)$ if $t_i = 0$; we see that $z'_{k-1} \in S_{k-1, \bar{\eta}, m'}(\beta') \cap U_{z_{k-1}}$. Then by (3.84) (in the former case) or (3.87) (in the latter), and using w' as defined in (3.135), we have

$$(3.141) \quad \mu_{k, \bar{\eta}, \alpha', z'_k} = \text{Ad}(vw') \circ \text{diag}(0_{(p+i-1)m_k}, \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k^{(i)}, z'_{k-1}}, 0_{m-(p-i)m_k})$$

For $z'_k = (z'_{k-1}, 0) \in R_\gamma$, by (3.87), we have

$$(3.142) \quad \mu_{k, \bar{\eta}, \alpha, z'_k} = \text{Ad}(vw') \circ \text{diag}(0_{(p+q)m_k}, \mu_{k-1, \bar{\eta}, \gamma, z'_{k-1}})$$

From these computations, we see that

$$\begin{aligned} & \sum_{z'_k \in S_{k, \bar{\eta}, m}(\alpha') \cap U_{z_k}} \mu_{k, \bar{\eta}, \alpha', z'_k}(\kappa) \\ &= \sum_{z'_{k-1} \in S_{k-1, \bar{\eta}, m'}(\beta') \cap U_{z_{k-1}}} \text{Ad}(v)(\text{diag}(0_{pm_k}, \mu_{k-1, \bar{\eta}, \beta', z'_{k-1}}(\kappa))) \\ & \stackrel{(3.125)}{\approx}_{\epsilon/3} \text{Ad}(v)(\text{diag}(0_{pm_k}, \mu_{k-1, \bar{\eta}, \beta, z_{k-1}}(\kappa))) \\ & \stackrel{(3.132)}{\approx}_{2\epsilon/3} \text{Ad}(u)(\text{diag}(0_{pm_k}, \mu_{k-1, \bar{\eta}, \beta, z_{k-1}}(\kappa))) \\ (3.143) & \stackrel{(3.87)}{=} \mu_{k, \bar{\eta}, \alpha, z_k}(\kappa). \end{aligned}$$

□

3.7. Construction of $\xi_{k,\bar{\eta},k'}$. For $k' < k$ and $f \in C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_{k'}})$, define $\check{\xi}_{k,\bar{\eta},k'}(f): X_k \times [0, \eta_k) \rightarrow M_{m_k}$ by

$$(3.144) \quad \check{\xi}_{k,\bar{\eta},k'}(f)(x_k, t) := \begin{cases} \sum_{z_{k-1}} \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}}(f((x_k, z_{k-1}), t)), & \text{if } t > 0; \\ \sum_{z_{k-1}} \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}}(f(z_{k-1}, 0)), & \text{if } t = 0. \end{cases}$$

where the sum is taken over all

$$(3.145) \quad z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k)) \cap Z_{k-1,\bar{\eta},k'}^{**}.$$

By Lemma 3.8 (i), the domain of $\mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}}$ is indeed $M_{m_{k'}}$ for such z_{k-1} .

Lemma 3.10. *For $k' < k$, $\check{\xi}_{k,\bar{\eta},k'}$ defines a *-homomorphism*

$$(3.146) \quad C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_{k'}}) \rightarrow C_0(X_k \times [0, \eta_k), M_{m_k})$$

Proof. That $\check{\xi}_{k,\bar{\eta},k'}$ is a *-homomorphism follows from the fact that in (3.144), the right hand side is a sum of *-homomorphisms (applied to f) with orthogonal ranges (by the last part of Lemma 3.7).

We need to show that $\check{\xi}_{k,\bar{\eta},k'}(f)$ is continuous and vanishes at infinity, for every $f \in C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_{k'}})$. For continuity, let $(x_k, t) \in X_k \times [0, \eta_k)$, $\epsilon > 0$. Let

$$(3.147) \quad S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k)) \cap Z_{k-1,\bar{\eta},k'}^{**} = \{z_{k-1}^{(1)}, \dots, z_{k-1}^{(p)}\}.$$

By Lemma 3.9, there exists a neighbourhood V of $\hat{\theta}_k(x_k)$ in $\text{Hom}(A_{k-1}, M_{m_k})$ and, for each $i = 1, \dots, p$, a neighbourhood U_i of $z_{k-1}^{(i)}$ in $Z_{k-1,\bar{\eta},k'}^{**}$ such that for $\alpha \in V$ and each contraction $\kappa \in M_{m_{k'}}$,

$$(3.148) \quad \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}^{(i)}}(\kappa) \approx_{\epsilon/p} \sum_{z'_{k-1}} \mu_{k-1,\bar{\eta},\alpha,z'_{k-1}}(\kappa),$$

where the sum is over all $z'_{k-1} \in S_{k-1,\bar{\eta},m_k}(\alpha) \cap U_i$.

By possibly shrinking the U_i , arrange that they are disjoint. By Lemma 3.6, if α is sufficiently close to $\hat{\theta}_k(x_k)$ then

$$(3.149) \quad S_{k-1,\bar{\eta},m_k}(\alpha) \cap Z_{k-1,\bar{\eta},k'}^{**} = \prod_{i=1}^p S_{k-1,\bar{\eta},m_k}(\alpha) \cap U_i.$$

By possibly shrinking V , arrange that (3.149) holds for all $\alpha \in V$.

Let W be a neighbourhood of (x_k, t) in $X_k \times [0, \eta_k)$ such that, for $(x'_k, t') \in W$, the following hold:

- (i) $\hat{\theta}_k(x'_k) \in V$;
- (ii) If $t > 0$ then $t' > 0$ and for $((x'_k, z'_{k-1}), t') \in Z_{k,\bar{\eta},k'}^{**}$, there exists i (necessarily unique) such that $z'_{k-1} \in U_i$ and

$$(3.150) \quad f((x_k, z_{k-1}^i), t) \approx_{\epsilon/p} f((x'_k, z'_{k-1}), t').$$

(iii) If $t = 0$ and $t' > 0$ then for $z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k))$, there exists i (necessarily unique) such that $z'_{k-1} \in U_i$ and

$$(3.151) \quad f(z_{k-1}^i, 0) \approx_{\epsilon/p} f((x'_k, z'_{k-1}), t').$$

(iv) If $t = 0$ and $t' = 0$ then for $z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k))$, there exists i (necessarily unique) such that $z'_{k-1} \in U_i$ and

$$(3.152) \quad f(z_{k-1}^i, 0) \approx_{\epsilon/p} f(z'_{k-1}, 0).$$

Now, for $(x'_k, t') \in W$, let us show that $\check{\xi}_{k, \bar{\eta}, k'}(f)(x_k, t) \approx_{3\epsilon} \check{\xi}_{k, \bar{\eta}, k'}(f)(x'_k, t')$. Suppose first that $t, t' > 0$. Then for each $i = 1, \dots, p$,

$$(3.148) \quad \begin{aligned} & \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}^{(i)}}(f((x_k, z_{k-1}^{(i)}), t)) \\ & \approx_{\epsilon/p} \sum_{z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k)) \cap U_i} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x'_k), z'_{k-1}}(f((x_k, z_{k-1}^{(i)}), t)) \\ & \approx_{\epsilon/p} \sum_{z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k)) \cap U_i} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x'_k), z'_{k-1}}(f((x'_k, z'_{k-1}), t')). \end{aligned}$$

Thus,

$$(3.144), (3.147) \quad \begin{aligned} & \check{\xi}_{k, \bar{\eta}, k'}(f)(x_k, t) \\ & \stackrel{(3.144), (3.147)}{=} \sum_{i=1}^p \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}^{(i)}}(f((x_k, z_{k-1}^{(i)}), t)) \\ & \stackrel{(3.153)}{\approx_{2\epsilon}} \sum_{i=1}^p \sum_{z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k)) \cap U_i} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x'_k), z'_{k-1}}(f((x'_k, z'_{k-1}), t')) \\ & \stackrel{(3.149)}{=} \sum_{z'_{k-1} \in S_{k-1, \bar{\eta}, m_k}(\hat{\theta}_k(x'_k)) \cap Z_{k-1, \bar{\eta}, k'}^{**}} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x'_k), z'_{k-1}}(f((x'_k, z'_{k-1}), t')) \\ & \stackrel{(3.144)}{=} \check{\xi}_{k, \bar{\eta}, k'}(f)(x'_k, t'). \end{aligned}$$

The argument is almost exactly the same in other cases (if $t = 0$ and $t' > 0$, or if $t = t' = 0$), using (3.151) or (3.152) in place of (3.150). This concludes the proof that $\check{\xi}_{k, \bar{\eta}, k'}(f)$ is continuous.

To see that $\check{\xi}_{k, \bar{\eta}, k'}(f)$ vanishes at infinity, note that

$$(3.155) \quad \lim_{t \rightarrow \eta_k} \|f|_{Z_{k, \bar{\eta}, k'}^{**}|_{\{t\}}}\| = 0,$$

and so, by (3.144),

$$(3.156) \quad \lim_{t \rightarrow \eta_k} \|\check{\xi}_{k,\bar{\eta},k'}(f)|_{X_k \times \{t\}}\| = 0,$$

as required. \square

Upon identifying $Z_{k,\bar{\eta},k}^{**}$ with $CX_k|_{(\eta_k,1]}$ for $k > 0$, we also define

$$(3.157) \quad \check{\xi}_{k,\bar{\eta},k}: C_0(Z_{k,\bar{\eta},k}^{**}, M_{m_k}) \rightarrow C(CX_k, M_{m_k})$$

to be the inclusion. Likewise, $\check{\xi}_{0,(),0}: C_0(Z_{0,(),0}^{**}, M_{m_0}) \rightarrow C(CX_0, M_{m_0})$ is defined to be the identity map.

Now, for $1 \leq k' \leq k$, we define

$$(3.158) \quad \begin{aligned} \xi_{k,\bar{\eta},k'}: C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_{k'}}) &\rightarrow A_{k'-1} \oplus \\ &C(CX_{k'}, M_{m_{k'}}) \oplus \cdots \oplus C(CX_k, M_{m_k}) \end{aligned}$$

by

$$(3.159) \quad \begin{aligned} \xi_{k,\bar{\eta},k'}(f) := &(0_{A_{k'-1}}, \check{\xi}_{k',\bar{\eta},k'}(f|_{Z_{k',\bar{\eta},k'}^{**}}), \\ &\check{\xi}_{k'+1,\bar{\eta},k'}(f|_{Z_{k'+1,\bar{\eta},k'}^{**}}), \dots, \check{\xi}_{k,\bar{\eta},k'}(f)), \end{aligned}$$

where we are viewing $Z_{k'',\bar{\eta},k'}^{**}$ as a subset of $Z_{k,\bar{\eta},k'}^{**}$, for $k'' = k', \dots, k-1$, by identifying

$$(3.160) \quad Z_{k-1,\bar{\eta},k'}^{**} \quad \text{with} \quad Z_{k,\bar{\eta},k'}^{**}|_{\{0\}}$$

and so on.

For $k' = 0$, we define $\xi_{k,\bar{\eta},k'}$ by more or less the same formula, except that its range is $C(CX_0, M_{m_0}) \oplus \cdots \oplus C(CX_k, M_{m_k})$.

Lemma 3.11. *The image of $\xi_{k,\bar{\eta},k'}$ is contained in A_k .*

Proof. The proof consists in showing, by induction on k , simultaneously both the statement of the lemma and that, for $m \in \mathbb{N}$, $\alpha \in \text{Hom}(A_k, M_m)$, and $f \in C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_{k'}})$,

$$(3.161) \quad \alpha(\xi_{k,\bar{\eta},k'}(f)) = \sum_{z_k} \mu_{k,\bar{\eta},\alpha,z_k}(f(z_k)),$$

where the sum is taken over all

$$(3.162) \quad z_k \in S_{k,\bar{\eta},m}(\alpha) \cap Z_{k,\bar{\eta},k'}^{**}.$$

The case $k = k'$ is trivial.

For the inductive step, note that

$$(3.163) \quad \xi_{k,\bar{\eta},k'}(f) = \xi_{k-1,\bar{\eta},k'}(f|_{Z_{k-1,\bar{\eta},k'}^{**}}) \oplus \check{\xi}_{k,\bar{\eta},k'}(f),$$

so that, to show that the image of $\xi_{k,\bar{\eta},k'}$ is contained in A_k , we must show that

$$(3.164) \quad \theta_k(\xi_{k-1,\bar{\eta},k'}(f|_{Z_{k-1,\bar{\eta},k'}^{**}})) = \check{\xi}_{k,\bar{\eta},k'}(f)|_{X_k \times \{0\}}.$$

For $x_k \in X_k$,

$$\begin{aligned}
 \check{\xi}_{k,\bar{\eta},k'}(f)(x_k, 0) &\stackrel{(3.144)}{=} \sum_{z_{k-1}} \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}}(f(z_{k-1}, 0)) \\
 &\stackrel{(3.161)}{=} \hat{\theta}_k(x_k)(\xi_{k-1,\bar{\eta},k'}(f|_{Z_{k-1,\bar{\eta},k'}^{**}})) \\
 (3.165) \quad &= \theta_k(\xi_{k-1,\bar{\eta},k'}(f|_{Z_{k-1,\bar{\eta},k'}^{**}}))(x_k),
 \end{aligned}$$

where we have used induction for the second line. This establishes (3.164).

Now, to show (3.161), let $\alpha \in \text{Hom}(A_k, M_m)$, and decompose it as

$$(3.166) \quad \alpha = \text{Ad}(u) \circ \text{diag}(\text{ev}_{v_1} \circ \sigma_k, \dots, \text{ev}_{v_p} \circ \sigma_k, \beta \circ \lambda_k),$$

where $v_1, \dots, v_p \in CX_k|_{(0,1]}$, and $\beta \in \text{Hom}(A_{k-1}, M_{m'})$ for some m' . Assume that v_1, \dots, v_p are arranged so that $v_1, \dots, v_{p'} \in CX_k|_{[\eta_k,1]}$ and $v_{p'+1}, \dots, v_p \in CX_k|_{(0,\eta_k)}$.

Fix $f \in C_0(Z_{k,\bar{\eta},k'}^{**}, M_{m_k'})$. For $i = 1, \dots, p'$,

$$(3.167) \quad \text{ev}_{v_i} \circ \sigma_k \circ \xi_{k,\bar{\eta},k'}(f) = 0.$$

For $i = p' + 1, \dots, p$, let $v_i = (x_k^{(i)}, t_i) \in X_k \times (0, \eta_k)$. Then

$$\begin{aligned}
 &\text{ev}_{v_i} \circ \sigma_k \circ \xi_{k,\bar{\eta},k'}(f) \\
 &\stackrel{(3.159)}{=} \check{\xi}_{k,\bar{\eta},k'}(f)(x_k^{(i)}, t_i) \\
 &\stackrel{(3.144)}{=} \sum_{z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k^{(i)})) \cap Z_{k,\bar{\eta},k'}^{**}} \mu_{k-1,\bar{\eta},\hat{\theta}_k(x_k),z_{k-1}}(f((x_k^{(i)}, z_{k-1}), t)), \\
 (3.168) \quad &
 \end{aligned}$$

so that, by (3.84),

$$\begin{aligned}
 &\text{diag}(0_{m_k}, \dots, 0_{m_k}, \text{ev}_{v_i} \circ \sigma_k \circ \xi_{k,\bar{\eta},k'}(f), 0_{m_k}, \dots, 0_{m_k}, 0_{m'}) \\
 (3.169) \quad &= \sum_{z_{k-1} \in S_{k-1,\bar{\eta},m_k}(\hat{\theta}_k(x_k^{(i)})) \cap Z_{k-1,\bar{\eta},k'}^{**}} \mu_{k,\bar{\eta},\alpha,((x_k^{(i)}, z_{k-1}), t_i)}(f(x_k, z_{k-1}), t).
 \end{aligned}$$

Summing over i gives

$$(3.170) \quad (1 - e')\alpha(\xi_{k,\bar{\eta},k'}(f)) = \sum_{z_k} \mu_{k,\bar{\eta},\alpha,z_k}(f(z_k)),$$

where $e' := \text{Ad}(u)(0_{m_k}, \dots, 0_{m_k}, 1_{m'})$ and the sum is taken over all

$$(3.171) \quad z_k \in S_{k,\bar{\eta},m}(\alpha) \cap Z_{k,\bar{\eta},k'}^{**}|_{(0,1]}.$$

By induction,

$$(3.172) \quad \beta(\xi_{k-1,\bar{\eta},k'}(f|_{Z_{k-1,\bar{\eta},k'}^{**}})) = \sum_{z_{k-1}} \mu_{k-1,\bar{\eta},\beta,z_{k-1}}(f(z_{k-1}, 0)),$$

where the sum is taken over all

$$(3.173) \quad z_{k-1} \in S_{k-1,\bar{\eta},m'}(\beta) \cap Z_{k-1,\bar{\eta},k'}^{**}.$$

By (3.87), it follows that

$$(3.174) \quad \begin{aligned} e' \alpha(\xi_{k,\bar{\eta},k'}(f)) &= (0_{m_k}, \dots, 0_{m_k}, \beta \circ \lambda_k \circ \xi_{k,\bar{\eta},k'}(f)) \\ &= \sum_{z_k} \mu_{k,\bar{\eta},\alpha,(z_k)}(f(z_k)), \end{aligned}$$

where the sum is taken over all

$$(3.175) \quad z_k \in S_{k,\bar{\eta},k'}(\alpha) \cap Z_{k,\bar{\eta},k'}^{**} |_{\{0\}}.$$

Combining (3.170) and (3.174) yields (3.161). \square

3.8. Maps related to changing parameters $\bar{\eta}$. Our construction will involve using different $Z_{k,\bar{\eta}}$ given by varying the parameters $\bar{\eta}$. We will need to relate these spaces, and for this we define maps between them.

Let $\eta_1, \dots, \eta_k, \eta'_1, \dots, \eta'_k \in (0, 1)$ with $\eta_i \geq \eta'_i$ for $i = 1, \dots, k$. Let $\bar{\eta}$ and $\bar{\eta}'$ denote (η_1, \dots, η_k) and $(\eta'_1, \dots, \eta'_k)$ respectively (or $(\eta_1, \dots, \eta_{k-1}), (\eta'_1, \dots, \eta'_{k-1})$, as appropriate). We shall now construct maps

$$(3.176) \quad \rho_{k,\bar{\eta},\bar{\eta}'} : Z_{k,\bar{\eta}} \rightarrow Z_{k,\bar{\eta}'}$$

For $k = 0$, set

$$(3.177) \quad \rho_{0,(),()} := \text{id}_{X^{(0)}}.$$

Having defined $\rho_{k-1,\bar{\eta},\bar{\eta}'}$, let us define $\rho_{k,\bar{\eta},\bar{\eta}'}$. We do this in cases:

(i) For $z_k \in Z_{k,\bar{\eta}}|_{[\eta_k,1]} = CX_k|_{[\eta_k,1]} = Z_{k,\bar{\eta}'}|_{[\eta_k,1]}$, simply set

$$(3.178) \quad \rho_{k,\bar{\eta},\bar{\eta}'}(z_k) := (z_k).$$

(ii) For $((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times [\eta'_k, \eta_k]$, set

$$(3.179) \quad \rho_{k,\bar{\eta},\bar{\eta}'}((x_k, z_{k-1}), t) := (x_k, t).$$

(ii)' For $((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta'_k)$, set

$$(3.180) \quad \rho_{k,\bar{\eta},\bar{\eta}'}((x_k, z_{k-1}), t) := ((x_k, \rho_{k-1,\bar{\eta},\bar{\eta}'}(z_{k-1})), t).$$

By (3.184) (with $k-1$ in place of k , i.e., using induction on k) we see that

$$(3.181) \quad \rho_{k,\bar{\eta},\bar{\eta}'}(x_k, \rho_{k-1,\bar{\eta},\bar{\eta}'}(z_{k-1})) \in Y_{k,\bar{\eta}'},$$

so that the right-hand side of (3.180) is in $Z_{k,\bar{\eta}'}$.

(iii) For $(z_{k-1}, 0) \in Z_{k-1,\bar{\eta}} \times \{0\}$, set

$$(3.182) \quad \rho_{k,\bar{\eta},\bar{\eta}'}(z_{k-1}, 0) := (\rho_{k-1,\bar{\eta},\bar{\eta}'}(z_{k-1}), 0).$$

Here are some easy facts about the maps $\rho_{k,\bar{\eta},\bar{\eta}'}$.

Lemma 3.12. *Let $k, \bar{\eta}, \bar{\eta}'$ be given as above.*

(i) $\rho_{k,\bar{\eta},\bar{\eta}'} : Z_{k,\bar{\eta}} \rightarrow Z_{k,\bar{\eta}'}$ is continuous.

(ii) If $\eta''_1, \dots, \eta''_k \in (0, 1)$ are such that $\eta''_i \leq \eta'_i$ for each i then

$$(3.183) \quad \rho_{k,\bar{\eta},\bar{\eta}''} = \rho_{k,\bar{\eta}',\bar{\eta}''} \circ \rho_{k,\bar{\eta},\bar{\eta}'}$$

(iii) For each m and $\alpha \in \text{Hom}(A_k, M_m)$,

$$(3.184) \quad \begin{aligned} \rho_{k, \bar{\eta}, \bar{\eta}'}(S_{k, \bar{\eta}, m}(\alpha)) &= S_{k, \bar{\eta}', m}(\alpha) \quad \text{and} \\ S_{k, \bar{\eta}, m}(\alpha) &= \rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(S_{k, \bar{\eta}', m}(\alpha)). \end{aligned}$$

Proof. (i), (ii): These are easy inductive arguments.

(iii) By induction on k ; the case $k = 0$ is easy.

For the inductive step, let $\alpha \in \text{Hom}(A_k, M_m)$. By induction, Cases (ii)' and (iii) of the definition of $\rho_{k, \bar{\eta}, \bar{\eta}'}$, and Cases (ii) and (iii) of the definition of $S_{k, \bar{\eta}, m}(\alpha)$, we see that (3.184) holds when both sides are restricted to $[0, \eta'_k]$. It is also clear, from Case (i) of the definition of $\rho_{k, \bar{\eta}, \bar{\eta}'}$ that (3.184) holds when both sides are restricted to $[\eta_k, 1]$.

Finally, Case (ii) of the definition of $\rho_{k, \bar{\eta}, \bar{\eta}'}$, together with Cases (i)' and (ii) of the definition of $S_{k, \bar{\eta}, m}(\alpha)$ shows that (3.184) holds when both sides are restricted to $[\eta'_k, \eta_k]$. \square

Lemma 3.13. *Let $\bar{\eta}, \bar{\eta}'$ be as above. Let $\alpha \in \text{Hom}(A_k, M_m)$ and $z'_k \in S_{k, \bar{\eta}', m}(\alpha)$. Then*

$$(3.185) \quad \mu_{k, \bar{\eta}', \alpha, z'_k}(1) = \sum_{z_k \in \rho^{-1}(\{z'_k\})} \mu_{k, \bar{\eta}, \alpha, z_k}(1).$$

Proof. We prove this by induction on k . In the base case, $k = 0$, it is trivial since $\bar{\eta} = () = \bar{\eta}'$.

For $k > 0$, consider cases.

(i) If $z'_k \in Z_{k, \bar{\eta}'}|_{[\eta_k, 1]} = Z_{k, \bar{\eta}}|_{[\eta_k, 1]}$ (see Case (i) of the definition of $\rho_{k, \bar{\eta}, \bar{\eta}'}$) then $\rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(\{z'_k\}) = \{z_k\}$ and $\mu_{k, \bar{\eta}', \alpha, z'_k} = \mu_{k, \bar{\eta}, \alpha, z_k}$, so the result holds in this case.

(ii) If $z'_k = (x_k, t) \in Z_{k, \bar{\eta}'}|_{[\eta'_k, \eta_k]}$, then by Case (ii) of the definition of $\rho_{k, \bar{\eta}, \bar{\eta}'}$,

$$(3.186) \quad \rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(\{z'_k\}) = \{(x_k, z_{k-1}), t) \mid z_{k-1} \in S_{k, \bar{\eta}, m_k}(\hat{\theta}_k(x_k))\}.$$

Let $\alpha': A_k \rightarrow M_m$ be the maximal subrepresentation of α that factors through $\text{ev}_{(x_k, t)} \circ \sigma_k$. Then by (3.82),

$$(3.187) \quad \alpha' = \mu_{k, \bar{\eta}', \alpha, z'_k} \circ \text{ev}_{(x_k, t)} \circ \sigma_k,$$

and so, by (3.84), for $z_{k-1} \in S_{k, \bar{\eta}, m_k}(\hat{\theta}_k(x_k))$,

$$(3.188) \quad \mu_{k, \bar{\eta}, \alpha, ((x_k, z_{k-1}), t)} = \mu_{k, \bar{\eta}', \alpha, z'_k} \circ \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}.$$

By the final statement of Lemma 3.7,

$$(3.189) \quad 1_{m_k} = \sum_{z_{k-1} \in S_{k, \bar{\eta}, m_k}(\hat{\theta}_k(x_k))} \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}(1),$$

and thus,

$$\begin{aligned}
& \mu_{k, \bar{\eta}', \alpha, z_k}(1_{m_k}) \\
&= \sum_{z_{k-1} \in S_{k, \bar{\eta}, m_k}(\hat{\theta}_k(x_k))} \mu_{k, \bar{\eta}', \alpha, z_k} \circ \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}(1) \\
(3.188) \quad & \stackrel{=}{=} \sum_{z_{k-1}} \mu_{k, \bar{\eta}, \alpha, ((x_k, z_{k-1}), t)}(1) \\
(3.190) \quad & \stackrel{(3.186)}{=} \sum_{z_k \in \rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(\{z'_k\})} \mu_{k, \bar{\eta}, \alpha, z_k, t}(1),
\end{aligned}$$

as required.

(iii) If $z'_k = ((x_k, z'_{k-1}), t) \in Y_{k, \bar{\eta}'} \times (0, \eta'_k)$ then, defining μ' as in (3.83), we have by (3.84)

$$(3.191) \quad \mu_{k, \bar{\eta}', z'_k} = \mu' \circ \mu_{k-1, \bar{\eta}', \hat{\theta}_k(x_k), z'_{k-1}}.$$

Every $z_k \in \rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(z'_k)$ is equal to $((x_k, z_{k-1}), t)$ for some $z_{k-1} \in \rho_{k-1, \bar{\eta}, \bar{\eta}'}^{-1}(z'_{k-1})$, and for such z_k we likewise have

$$(3.192) \quad \mu_{k, \bar{\eta}, z_k} := \mu' \circ \mu_{k-1, \bar{\eta}, \hat{\theta}_k(x_k), z_{k-1}}.$$

Therefore, in this case, the conclusion follows from the inductive hypothesis.

(iv) If $z'_k = (z'_{k-1}, 0) \in Z_{k-1, \bar{\eta}'} \times \{0\}$ then defining α' and μ' as in Case (iii) of the definition of $\mu_{k, \bar{\eta}, \alpha, z_k}$, we have by (3.87),

$$(3.193) \quad \mu_{k, \bar{\eta}', z'_k} := \mu' \circ \mu_{k-1, \bar{\eta}', \alpha', z'_{k-1}}.$$

Every $z_k \in \rho_{k, \bar{\eta}, \bar{\eta}'}^{-1}(z'_k)$ is of the form $(z_{k-1}, 0)$ for $z_{k-1} \in \rho_{k-1, \bar{\eta}, \bar{\eta}'}^{-1}(z'_{k-1})$, and for such z_k we likewise have

$$(3.194) \quad \mu_{k, \bar{\eta}, z_k} := \mu' \circ \mu_{k-1, \bar{\eta}, \alpha', z_{k-1}}.$$

So once again, in this case, the conclusion follows from the inductive hypothesis. \square

3.9. Open covers. Now fix $\eta \in (0, 1/2)$ and let us use $\bar{\eta}$ or $\bar{\eta}^k$ to denote (η, \dots, η) . Further, write

$$(3.195) \quad \overline{2\eta}^k := (2\eta, \dots, 2\eta) \in (0, 1)^k$$

and

$$(3.196) \quad \bar{\eta}^{k'} \overline{2\eta}^{k''} := (\eta, \dots, \eta, 2\eta, \dots, 2\eta) \in (0, 1)^{k'+k''},$$

where η appears k' times and 2η appears k'' times.

We shall construct open covers \mathcal{U}_k of $Z_{k, \bar{\eta}}$. These will depend on a prescribed finite subset \mathcal{F} of $A_{k_0, \bar{\eta}}$ and a tolerance $\delta > 0$, although this dependence is suppressed in the notation. For simplicity, all open covers will be hereditary, i.e., closed under taking open subsets.

First, for each k , choose a hereditary open cover \mathcal{V}_k of CX_k such that, for each $a \in \mathcal{F}$, the function $\sigma_k \circ \lambda_{k_0}^k(a)$ is approximately constant, to within δ , on each set in \mathcal{V}_k , i.e., for $V \in \mathcal{V}_k$ and $v_k, v'_k \in V$,

$$(3.197) \quad \sigma_k \circ \lambda_{k_0}^k(a)(v_k) \approx_\delta \sigma_k \circ \lambda_{k_0}^k(a)(v'_k).$$

Let us define \mathcal{U}_k recursively over k . For $k = 0$, set $\mathcal{U}_0 := \mathcal{V}_0$.

For $k \geq 1$, set $\mathcal{U}_k^{(1)}$ to the set of all open sets $U \in \mathcal{V}_k$ which are subsets of $CX_k|_{(\eta,1]}$. Identifying $CX_k|_{(\eta,1]}$ with $Z_{k,\bar{\eta}}|_{(\eta,1]}$, let us view $\mathcal{U}_k^{(1)}$ as a family of open subsets of $Z_{k,\bar{\eta}}$.

Set $\mathcal{U}_k^{(2)}$ equal to the collection of all open sets $U \subset Z_{k,\bar{\eta}}|_{[0,2\eta]}$ such that the following decomposition holds: We may decompose

$$(3.198) \quad \rho_{k,(\eta,\dots,\eta,2\eta),\bar{\eta}}^{-1}(U) = V_1 \amalg \dots \amalg V_p$$

such that V_1, \dots, V_p are open in $Z_{k,(\eta,\dots,\eta,2\eta)}$, and for each $i = 1, \dots, p$,

$$(3.199) \quad V_i(0) \cup \{z_{k-1} \mid ((x_k, z_{k-1}), t) \in V_i, \text{ some } x_k \in X_k, t \in (0, 2\eta)\} \in \mathcal{U}_{k-1}.$$

Now set

$$(3.200) \quad \mathcal{U}_k := \mathcal{U}_k^{(1)} \cup \mathcal{U}_k^{(2)}.$$

Lemma 3.14. *For $k \geq 1$, $\mathcal{U}_k^{(1)}$ (respectively $\mathcal{U}_k^{(2)}$) is a hereditary open cover of $Z_{k,\bar{\eta}}|_{(\eta,1]}$ (respectively $Z_{k,\bar{\eta}}|_{[0,2\eta]}$). Therefore, \mathcal{U}_k is a hereditary open cover of $Z_{k,\bar{\eta}}$.*

Proof. Using the fact that \mathcal{V}_k is a hereditary open cover of CX_k , we see that $\mathcal{U}_k^{(1)}$ is an open cover of $Z_{k,\bar{\eta}}|_{(\eta,1]}$. The rest of the lemma is proven by induction; at the outset, we know that \mathcal{U}_0 is an open cover of $Z_{0,()}$.

Now let us assume that \mathcal{U}_{k-1} is an open cover of $Z_{k-1,\bar{\eta}}$, and prove that $\mathcal{U}_k^{(2)}$ is an open cover of $Z_{k,\bar{\eta}}|_{[0,2\eta]}$. Fix $z_k \in Z_{k,\bar{\eta}}|_{[0,2\eta]}$, and let us argue that it is contained in a set in $\mathcal{U}_k^{(2)}$ in cases as follows:

(i) If $z_k = (x_k, t) \in X_k \times [\eta, 2\eta]$, write $S_{k,\bar{\eta},m_k}(\hat{\theta}_k(x)) = \{z_{k-1}^{(1)}, \dots, z_{k-1}^{(p)}\} \subset Z_{k-1,\bar{\eta}}$. By the inductive hypothesis, choose $W_1, \dots, W_p \in \mathcal{U}_{k-1}$ that are disjoint, with $z_{k-1}^{(i)} \in W_i$ for each i . By continuity of $S_{k,\bar{\eta},m_k} \circ \hat{\theta}_k$, there is a neighbourhood \hat{U} of (x_k, t) in CX_k such that

$$(3.201) \quad \bigcup_{(x'_k, t') \in \hat{U}} S_{k,\bar{\eta},m_k}(\hat{\theta}_k(x'_k, t')) \subseteq W_1 \cup \dots \cup W_p.$$

Set

$$(3.202) \quad U := \{((x'_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta) \mid (x'_k, t) \in \hat{U}\} \cup (\hat{U} \cap X_k \times [\eta, 2\eta]).$$

Then, by construction, $U \in \mathcal{U}_k^{(2)}$ ($p_U = p$ and the sets $V_{U,1}, \dots, V_{U,p_U}$ are preimages of W_1, \dots, W_p) and $(x_k, t) \in U$.

(ii) If $z_k = ((x_k, z_{k-1}), t) \in Y_{k,\bar{\eta}} \times (0, \eta)$, we do the exact same thing as in (i).

(iii) For $(z_{k-1}, 0) \in Z_{k-1, \bar{\eta}}$, by the inductive hypothesis, there exists a neighbourhood W of z_{k-1} such that $W \in \mathcal{U}_{k-1}$. Set

$$(3.203) \quad U := W \times \{0\} \cup \{(x_k, z'_{k-1}), t) \in Y_{k, \bar{\eta}} \times [0, \eta] \mid z'_{k-1} \in W\}.$$

By construction, $U \in \mathcal{U}_k^{(2)}$ ($p_U = 1$) and $(z_{k-1}, 0) \in U$. \square

Lemma 3.15. *Let $U \in \mathcal{U}_k$. Then we have a decomposition*

$$(3.204) \quad \rho_{k, \overline{2\eta}^k, \bar{\eta}^k}^{-1}(U) = W_1 \amalg \cdots \amalg W_q$$

where, for each $i = 1, \dots, q$, there is $k(i) \in \{0, \dots, k\}$ such that the set

$$(3.205) \quad \hat{W}_i := \rho_{k, \overline{2\eta}^k, \bar{\eta}^{k(i)} \overline{2\eta}^{k-k(i)}}^{-1}(W_i)$$

is open component of $\rho_{k, \bar{\eta}^{k(i)} \overline{2\eta}^{k-k(i)}, \bar{\eta}^k}^{-1}(U)$,

$$(3.206) \quad \hat{W}_i \subseteq Z_{k, \bar{\eta}^{k(i)} \overline{2\eta}^{k-k(i)}, k(i)}^{**},$$

and for $a \in \mathcal{F}$ and $z_k, z'_k \in \hat{W}_i$

$$(3.207) \quad \nu_{k, \bar{\eta}^{k(i)} \overline{2\eta}^{k-k(i)}, z_k}(\lambda_{k_0}^k(a)) \approx_\epsilon \nu_{k, \bar{\eta}^{k(i)} \overline{2\eta}^{k-k(i)}, z'_k}(\lambda_{k_0}^k(a))$$

in $M_{m_{k(i)}}$.

Proof. Let us use induction on k . For $k = 0$, the conclusion is trivial (with $q = 1$). For $k \geq 1$, we consider cases as follows.

(i) If $U \in \mathcal{U}_k^{(1)}$ then by definition, $U \in Z_{k, \bar{\eta}^k, k}^{**}$. We may therefore set

$$(3.208) \quad q = 1, \quad W_1 = U, \quad k(1) = k.$$

(ii) If $U \in \mathcal{U}_k^{(2)}$, then let V_1, \dots, V_p be as in (3.198). Fix i for the moment. Set

$$(3.209) \quad V'_i := V_i(0) \cup \{(x_k, z_{k-1}), t) \in V_i, \text{ some } x_k \in X_k, t \in (0, 2\eta)\} \in \mathcal{U}_{k-1}.$$

i.e., exactly the set appearing in (3.199). By induction, we have the decomposition

$$(3.210) \quad \rho_{k-1, \overline{2\eta}^{k-1}, \bar{\eta}^{k-1}}^{-1}(V'_i) = W'_{i,1} \amalg \cdots \amalg W'_{i,q_i},$$

and we can find $k(i, j) \in \{0, \dots, k-1\}$ for $j = 1, \dots, q_i$ such that the set

$$(3.211) \quad \hat{W}'_{i,j} := \rho_{k-1, \overline{2\eta}^{k-1}, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}}^{-1}(W'_{i,j})$$

is an open component of $\rho_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \bar{\eta}^{k-1}}^{-1}(V'_i)$,

$$(3.212) \quad \hat{W}'_{i,j} \subseteq Z_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, k(i,j)}^{**},$$

and for $a \in \mathcal{F}$, $z_{k-1}, z'_{k-1} \in \hat{W}'_{i,j}$,

$$(3.213) \quad \nu_{k-1, \bar{\eta}^{k(i)} \overline{2\eta}^{k-1-k(i)}, z_{k-1}}(\lambda_{k_0}^{k-1}(a)) \approx_\epsilon \nu_{k-1, \bar{\eta}^{k(i)} \overline{2\eta}^{k-1-k(i)}, z'_{k-1}}(\lambda_k(a)).$$

Define $\hat{W}_{i,j} \subset Z_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)}}$ to consist of all

$$(3.214) \quad (z_{k-1}, 0) \in \rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},(\eta,\dots,\eta,2\eta)}^{-1}(V_i)|_{\{0\}}$$

such that $z_{k-1} \in \hat{W}'_{i,j}$, together with all

$$(3.215) \quad ((x_k, z_{k-1}), t) \in \rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},(\eta,\dots,\eta,2\eta)}^{-1}(V_i)|_{(0,\eta)}$$

such that $z_{k-1} \in \hat{W}'_{i,j}$.

Then $\hat{W}_{i,j}$ is the preimage of a component of

$$(3.216) \quad \rho_{k-1,\bar{\eta}^k(i,j)\overline{2\eta}^{k-1-k(i,j)},\bar{\eta}^{k-1}}^{-1}(V'_i)$$

under the continuous map $Z_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)}}|_{[0,\eta]} \rightarrow Z_{k-1,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)-1}}$ defined by $(z_{k-1}, 0) \mapsto z_{k-1}$ and $((x_k, z_{k-1}), t) \mapsto z_{k-1}$. The set $\hat{W}_{i,j}$ is therefore a component of

$$(3.217) \quad \rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},(\eta,\dots,\eta,2\eta)}^{-1}(V_i).$$

Since V_i is a component of $\rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},(\eta,\dots,\eta,2\eta)}^{-1}(U)$, it follows that $\hat{W}_{i,j}$ is a component of

$$(3.218) \quad \rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},(\eta,\dots,\eta,2\eta)}^{-1}(U).$$

Moreover, since $\hat{W}_{i,j} \subseteq Z_{k,(\eta,\dots,\eta,2\eta)}|_{[0,\eta]} = Z_{k,\bar{\eta}^k}|_{[0,\eta]}$, it follows that $\hat{W}_{i,j}$ is a component of

$$(3.219) \quad \rho_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},\bar{\eta}^k}^{-1}(U).$$

By the recursive definition (3.110) of $Z_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},k(i,j)}^{**}$, it follows that

$$(3.220) \quad \hat{W}_{i,j} \subseteq Z_{k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)},k(i,j)}^{**}.$$

For $z_k, z'_k \in \hat{W}_{i,j}$, these correspond to points z_{k-1}, z'_{k-1} according to the definition of $\hat{W}_{i,j}$ ((3.214) and (3.215)), so that by the definition (3.79) of $\nu_{k,\bar{\eta},z_k}$,

$$(3.221) \quad \nu_{k,\bar{\eta}^k(i)\overline{2\eta}^{k-k(i)},z_k} = \nu_{k-1,\bar{\eta}^k(i)\overline{2\eta}^{k-1-k(i)},z_{k-1}} \circ \lambda_k,$$

and

$$(3.222) \quad \nu_{k,\bar{\eta}^k(i)\overline{2\eta}^{k-k(i)},z'_k} = \nu_{k-1,\bar{\eta}^k(i)\overline{2\eta}^{k-1-k(i)},z'_{k-1}} \circ \lambda_k.$$

Using this, and (3.213), it follows that for $a \in \mathcal{F}$,

$$(3.223) \quad \nu_{k,\bar{\eta}^k(i)\overline{2\eta}^{k-k(i)},z_k}(\lambda_{k_0}^k(a)) \approx_\epsilon \nu_{k,\bar{\eta}^k(i)\overline{2\eta}^{k-k(i)},z'_k}(\lambda_{k_0}^k(a)).$$

Define

$$(3.224) \quad W_{i,j} := \rho_{k,\overline{2\eta}^k,\bar{\eta}^k(i,j)\overline{2\eta}^{k-k(i,j)}}^{-1}(\hat{W}_{i,j}) \subseteq Z_{k,\overline{2\eta}^k}.$$

We have shown that $W_{i,j}$ has the properties required of each W_i in the statement of the lemma. Moreover, since

$$(3.225) \quad \rho_{k-1, 2\bar{\eta}^{k-1}, \bar{\eta}^{k-1}}^{-1}(V'_i) = \coprod_j W'_{i,j},$$

it follows that

$$(3.226) \quad \rho_{k, 2\bar{\eta}^k, \bar{\eta}^{k-1}}^{-1}(V_i) = \coprod_j W_{i,j}.$$

Consequently, combining these for all i , we have

$$(3.227) \quad \begin{aligned} \rho_{k, 2\bar{\eta}^k, \bar{\eta}^k}^{-1}(U) &= \prod_{i=1}^p \rho_{k, 2\bar{\eta}^k, \bar{\eta}^k}^{-1}(V_i) \\ &= \prod_{i=1}^q \prod_{j=1}^{q_i} W_{i,j}. \end{aligned}$$

We conclude by relabelling the family $(W_{i,j})_{i,j}$ as W_1, \dots, W_q . \square

3.10. What to do with a partition of unity. Let $(e_i)_{i=1}^p$ be a finite family of positive functions in $C(Z_{k_0, \bar{\eta}}, D)$ for some unital C^* -algebra D , subordinate to the open cover \mathcal{U}_{k_0} , which means that for each $i = 1, \dots, p$,

$$(3.228) \quad U_i := \{z_{k_0} \in Z_{k_0, \bar{\eta}} \mid e_i(z_{k_0}) \neq 0\} \in \mathcal{U}_{k_0}.$$

(We will soon ask that e_i is an approximate partition of unity, although this is not required yet.)

By Lemma 3.15, for each i , we have the decomposition

$$(3.229) \quad \rho_{\frac{1}{2\bar{\eta}^{k_0}}, \bar{\eta}^{k_0}}^{-1}(U_i) = W_{i,1} \amalg \dots \amalg W_{i,q_i}$$

where, for each $j = 1, \dots, q_i$, there is $k(i, j) \in \{0, \dots, k_0\}$ such that the open set

$$(3.230) \quad \hat{W}_{i,j} := \rho_{k_0, 2\bar{\eta}^{k_0}, \bar{\eta}^{k(i,j)} 2\bar{\eta}^{k_0 - k(i,j)}}^{-1}(W_{i,j})$$

is a component of $\rho_{k_0, \bar{\eta}^{k(i,j)} 2\bar{\eta}^{k_0 - k(i,j)}}^{-1}(U)$, and

$$(3.231) \quad \hat{W}_{i,j} \subseteq Z_{k_0, \bar{\eta}^{k(i,j)} 2\bar{\eta}^{k_0 - k(i,j)}, k(i,j)}^{**}.$$

Define

$$(3.232) \quad e_{i,j} := (e_i \circ \rho_{k_0, \bar{\eta}^{k(i,j)} 2\bar{\eta}^{k_0 - k(i,j)}, \bar{\eta}^{k_0}})|_{\hat{W}_{i,j}} \in C_0(\hat{W}_{i,j}, D)_+,$$

and use this to define a c.p.c. order zero map

$$(3.233) \quad \phi_{i,j} : M_{m_{k(i,j)}} \rightarrow A_{k_0} \otimes D$$

by

$$(3.234) \quad \phi_{i,j}(\kappa) := \xi_{k_0, \bar{\eta}^{k(i,j)} 2\bar{\eta}^{k_0 - k(i,j)}, k(i,j)}(\kappa \otimes e_{i,j}).$$

Also, pick some

$$(3.235) \quad z_{k_0}^{(i,j)} \in \widehat{W}_{i,j}$$

and set

$$(3.236) \quad \psi_{i,j} := \nu_{k_0, \overline{\eta}^{k(i,j)} \overline{2\eta}^{k_0 - k(i,j)}, z_{k_0}^{(i,j)}} : A_{k_0} \rightarrow M_{m_{k(i,j)}}$$

(the range is correct by (3.231) and Lemma 3.8 (i)). Also set

$$(3.237) \quad \phi_i := \sum_{j=1}^{q_i} \phi_{i,j} : \bigoplus_{j=1}^{q_i} M_{m_{k(i,j)}} \rightarrow A_{k_0} \otimes D.$$

To save space, let us write σ_k to mean $(\sigma_k \otimes \text{id}_D) : A_k \otimes D \rightarrow C(CX_k, M_{m_k}) \otimes D$, and likewise for other maps such as $\lambda_{k'}^k$ and $\mu_{k,\overline{\eta},\alpha,z_k}$.

Lemma 3.16. *Let $i = 1, \dots, p$, $j = 1, \dots, q_i$, and $k = 0, \dots, k_0$. Consider the c.p.c. map $\sigma_k \circ \lambda_{k_0}^k \circ \phi_{i,j} : M_{m_{k(i,j)}} \rightarrow C(CX_k, M_{m_k}) \otimes D$.*

(i) *If $k < k(i, j)$ then*

$$(3.238) \quad \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}})) = 0.$$

(ii) *If $k = k(i, j)$ then*

$$(3.239) \quad \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}})) = 1_{m_k} \otimes e_{i,j}|_{CX_k|_{(\eta,1]}},$$

where we canonically identify $M_{m_k} \otimes C_0(CX_k|_{(\eta,1]}, D)$ (to which the right-hand side belongs) with a subalgebra of $C(CX_k, M_{m_k}) \otimes D$ (for the left-hand side).

(iii) *If $k \geq k(i, j)$ and $(x_k, t) \in X_k \times (0, 2\eta)$ then*

$$(3.240) \quad \begin{aligned} & \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}}))(x_k, t) \\ &= \sum_{z_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z_{k-1}} (1_{m_{k(i,j)}}) \\ & \otimes e_i \circ \rho_{k, \overline{\eta}^k, \overline{2\eta}^k} ((x_k, z_{k-1}), t). \end{aligned}$$

where the sum is over all

$$(3.241) \quad z_{k-1} \in S_{k-1, \overline{2\eta}^{k-1}, m_k} (\hat{\theta}_k(x_k))$$

for which $((x_k, z_{k-1}), t) \in W_{i,j}$.

(iv) *If $k > k(i, j)$ then*

$$(3.242) \quad \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}}))$$

vanishes on $CX_k|_{[2\eta,1]}$.

Proof. We compute

$$\begin{aligned}
& \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}})) \\
& \stackrel{(3.234)}{=} \sigma_k \circ \lambda_{k_0}^k \circ \xi_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, k(i,j)} (1_{m_{k(i,j)}} \otimes e_{i,j}) \\
(3.243) \quad & \stackrel{(3.159)}{=} \begin{cases} 0, & \text{if } k < k(i,j), \\ \check{\xi}_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}, k(i,j)} (1_{m_{k(i,j)}} \otimes e_{i,j})|_{Z_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}}^{**}}, & \text{if } k \geq k(i,j), \end{cases}
\end{aligned}$$

where, in the latter case, as in (3.159), we view $Z_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}}^{**}$ as a subset of $Z_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}}$. The former case proves (i). Point (iv) follows from the latter case and Lemma 3.10.

Picking up in the latter case, for $k = k(i,j)$, we have

$$\begin{aligned}
& \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}})) \\
& = \check{\xi}_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, k(i,j)} (1_{m_{k(i,j)}} \otimes e_{i,j})|_{Z_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}}^{**}} \\
(3.244) \quad & \stackrel{(3.157)}{=} 1_{m_{k(i,j)}} \otimes e_{i,j}|_{cX_k|_{(\eta,1]}},
\end{aligned}$$

thus establishing (ii).

On the other hand, for $k > k(i,j)$ and for $(x_k, t) \in X_k \times (0, 2\eta)$,

$$\begin{aligned}
& \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1_{m_{k(i,j)}}))(x_k, t) \\
& \stackrel{(3.243)}{=} \check{\xi}_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}, k(i,j)} (1_{m_{k(i,j)}} \otimes e_{i,j})|_{Z_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}}^{**}}(x_k, t) \\
& \stackrel{(3.144)}{=} \sum_{z_{k-1}} \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (1_{m_{k(i,j)}}) \\
(3.245) \quad & \otimes e_{i,j}((x_k, z_{k-1}), t),
\end{aligned}$$

where the sum is taken over all

$$\begin{aligned}
& z_{k-1} \in S_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, m_k} (\hat{\theta}_k(x_k)) \cap \\
(3.246) \quad & Z_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, k(i,j)}^{**}.
\end{aligned}$$

For z_{k-1} as in (3.246), by Lemma 3.13,

$$\begin{aligned}
& \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (1_{m_{k(i,j)}}) \\
(3.247) \quad & = \sum_{z'_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z'_{k-1}} (1),
\end{aligned}$$

where the sum is over all

$$\begin{aligned}
& z'_{k-1} \in \rho_{k-1, \overline{2\eta}^{k-1}, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}}^{-1} (\{z_{k-1}\}) \\
(3.248) \quad & \subset Z_{k-1, \overline{2\eta}^{k-1}}.
\end{aligned}$$

For such z_{k-1} and z'_{k-1} ,

$$\begin{aligned}
 & e_{i,j}((x_k, z_{k-1}), t) \\
 = & e_{i,j}((x_k, \rho_{k-1, \overline{2\eta}^{k-1}, \overline{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}}(z'_{k-1})), t) \\
 \stackrel{(3.182), (3.183), (3.232)}{=} & \begin{cases} e_i \circ \rho_{k_0, \overline{2\eta}^{k_0}, \overline{\eta}^{k_0}} & ((x_k, z'_{k-1}), t), \\ & \text{if } ((x_k, z_{k-1}), t) \in \hat{W}_{i,j}; \\ 0, & \text{otherwise} \end{cases} \\
 (3.249) \stackrel{(3.183), (3.230)}{=} & \begin{cases} e_i \circ \rho_{k_0, \overline{2\eta}^{k_0}, \overline{\eta}^{k_0}} & ((x_k, z'_{k-1}), t), \\ & \text{if } ((x_k, z'_{k-1}), t) \in W_{i,j}; \\ 0, & \text{otherwise,} \end{cases}
 \end{aligned}$$

where we are viewing $Z_{k, \overline{2\eta}^k}$ as a subset of $Z_{k_0, \overline{2\eta}^{k_0}}$. Putting together (3.245), (3.247), and (3.249), and using (3.231), we obtain

$$\begin{aligned}
 & \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j}(1_{m_{k(i,j)}}))(x_k, t) \\
 = & \sum_{z_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z_{k-1}}(1_{m_{k(i,j)}}) \\
 (3.250) \quad & \otimes e_i \circ \rho_{k, \overline{\eta}^k, \overline{2\eta}^k}((x_k, z_{k-1}), t),
 \end{aligned}$$

where the sum is over all

$$(3.251) \quad z_{k-1} \in S_{k-1, \overline{2\eta}^{k-1}, m_k}(\hat{\theta}_k(x_k))$$

for which $((x_k, z_{k-1}), t) \in W_{i,j}$. This establishes (iii), in the case $k > k(i, j)$.

In the case $k = k(i, j)$, (iii) follows from (ii) and Lemma 3.13. \square

Definition 3.17. Let F, A be C^* -algebras, with F finite dimensional, and let $\phi: F \rightarrow A$ be a c.p. map. The map ϕ is $(n+1)$ -**colourable** if F decomposes as a direct sum of subalgebras, $F = F_0 \oplus \cdots \oplus F_n$, such that $\phi|_{F_i}$ is orthogonality preserving (i.e. order zero), $i = 0, \dots, n$. Likewise, a finite family $(e_j)_{j \in I}$ of positive contractions is $(n+1)$ -**colourable** means that we can decompose $I = I_0 \amalg \cdots \amalg I_n$ such that $(e_j)_{j \in I_i}$ is a pairwise orthogonal family, $i = 0, \dots, n$.

Lemma 3.18. (i) For each i , the map ϕ_i is c.p.c. order zero.

(ii) If e_{i_1} and e_{i_2} are orthogonal, then so are the ranges of ϕ_i and ϕ'_i . Therefore, if the elements $(e_i)_i$ are $(n+1)$ -colourable then so also is the map $\phi = \sum_{i,j} \phi_{i,j}$.

(iii) If $(e_i)_i$ is an exact partition of unity then $\sum_{i,j} \phi_i(1_{m_{k(i,j)}}) = 1_B$.

(iii)' If $\sum_i e_i \approx_\epsilon 1$ (i.e., $(e_i)_i$ is an ϵ -**approximate partition of unity**) then $\sum_{i,j} \phi_{i,j}(1_{m_{k(i,j)}}) \approx_\epsilon 1_B$.

Proof. (i) and (ii): $\phi_{i,j}$ is quite clearly a c.p.c. order zero map. To show that ϕ_i is order zero, it therefore suffices to show that

$$(3.252) \quad \phi_{i,j_1}(1_{m_{k(i,j_1)}}) \quad \text{and} \quad \phi_{i,j_2}(1_{m_{k(i,j_2)}})$$

are orthogonal for $j_1 \neq j_2$. The key fact to be used here is

$$(3.253) \quad W_{i,j_1} \cap W_{i,j_2} = \emptyset.$$

Similarly, for (ii), we need to show that, for all j_1, j_2 ,

$$(3.254) \quad \phi_{i_1,j_1}(1_{m_k(i_1,j_1)}) \quad \text{and} \quad \phi_{i_1,j_2}(1_{m_k(i_1,j_2)})$$

are orthogonal, where we can assume that

$$(3.255) \quad W_{i_1} \cap W_{i_2} = \emptyset.$$

Thus, (i) and (ii) both reduce to showing that, if

$$(3.256) \quad W_{i_1,j_1} \cap W_{i_2,j_2} = \emptyset$$

then $\phi_{i_1,j_1}(1)$ and $\phi_{i_2,j_2}(1)$ are orthogonal, for which it is sufficient to show that they have orthogonal images in each irreducible representation, i.e., that for $k = 0, \dots, k_0$ and $v_k \in CX_k|_{(0,1]}$, the two operators

$$(3.257) \quad \sigma_k \circ \lambda_{k_0}^k(\phi_{i_s,j_s}(1))(v_k), \quad s = 1, 2,$$

are orthogonal.

Three cases naturally arise.

(a) $k(i_s, j_s) > k$ for some s : By Lemma 3.16 (i), one of the elements in (3.257) is already 0.

(b) $k(i_s, j_s) = k$ for $s = 1, 2$: In this case, e_{i_1,j_1} and e_{i_2,j_2} are orthogonal (their supports lie in disjoint sets W_{i_1,j_1} and W_{i_2,j_2} respectively), and so by Lemma 3.16 (ii), the elements in (3.257) are orthogonal.

(c) $k(i_s, j_s) \leq k$ for $s = 1, 2$, with at least one inequality strict: If $v_k \in CX_k|_{[2\eta,0]}$ then by Lemma 3.16 (iv), one of the elements in (3.257) is already zero. So assume that $v_k \in CX_k|_{(0,2\eta]}$, so that the elements in (3.257) are both described in Lemma 3.16 (iii).

The sum in Lemma 3.16 (iii) is taken over a subset of $W_{i,j}$, and therefore, for $(i, j) = (i_1, j_1)$ and $(i, j) = (i_2, j_2)$, it is taken over disjoint sets. Since the ranges of $\mu_{k-1, 2\eta^{k-1}, \hat{\theta}_k(x_k), z_{k-1}}$ are orthogonal as z_{k-1} varies, it follows that the two elements in (3.257) are orthogonal.

(iii) follows from (iii)', which we do presently.

(iii)': We work again in each irreducible representation, that is, we will show that for $k = 0, \dots, k_0$ and $v_k \in CX_k|_{(0,1]}$,

$$(3.258) \quad \sum_{i,j} \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j}(1))(v_k) \approx_\epsilon 1_{m_k}.$$

We must break our analysis into two cases.

(a) $v_k \in CX_k|_{[2\eta,1]}$: By Lemma 3.16 (i) and (iv), the only contribution to the sum in (3.258) comes from (i, j) , for which $k(i, j) = k$, and then by Lemma 3.16 (ii), we get

$$(3.259) \quad \begin{aligned} \sum_{i,j} \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j}(1))(v_k) &= \sum_{i,j} 1_{m_k} \otimes e_{i,j}(v_k) \\ &\approx_\epsilon 1_{m_k} \otimes 1_D, \end{aligned}$$

(by hypothesis).

(b) $v_k = (x_k, t) \in CX_k|_{(0,2\eta)}$: By Lemma 3.16 (i), the only contributions to the sum in (3.258) come from those (i, j) for which $k(i, j) \geq k$. From Lemma 3.16 (iii) and (3.229), we derive that

$$(3.260) \quad \begin{aligned} & \sigma_k \circ \lambda_{k_0}^k(\phi_i(1))(x_k, t) \\ &= \sum_{z_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z_{k-1}}(e_i \circ \rho_{k, \overline{\eta}^k, \overline{2\eta}^k}((x_k, z_{k-1}), t)), \end{aligned}$$

where the sum is over all

$$(3.261) \quad z_{k-1} \in S_{k-1, \overline{2\eta}^{k-1}, m_k}(\hat{\theta}_k(x_k)).$$

Thus,

$$(3.262) \quad \begin{aligned} & \sum_{i,j} \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j}(1))(v_k) \\ &= \sum_{i, z_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z_{k-1}}(e_i \circ \rho_{k, \overline{\eta}^k, \overline{2\eta}^k}((x_k, z_{k-1}), t)) \\ &\approx_\epsilon \sum_{z_{k-1}} \mu_{k-1, \overline{2\eta}^{k-1}, \hat{\theta}_k(x_k), z_{k-1}}(1) \\ &\stackrel{\text{Lemma 3.7}}{=} 1, \end{aligned}$$

where the approximation to within ϵ is possible since the summands are orthogonal. \square

Recall that the open cover \mathcal{U}_{k_0} depended on open covers \mathcal{V}_k of CX_k which in turn depended on an open set \mathcal{F} and a tolerance $\delta > 0$, by the fact that $\sigma_k \circ \lambda_{k_0}^k(a)$ is approximately constant on each set in \mathcal{V}_k , for $a \in \mathcal{F}$. We exploit this relationship in proving the following.

Lemma 3.19. *Suppose that $(e_i)_i$ is an $(n+1)$ -colourable ϵ -approximate partition of unity. Let $a \in \mathcal{F}$. Then*

$$(3.263) \quad \sum_{i,j} \phi_{i,j} \circ \psi_{i,j}(a) \approx_{(n+1)\delta+\epsilon} a \otimes 1_D.$$

Proof. Let us first show that

$$(3.264) \quad \phi_{i,j}(1) \cdot (a \otimes 1_D) = (a \otimes 1_D) \cdot \phi_{i,j}(1) \approx_\delta \phi_{i,j} \circ \psi_{i,j}(a)$$

for every i, j , by showing that this holds in every irreducible representation.

To this end, let $k = 0, \dots, k_0$, let $v_k \in CX_k|_{(0,1]}$ and consider images under the representation

$$(3.265) \quad \text{ev}_{v_k} \circ \sigma_k \circ \lambda_{k_0}^k.$$

Set

$$(3.266) \quad a_k := \lambda_{k_0}^k(a) \in A_k.$$

The analysis divides naturally into three cases.

(i) $k < k(i, j)$: In this case, everything in (3.264) is zero in A_k , by Lemma 3.16 (i).

(ii) $k = k(i, j)$: By Lemma 3.16 (ii),

$$\begin{aligned}
& \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j}(1) \cdot (a \otimes 1_D))(v_k) \\
&= (1_{m_k} \otimes e_{i,j}(v_k)) \cdot (\sigma_k(a_k)(v_k) \otimes 1_D) \\
&= (\sigma_k(a_k)(v_k) \otimes e_{i,j}(v_k)) \\
&= \nu_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, v_k}(a) \otimes e_{i,j}(v_k) \\
&\stackrel{(3.207), (3.232), (3.235)}{\approx_\delta} \nu_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, z_{k_0}^{(i,j)}}(a) \otimes e_{i,j}(v_k) \\
&\stackrel{(3.236)}{=} (\psi_{i,j}(a) \otimes e_{i,j}(v_k)) \\
(3.267) \quad &= \sigma_k \circ \lambda_{k_0}^k(\phi_{i,j} \circ \psi_{i,j}(a)).
\end{aligned}$$

We clearly get the same thing in the second line if we start with the expression

$$(3.268) \quad \sigma_k \circ \lambda_{k_0}^k((a \otimes 1_D) \cdot \phi_{i,j}(1))(v_k),$$

and thus we have established (3.264) in this irreducible representation.

(iii) $k > k(i, j)$: By Lemma 3.16 (iv) and the fact that $\phi_{i,j}$ is order zero, everything in (3.264) is zero in this representation for $v_k \in CX_k|_{[2\eta, 1]}$. For $v_k = (x_k, t) \in CX_k|_{(0, 2\eta)}$, note that

$$(3.269) \quad \sigma_k(a_k) = \hat{\theta}_k(x_k)(a_{k-1}).$$

As in Lemma 3.16 (iii), we have

$$\begin{aligned}
 & \sigma_k \circ \lambda_{k_0}^k (\phi_{i,j}(1) \cdot (a \otimes 1_D))(v_k) \\
 (3.234) \quad & \stackrel{=}{=} \sigma_k \circ \lambda_{k_0}^k (\xi_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, k(i,j)} (1_{m_{k(i,j)}} \otimes e_{i,j})(v_k) \\
 & \quad \cdot (\sigma_k(a_k) \otimes 1_D)(v_k)) \\
 (3.159) \quad & \stackrel{=}{=} \check{\xi}_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}, k(i,j)} (1 \otimes e_{i,j} |_{Z_{k, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-k(i,j)}}^{**}})(v_k) \\
 & \quad \cdot (\sigma_k(a_k) \otimes 1_D)(v_k) \\
 (3.144), (3.232) \quad & \stackrel{=}{=} \sum_{z_{k-1} \in S_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, m_k}} (\hat{\theta}_k(x_k)) \cap \hat{W}_{i,j} \\
 & (\mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (1) \otimes e_{i,j}(v_k)) \cdot (\sigma_k(a_{k-1}) \otimes 1_D)(v_k) \\
 (3.269) \quad & \stackrel{=}{=} \sum_{z_{k-1} \in S_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, m_k}} (\hat{\theta}_k(x_k)) \cap \hat{W}_{i,j} \\
 & (\mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (1) \otimes e_{i,j}(v_k)) \cdot (\hat{\theta}_k(x_k)(a_{k-1}) \otimes 1_D) \\
 & = \sum_{z_{k-1}} (\mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (1) \cdot \hat{\theta}_k(x_k)(a_{k-1})) \\
 & \quad \otimes e_{i,j}(v_k) \\
 (3.270) \quad (3.89) \quad & \stackrel{=}{=} \sum_{z_{k-1}} \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} \\
 & \quad \circ \nu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, z_{k-1}} (a_{k-1}) \otimes e_{i,j}(v_k) \\
 (3.79) \quad & \stackrel{=}{=} \sum_{z_{k-1}} \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} \\
 & \quad \circ \nu_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, ((x_k, z_{k-1}), t)} (a) \otimes e_{i,j}(v_k) \\
 (3.207), (3.232), \quad & \\
 (3.235) \quad & \stackrel{\approx}{\approx} \delta \sum_{z_{k-1}} \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} \\
 & \quad \circ \nu_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k_0-k(i,j)}, z_{k_0}^{(i,j)}} (a) \otimes e_{i,j}(v_k) \\
 (3.236) \quad & \stackrel{=}{=} \sum_{z_{k-1}} \mu_{k-1, \bar{\eta}^{k(i,j)} \overline{2\eta}^{k-1-k(i,j)}, \hat{\theta}_k(x_k), z_{k-1}} (\psi_{i,j}(a)) \\
 & \quad \otimes e_{i,j}(v_k) \\
 (3.144) \quad & \stackrel{=}{=} \check{\xi}_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}, k} (\psi_{i,j}(a) \otimes e_{i,j}(v_k)) \\
 (3.159) \quad & \stackrel{=}{=} (\sigma_k \circ \lambda_{k_0}^k \circ \xi_{k_0, \bar{\eta}^{k(i,j)} \overline{2\eta}, k} (\psi_{i,j}(a) \otimes e_{i,j})) (v_k) \\
 (3.271) \quad (3.234) \quad & \stackrel{=}{=} \sigma_k \circ \lambda_{k_0}^k \circ \phi_{i,j} \circ \psi_{i,j}(a),
 \end{aligned}$$

where we have used the fact that the summands are orthogonal to get the approximation to within δ . We clearly get the same thing in (3.270)

if we start with the expression

$$(3.272) \quad \sigma_k \circ \lambda_{k_0}^k((a \otimes 1_D) \cdot \phi_{i,j}(1))(v_k),$$

and thus we have shown (3.264) in this irreducible representation.

We have finished establishing (3.264).

By Lemma 3.18 (ii), we may colour (i.e., partition) the set of all indices (i, j) into

$$(3.273) \quad I_0 \amalg \cdots \amalg I_n,$$

in such a way that for $c = 0, \dots, n$,

$$(3.274) \quad (\phi_{i,j})_{(i,j) \in I_c}$$

have orthogonal ranges. It follows from (3.264) that

$$(3.275) \quad \sum_{(i,j) \in I_c} \phi_{i,j}(1)(a \otimes 1_D) \approx_\delta \sum_{(i,j) \in I_c} \phi_{i,j} \circ \psi_{i,j}(a).$$

Therefore,

$$(3.276) \quad \begin{aligned} a \otimes 1_D &\stackrel{\text{Lemma 3.18 (iii)'}}{\approx_\epsilon} \sum_{i,j} \phi_{i,j}(1)(a \otimes 1_D) \\ &= \sum_{c=0}^n \sum_{(i,j) \in I_c} \phi_{i,j}(1)(a \otimes D) \\ &\stackrel{(3.275)}{\approx_{(n+1)\delta}} \sum_{c=0}^n \sum_{(i,j) \in I_c} \phi_{i,j} \circ \psi_{i,j}(a) \\ &= \sum_{i,j} \phi_{i,j} \circ \psi_{i,j}(a), \end{aligned}$$

as required. \square

3.11. Proof of Theorem 3.1 and applications.

Proof of Theorem 3.1. Since A is an NC cell complex, we may assume it is A_{k_0} .

Since A_{k_0} is locally approximated by $A_{k_0, \bar{\eta}}$, it suffices to show that

$$(3.277) \quad \text{dr}(A_{k_0, \bar{\eta}} \otimes 1_D \subseteq A_{k_0} \otimes D) \leq n.$$

Let $\mathcal{F} \subseteq A_{k_0, \bar{\eta}}$ and let $\epsilon > 0$. Using this \mathcal{F} and $\delta := \frac{\epsilon}{2(n+1)}$, form the open cover \mathcal{U}_{k_0} as in Section 3.9. By the hypothesis, applied to $Z := Z_{k_0, \bar{\eta}}$, and by [40, Proposition 3.2], there exists an $(n+1)$ -colourable $\epsilon/2$ -approximate partition of unity $(e_i)_i$ in $C(Z_{k_0, \bar{\eta}}, D)$ subordinate to \mathcal{U}_{k_0} . Form the c.p.c. maps $\phi_{i,j}$ and $\psi_{i,j}$ as in (3.234), (3.236). By Lemma 3.18 (ii), $\phi = \sum_{i,j} \phi_{i,j}$ is $(n+1)$ -colourable, and by Lemma 3.18 (iii)', $\|\phi\| \leq 1 + \epsilon/2$. Rescaling, we may arrange that ϕ is c.p.c.

Finally, by Lemma 3.19,

$$(3.278) \quad \sum_{i,j} \phi_{i,j} \circ \psi_{i,j}(a) \approx_{\epsilon} a \otimes 1_D$$

for all $a \in \mathcal{F}$. □

Corollary 3.20 (Theorem A). *$\text{dr}A \leq 2$ for any \mathcal{Z} -stable, locally subhomogeneous algebra.*

Proof. Since locally subhomogeneous algebras are locally approximated by NC cell complexes, and by [40, Proposition 2.6], it suffices to show, for any NC cell complex A , that the inclusion

$$(3.279) \quad A \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}} \subset A \otimes \mathcal{Z} \otimes \mathcal{Z}$$

has decomposition rank at most 2. By approximating, it suffices to show this for the inclusion

$$(3.280) \quad A \otimes \mathcal{Z}_{p,q} \otimes 1_{\mathcal{Z}} \subset A \otimes \mathcal{Z}_{p,q} \otimes \mathcal{Z}.$$

Note that $A \otimes \mathcal{Z}_{p,q}$ is itself an NC cell complex, so that this inclusion has decomposition rank at most 2 follows from Theorem 3.1 and [40]. □

Theorem 3.1 also gives a different proof of the following result, the main theorem of [45].

Corollary 3.21 ([45]). *$\text{dr}A \leq n$ for any separable subhomogeneous algebra, provided that for each k , $\text{Prim}_k(A)$ has dimension at most n .*

Proof. By Theorem 2.15, A is locally approximated by NC cell complexes where the cells have dimension at most n . (Note that this does not use the result of [45], since the proof of Theorem 2.15 only uses the fact that $\text{Prim}_k(A)$ has dimension at most n .) Therefore, by Proposition 1.6, we may assume that A is such an NC cell complex.

This means that A has the form given in Section 3.2, where $\dim X_i \leq n - 1$ for all $i = 0, \dots, k$.

In order to apply Theorem 3.1, we must show that $Z_{k,\bar{\eta}}$ has dimension at most n . Evidently, there exists a finite-to-one continuous map $Y_{k,\bar{\eta}} \rightarrow X_i$ and so by [12, Theorem 1.12.4],

$$(3.281) \quad \dim Y_{k,\bar{\eta}} \leq \dim X_i \leq n$$

for every k . Then since $Z_{k,\bar{\eta}}$ can be decomposed into a closed set homeomorphic to $Z_{k-1,\bar{\eta}}$, an open set homeomorphic to $Y_{k,\bar{\eta}} \times (0, \eta)$, and a closed set homeomorphic to CX_k , we can show inductively that

$$(3.282) \quad \dim Z_{k,\bar{\eta}} \leq n.$$

□

4. MINIMAL \mathbb{Z} -CROSSED PRODUCTS

Let us now consider crossed products associated to minimal homeomorphisms $\alpha: X \rightarrow X$, for compact metrizable spaces X . Using $C(X) \rtimes_{\alpha} \mathbb{Z}$ to denote the crossed product, we show the following result.

Theorem 4.1. *Let X be a compact metrizable space, let $\alpha: X \rightarrow X$ be a minimal homeomorphism, and let U be an infinite dimensional UHF algebra. Then*

$$(4.1) \quad (C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes U$$

is locally subhomogeneous.

In particular, this result shows that Theorem A applies to $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes U$, and then on using [4, Lemma 3.1], it follows that $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}$ has finite decomposition rank, as explained in Corollary 4.8, below. In fact, using the heavy classification machinery of Gong, Lin, and Niu [14], Lin has shown that $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}$ is approximately subhomogeneous (see Remark 4.9) [21].

It would be nice to have a direct proof (such as the proof of Theorem 4.1) that \mathcal{Z} -stable (or even all) minimal \mathbb{Z} -crossed products are locally subhomogeneous, not using lengthy classification results.

Let X, α as in Theorem 4.1, and let $x \in X$. Let $u \in C(X) \rtimes_{\alpha} \mathbb{Z}$ denote the canonical unitary, so that

$$(4.2) \quad u^* f u = f \circ \alpha \text{ for } f \in C(X) \subset C(X) \rtimes_{\alpha} \mathbb{Z}.$$

Define

$$(4.3) \quad A_x := C^*(C(X) \cup uC_0(X \setminus \{x\})) \subset C(X) \rtimes_{\alpha} \mathbb{Z}.$$

This algebra is known to be locally subhomogeneous (in fact, an inductive limit of fairly explicit subhomogeneous algebras), as well as simple; these facts are largely due to Q. Lin [23]; see [30, Theorem 4.1] for a concise account. Many times has the structure of $C(X) \rtimes_{\alpha} \mathbb{Z}$ been studied using A_x [7, 9, 13, 22, 30, 32, 33, 24, 44, 43]! In Lemma 4.7, below, Berg's technique is used to show a new link: up to UHF stabilization, $C(X) \rtimes_{\alpha} \mathbb{Z}$ is locally approximated by the direct sum of a corner of A_x and a circle-matrix algebra. From this, Theorem 4.1 will immediately follow. First, a few (known) preparatory lemmas.

In the following, for positive elements a, e , we write $a \triangleleft e$ to mean $ea = ae = a$.

Lemma 4.2. *Let A be a C^* -algebra with stable rank one. Suppose that $a, b, c \in A_+$ are positive contractions and $p \in A$ is a projection such that*

$$(4.4) \quad \begin{aligned} & a \triangleleft b \triangleleft c \quad \text{and} \\ & [b] \leq [p] \leq [c] \end{aligned}$$

in the Cuntz semigroup $W(A)$. Then there exists a projection $q \in A$ such that $a \triangleleft q \triangleleft c$ and $[p] = [q]$.

Proof. Using [35, Proposition 2.4], we may replace p by a Murray-von Neumann equivalent projection in $A \cap \{1 - c\}^\perp$. Therefore, without loss of generality, $p \triangleleft c$.

Since stable rank one passes to hereditary subalgebras, we may apply [35, Proposition 2.4 (iv)] to obtain a unitary $u \in (A \cap \{1 - c\}^\perp)^\sim$ such that

$$(4.5) \quad uau^* \in \text{her}(p).$$

Therefore, $q = upu^*$ satisfies the conclusion. \square

Lemma 4.3. *Let X, α be as in Theorem 4.1, let $x \in X$, $h \in C(X)$ and $k \in \mathbb{N}$.*

- (i) *If $h(\alpha^{-i}(x)) = 0$ for $i = 0, \dots, k - 1$ then $u^k h \in A_x$.*
- (ii) *If $h(\alpha^i(x)) = 0$ for $i = 1, \dots, k$ then $hu^k \in A_x$.*

Proof. (i): It suffices to show that $u^k h$ is approximately contained in A_x . Therefore, perturbing h , we may assume that h is zero in a neighbourhood of $\alpha^{-i}(x)$ for each $i = 0, \dots, k - 1$. Then there exists $e \in C_0(X \setminus \{x\})$ such that

$$(4.6) \quad (e \circ \alpha^i)h = h$$

for $i = 0, \dots, k - 1$. Thus,

$$(4.7) \quad \begin{aligned} u^k h &= u^k (e \circ \alpha^{k-1}) \cdots (e \circ \alpha) e h \\ &= (ue)^k h \in A_x. \end{aligned}$$

(ii): Note that $hu^k = u^k (h \circ \alpha^k)$. In view of this, (ii) follows directly from (i). \square

Lemma 4.4. *Let X, α be as in Theorem 4.1, let $x \in X$, and let U be an infinite dimensional UHF algebra. Let $\mathcal{G} \subset C(X)$ be a finite set, let $\eta > 0$, and let $k \in \mathbb{N}$. Then there exist $m \in \mathbb{N}$ with $k < m$, a finite set $\mathcal{H} \subset C(X)$, and orthogonal projections*

$$(4.8) \quad p_{-k}, p_{-k+1}, \dots, p_{m-1}, p_m \in A_x \otimes U \cap \mathcal{H}'$$

such that:

- (i) $\mathcal{G} \subset_\eta \mathcal{H}$;
- (ii) $1 - p_0 \in \text{her}(C_0(X \setminus \{x\}))$;
- (iii) for $j \neq 0$, $p_j \in \text{her}(C_0(X \setminus \{x\}))$;
- (iv) for $j = -k, \dots, m - 1$,

$$(4.9) \quad p_{j+1} = up_j u^*.$$

- (v) for $f \in \mathcal{H}$ and $j = -k, \dots, m$, there exists $\lambda_{f,j} \in \mathbb{C}$ such that

$$(4.10) \quad fp_j = \lambda_{f,j} p_j;$$

(vi) for $f \in \mathcal{H}$ and $j = 0, \dots, k$,

$$(4.11) \quad \lambda_{f,-j} \approx_\eta \lambda_{f,m-j}.$$

Proof. The crux of this argument is the use of Berg's techniques, inspired by the proof of [22, Lemma 4.2].

Fix a compatible metric d on X and let $\delta > 0$ be such that, for $x, y \in X$, if $d(x, y) < \delta$ then

$$(4.12) \quad |f(x) - f(y)| < \eta, \text{ for all } f \in \mathcal{G}.$$

By minimality, there exists $m > k$ be such that

$$(4.13) \quad d(\alpha^{m-j}(x), \alpha^{-j}(x)) < \delta$$

for all $j = 0, \dots, k$.

Let V be a neighbourhood of x such that

$$(4.14) \quad \alpha^{-k}(V), \alpha^{-k+1}(V), \dots, \alpha^m(V)$$

are disjoint. By possibly shrinking V , each function $f \in \mathcal{G}$ can be perturbed by at most η to a function f' which is constant on each set in (4.14). Set

$$(4.15) \quad \mathcal{H} := \{f' \mid f \in \mathcal{G}\},$$

and we see that (i) holds. For $f \in \mathcal{H}$ and $j = -k, \dots, m$, set $\lambda_{f,j}$ equal to the complex number satisfying

$$(4.16) \quad f|_{\alpha^j(V)} \equiv \lambda_{f,j}.$$

It follows from (4.12) and (4.13) that

$$(4.17) \quad |\lambda_{f,-j} - \lambda_{f,m-j}| < \eta$$

for all $f \in \mathcal{H}$, $j = 0, \dots, k$, establishing (vi).

Let $h_0 \in C_0(V)_+$ be a function that is constant equal to 1 in a neighbourhood of x . Since $(A_x \otimes U) \cap \{1 - h_0\}^\perp$ is a non-zero full hereditary subalgebra of the UHF-stable, unital algebra $A_x \otimes U$, there exists a non-zero projection

$$(4.18) \quad p'_0 \in (A_x \otimes U) \cap \{1 - h_0\}^\perp.$$

Since α is minimal, we may find $g'_0, g_0 \in C(X)_+ \cap \{1 - h_0\}^\perp$ such that

$$(4.19) \quad d_\tau(g'_0) < \tau(p'_0) \text{ for all } \tau \in T(A_x),$$

$g_0 \triangleleft g'_0$, and g_0 is constantly equal to 1 in a neighbourhood of x . By strict comparison, it follows that $[g'_0] \leq [p'_0]$ in the Cuntz semigroup. Therefore, by Lemma 4.2, there exists a projection

$$(4.20) \quad p_0 \in A_x \otimes U$$

such that

$$(4.21) \quad g_0 \triangleleft p_0 \triangleleft h_0.$$

(We can now forget g'_0 and p'_0 .)

For $j = -k, \dots, m$, set

$$(4.22) \quad p_j := u^j p_0 u^{-j}.$$

Thus, we have defined projections $p_{-k}, \dots, p_m \in A \otimes U$, and it is quite clearly the case that (iii)–(v) hold. From (v), it follows that each p_j commutes with \mathcal{H} . Point (ii) follows since

$$(4.23) \quad 0 \leq 1 - p_0 \leq 1 - g_0 \in C_0(X \setminus \{x\}).$$

The projections are orthogonal by disjointness of $(\alpha^j(V))_{j=-k, \dots, m}$.

Let us show that $p_j \in A_x \otimes U$ for $j = -k, \dots, m$. For $j = 0$, this is true by our choice of p_0 .

For $j = -k, \dots, 0$,

$$(4.24) \quad p_j = u^j p_0 u^{-j} = (h_0 u^{-j})^* p_0 (h_0 u^{-j}) \in A_x \otimes U$$

by Lemma 4.3 (ii).

$j = 1$ is a special case; here, we note that since $h_0 - g_0$ vanishes in a neighbourhood of x , and by (4.21), it follows that there exists $e \in C_0(X \setminus \{x\})_+$ such that $p_0 - g_0 \triangleleft e$. Therefore,

$$(4.25) \quad \begin{aligned} p_1 &= u p_0 u^* \\ &= u e (p_0 - g_0) (u e)^* + u g_0 u^* \in A_x \otimes U. \end{aligned}$$

Now, for $j = 2, \dots, m$,

$$(4.26) \quad p_j = u^{j-1} p_1 u^{-(j-1)} = (u^{j-1} (1 - h_0)) p_1 (u^{j-1} (1 - h_0))^* \in A_x \otimes U$$

by Lemma 4.3 (i). □

Lemma 4.5. *Let A be a C^* -algebra. There exists a uniformly continuous map*

$$(4.27) \quad \gamma: \{v \in A \mid v \text{ is a partial isometry and } v^2 = 0\} \times [0, 1] \rightarrow A$$

such that, for each v , the assignment $t \mapsto \gamma(v, t)$ defines a path of projections in $C^*(v)$, from vv^* to v^*v . Also, for any unitary $u \in A$,

$$(4.28) \quad u \gamma(v, t) u^* = \gamma(u v u^*, t).$$

Proof. Simply set

$$(4.29) \quad \gamma(v, t) := t^2 v^* v + t(1-t)(v + v^*) + (1-t)^2 v v^*.$$

In matrix form, this is

$$(4.30) \quad \begin{pmatrix} t^2 & t(1-t) \\ t(1-t) & (1-t)^2 \end{pmatrix}.$$

The desired properties are easy to verify. □

Lemma 4.6. *The universal C^* -algebra generated by partial isometries w_1, \dots, w_m such that*

$$(4.31) \quad \begin{aligned} w_i^* w_i &= w_{i+1} w_{i+1}^*, & i = 1, \dots, m-1, \\ w_m^* w_m &= w_1 w_1^*, & \text{and} \\ w_i^* w_j &= 0, & i \neq j \end{aligned}$$

is isomorphic to $C(\mathbb{T}) \otimes M_m$, via an isomorphism sending w_i to $1 \otimes e_{i,i+1}$ for $i = 1, \dots, m-1$ and w_m to $z \otimes e_{m,1}$ where $z \in C(\mathbb{T})$, is a generating unitary.

Proof. Let A be the universal C^* -algebra generated by the w_i satisfying the relations (4.31). It is easy to see that the described map, call it π , is a surjective homomorphism. To see that π is injective, one derives from the relations that

- (i) $\sum_{i=1}^m w_i^* w_i$ is a unit for A ;
- (ii) $u := \sum_{i=1}^m w_i w_{i+1} \cdots w_m w_1 \cdots w_{i-1}$ is a unitary in the centre of A , so that under any irreducible representation, the image of u is a scalar of modulus 1;
- (iii) $\{w_i \mid i = 0, \dots, m-1\}$ generates a unital copy of M_m in A (therefore also in any irreducible representation); and
- (iv) $w_m = w_{m-1}^* \cdots w_1^* u$.

From this, one sees that every irreducible representation of A is equivalent to $\text{ev}_z \circ \pi$ for some point $z \in \mathbb{T}$, and therefore, π is injective. \square

Lemma 4.7. *Let X, α be as in Theorem 4.1, let $x \in X$, and let U be an infinite dimensional UHF algebra. Let \mathcal{C} denote the class of all C^* -algebras of the form*

$$(4.32) \quad p(A_x \otimes U)p \oplus C(\mathbb{T}, M_k),$$

where $p \in A_x \otimes U$ is a projection and $k \in \mathbb{N}$.

Then $(C(X) \rtimes_\alpha \mathbb{Z}) \otimes U$ is locally \mathcal{C} .

Proof. This proof uses ideas from [22, Lemma 4.2].

Let $\mathcal{F} \subset (C(X) \rtimes_\alpha \mathbb{Z}) \otimes U$ be a finite subset to be approximated and let $\epsilon > 0$ be a specified tolerance. By approximating within $(C(X) \rtimes_\alpha \mathbb{Z}) \otimes M$ for some matrix subalgebra M of U , and then observing that it suffices to approximate the matrix entries in $(C(X) \rtimes_\alpha \mathbb{Z}) \otimes (U \cap M')$, we see that without loss of generality, $\mathcal{F} \subset (C(X) \rtimes_\alpha \mathbb{Z}) \otimes 1_U$, which we hereby identify with $C(X) \rtimes_\alpha \mathbb{Z}$. Without loss of generality again, $\mathcal{F} = \{u\} \cup \mathcal{G}$ where $\mathcal{G} \subset C(X)$.

Let $k \in \mathbb{N}$ be such that, for the map γ defined in Lemma 4.5 (for $A = (C(X) \rtimes_\alpha \mathbb{Z}) \otimes U$), if $|t - t'| < 1/k$ then

$$(4.33) \quad \|\gamma(v, t) - \gamma(v, t')\| < \epsilon/16.$$

Apply Lemma 4.4 with $\eta := \epsilon/16$, to obtain m, \mathcal{H} , and $p_{-k}, \dots, p_m \in A_x \otimes U$ satisfying the conclusions of that lemma.

Note that $p_m = u^m p_0 u^{-m} \sim p_0$ in $A \otimes U$. Since the embedding of $A_x \otimes U$ in $A \otimes U$ induces an isomorphism between ordered K_0 -groups [30, Theorem 4.1 (3)], and since $A_x \otimes U$ has cancellation of projections, there exists $v_0 \in A_x \otimes U$ such that

$$(4.34) \quad p_0 = v_0 v_0^* \quad \text{and} \quad v_0^* v_0 = p_m.$$

For $j = k, \dots, 1$, set

$$(4.35) \quad v_j := u^{-j} v_0 u^j = u^{-j} h_0 v_0 h_m u^j.$$

By Lemma 4.3 $v_j \in A_x \otimes U$. Also, an easy calculation yields

$$(4.36) \quad p_{-j} = v_j v_j^* \quad \text{and} \quad v_0^* v_0 = p_{m-j}.$$

Using γ from Lemma 4.5, set

$$(4.37) \quad \tilde{p}_{-j} := \gamma(v_j, j/k) \in (A_x \otimes U) \cap \text{her}(p_{-j} + p_{m-j}).$$

From the definition,

$$(4.38) \quad \tilde{p}_0 = p_0, \quad \tilde{p}_{-k} = p_{m-k},$$

and for each $j = 1, \dots, k$,

$$(4.39) \quad \begin{aligned} \|u \tilde{p}_{-j} u^* - \tilde{p}_{-j+1}\| &= \|u \gamma(v_{-j}, j/k) u^* - \gamma(v_{-j+1}, (j-1)/k)\| \\ &= \|\gamma(u v_{-j} u^*, j/k) - \gamma(v_{-j+1}, (j-1)/k)\| \\ &= \|\gamma(v_{-j+1}, j/k) - \gamma(v_{-j+1}, (j-1)/k)\| \\ &\stackrel{(4.33)}{<} \epsilon/16, \end{aligned}$$

and, therefore, there exists $w_{-j} \in A \otimes U$ such that

$$(4.40) \quad w_{-j}^* w_{-j} = \tilde{p}_{-j+1}, \quad w_{-j} w_{-j}^* = \tilde{p}_{-j}, \quad \text{and} \quad \|w_{-j} - \tilde{p}_{-j} u^*\| < \epsilon/4.$$

For $j = 0, \dots, m-k-1$, set

$$(4.41) \quad w_j := p_j u^* \in A \otimes U.$$

Now, set

$$(4.42) \quad C := C^*(\{w_{-k}, \dots, w_{m-k-1}\}) \subseteq A \otimes U.$$

To complete the proof, we shall show that:

- (i) $C \cong C(\mathbb{T}, M_m)$;
- (ii) $1_C \in A_x \otimes U$ and therefore $1_{A \otimes U} - 1_C \in A_x \otimes U$;
- (iii) $\|[1_C, \mathcal{F}]\| < \epsilon/2$;
- (iv) $\mathcal{F}(1_{A \otimes U} - 1_C) \subset A_x \otimes U$; and
- (v) $\mathcal{F}1_C \approx_{\epsilon/2} C$.

(i): Lemma 4.6 shows that there is a surjective map from $C(\mathbb{T}) \otimes M_m$ to C . Moreover, since $u1_C$ has non-zero K_1 -class (in $1_C((C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes U)1_C$) and is approximately contained in C (as will be shown in (v)), $K_1(C) \neq 0$, and, therefore, this map must be surjective.

(ii): We have already seen that $\tilde{p}_{-k}, \dots, \tilde{p}_0, p_1, \dots, p_{m-k-1} \in A_x$. It is not hard to see that the sum of these projections is 1_C .

(iii): For $g \in \mathcal{G}$, let $f \in \mathcal{H}$ be such that $g \approx_\eta f$. We have

$$(4.43) \quad \begin{aligned} [1_C, f] &= \left[\sum_{j=0}^k \tilde{p}_{-j} + \sum_{j=1}^{m-k} p_j, f \right] \\ &= \sum_{j=0}^k [\tilde{p}_{-j}, f] + \sum_{j=1}^{m-k-1} [p_j, f]. \end{aligned}$$

From Lemma 4.4, each p_j commutes with f , so that the second term vanishes. By Lemma 4.4 (v) and (vi), and since \tilde{p}_{-j} is in the hereditary subalgebra generated by $p_{-j} + p_{m-j}$, we see that for $f \in \mathcal{F}'$,

$$(4.44) \quad \|[\tilde{p}_{-j}, f]\| < 2\eta.$$

Also, since $p_{-j} + p_{m-j}$ commutes with f , $[\tilde{p}_{-j}, f]$ is in the hereditary subalgebra generated by $p_{-j} + p_{m-j}$, and therefore, the first sum in (4.43) is orthogonal. Hence,

$$(4.45) \quad \|[1_C, f]\| = \max_{j=0, \dots, k} \|[\tilde{p}_{-j}, f]\|$$

$$(4.46) \quad \stackrel{(4.44)}{<} 2\eta,$$

and thus $\|[1_C, g]\| < 4\eta < \epsilon/2$.

Next, we compute

$$(4.38) \quad \begin{aligned} & u1_C u^* - 1_C \\ &= \sum_{j=0}^k (u\tilde{p}_{-j}u^* - \tilde{p}_{-j}) + \sum_{j=1}^{m-k-1} (up_ju^* - p_j) \\ &\stackrel{(4.38)}{=} \sum_{j=1}^k (u\tilde{p}_{-j}u^* - \tilde{p}_{-j+1}) + up_0u^* - p_{m-k} + \sum_{j=1}^{m-k-1} (up_ju^* - p_j) \\ &= \sum_{j=1}^k (u\tilde{p}_{-j}u^* - \tilde{p}_{-j+1}) + \sum_{j=0}^{m-k-1} (up_ju^* - p_{j+1}) \\ &\stackrel{\text{Lemma 4.4 (iv)}}{(4.47)} \sum_{j=1}^k (u\tilde{p}_{-j}u^* - \tilde{p}_{-j+1}). \end{aligned}$$

Note that \tilde{p}_{-j} is in the hereditary subalgebra generated by $p_{-j} + p_{m-j}$, so that by Lemma 4.4 (iv), $u\tilde{p}_{-j}u^*$ is in the hereditary subalgebra generated by $p_{-j+1} + p_{m-j+1}$, and, therefore, this last sum is orthogonal. Consequently, we obtain

$$(4.48) \quad \begin{aligned} \|u1_C u^* - 1_C\| &= \max_{j=1, \dots, k} \|u\tilde{p}_{-j}u^* - \tilde{p}_{-j+1}\| \\ &\stackrel{(4.39)}{<} \epsilon/16, \end{aligned}$$

as required.

(iv): Since $(1 - 1_C) \in A_x \otimes U$, it follows that $\mathcal{G}(1 - 1_C) \subset A_x \otimes U$. The non-trivial part is showing $u(1 - 1_C) \in A_x \otimes U$.

For $j = 1, \dots, m$, since $\tilde{p}_{-j} \in \text{her}(p_{-j} + p_{m-j})$ and by Lemma 4.4 (iii), we see that $\tilde{p}_{-j} \in \text{her}(C_0(X \setminus \{x\}))$. Using this and Lemma 4.4 (ii) and (iii), we find that

$$(4.49) \quad 1 - 1_C = 1 - p_0 + \sum_{j=1}^k \tilde{p}_{-j} + \sum_{j=1}^{m-k-1} p_j \in \text{her}(C_0(X \setminus \{x\})).$$

Therefore, for $\delta > 0$, there exists $e \in C_0(X \setminus \{x\})$ such that $1 - 1_C \approx_\delta e(1 - 1_C)$. Hence,

$$(4.50) \quad u(1 - 1_C) \approx_\delta ue(1 - 1_C) \in A_x \otimes U.$$

Since δ is arbitrary, $u(1 - 1_C) \in A_x \otimes U$ as required.

(v): For $g \in \mathcal{G}$, let $f \in \mathcal{H}$ be such that $g \approx_\eta f$. We shall show that

$$(4.51) \quad g1_C \approx_{\epsilon/2} \sum_{j=1}^k \lambda_{f,-j} \tilde{p}_{-j} + \sum_{j=0}^{m-k-1} \lambda_{f,j} p_j.$$

Of course, it suffices to show this with f in place of g and η in place of $\epsilon/2$ (since $\eta < \epsilon/2$).

We have

$$(4.52) \quad \begin{aligned} & f1_C - \left(\sum_{j=1}^k \lambda_{f,-j} \tilde{p}_{-j} + \sum_{j=0}^{m-k-1} \lambda_{f,j} p_j \right) \\ &= \sum_{j=1}^k (f - \lambda_{f,-j}) \tilde{p}_{-j} + \sum_{j=0}^{m-k-1} (f - \lambda_{f,j}) p_j \\ &\stackrel{\text{Lemma 4.4 (v)}}{=} \sum_{j=1}^k (f - \lambda_{f,-j}) \tilde{p}_{-j} \end{aligned}$$

Since $\tilde{p}_{-j} \in \text{her}(p_{-j} + p_{m-j})$, by Lemma 4.4 (v) and (vi),

$$(4.53) \quad \|(f - \lambda_{f,-j}) \tilde{p}_{-j}\| < \eta.$$

Moreover since f commutes with $p_{-j} + p_{m-j}$, it follows that $(f - \lambda_{f,-j}) \tilde{p}_{-j} \in \text{her}(p_{-j} + p_{m-j})$, whence the sum in (4.52) is an orthogonal sum, so that

$$(4.54) \quad \begin{aligned} & \left\| f1_C - \left(\sum_{j=1}^k \lambda_{f,-j} \tilde{p}_{-j} + \sum_{j=0}^{m-k-1} \lambda_{f,j} p_j \right) \right\| \\ &= \max_{j=1, \dots, k} \|(f - \lambda_{f,-j}) \tilde{p}_{-j}\| \\ &< \eta. \end{aligned}$$

Finally, we have

$$\begin{aligned}
(4.55) \quad & \|u1_C - (w_{-k}^* + \cdots + w_{m-k-1}^*)\| \\
& \stackrel{(4.41)}{=} \|(u\tilde{p}_{-k} - w_{-k}^*) + \cdots + (u\tilde{p}_{-1} - w_{-1}^*)\| \\
& \leq \left\| \sum_{j \text{ even}} (u\tilde{p}_{-j} - w_{-j}^*) \right\| + \left\| \sum_{j \text{ odd}} (u\tilde{p}_{-j} - w_{-j}^*) \right\|.
\end{aligned}$$

Now, note that $p_{-j+1} + p_{m-j+1}$ (respectively, $p_{-j} + p_{m-j}$) acts as the unit on $u\tilde{p}_{-j} - w_{-j}^*$ on the left (respectively, right). Therefore, we can see that each of the two sums in the last inequality above is an orthogonal sum, and we continue,

$$\begin{aligned}
(4.56) \quad & \|u1_C - (w_{-k}^* + \cdots + w_{m-k-1}^*)\| \leq 2 \max_{j=1, \dots, k} \|(u\tilde{p}_{-k} - w_{-k}^*)\| \\
& \stackrel{(4.40)}{\leq} \epsilon/2,
\end{aligned}$$

as required. \square

Proof of Theorem 4.1. In the context of Lemma 4.7, A_x is locally subhomogeneous, and therefore so also is any unital corner of A_x (any subhomogeneous approximation that approximately contains a projection p will exactly contain a projection p' which is near to p). Thus, it follows from Lemma 4.7 that $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes U$ is locally subhomogeneous. \square

Corollary 4.8. *Let X be an infinite compact metrizable space, let $\alpha: X \rightarrow X$ be a minimal homeomorphism, and let U be an infinite dimensional UHF algebra. Then*

$$(4.57) \quad \text{dr}((C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes U) \leq 2 \quad \text{and}$$

$$(4.58) \quad \text{dr}((C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}) \leq 5.$$

In particular, if α has mean dimension zero then $\text{dr}(C(X) \rtimes_{\alpha} \mathbb{Z}) \leq 5$.

Remark 4.9. In fact, in [21, Corollary 5.3], Lin has very recently proven that when $\alpha: X \rightarrow X$ is a minimal homeomorphism (with X infinite), $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}$ belongs to the class of C^* -algebras classified in [14, Theorem 29.4]. Combined with the range of invariant result [14, Theorem 13.41], it follows that $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}$ is approximately subhomogeneous and has decomposition rank at most two.

Lin's proof uses the present corollary, and thus relies on Theorem A (specifically, that a UHF-stabilization of $C(X) \rtimes_{\alpha} \mathbb{Z}$ has finite nuclear dimension—which is weaker than finite decomposition rank). (Thus, Lin's proof uses Theorem A—and therefore also Theorem B—, as well as Theorem 4.1, of the present paper.)

Proof. The first inequality follows directly from Theorems 4.1 and A, upon noting that U is itself \mathcal{Z} -stable. For the second inequality, note

that by [36, Theorem 3.4], $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}$ is an inductive limit of copies of the C^* -algebra

$$(4.59) \quad (C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z}_{2^{\infty}, 3^{\infty}},$$

which is a continuous field over $[0, 1]$, whose fibres are each equal to $C(X) \rtimes_{\alpha} \mathbb{Z} \otimes U$ for a UHF algebra U . Therefore, the second line follows from the first line and [4, Lemma 3.1]. The last statement follows from the main result of [9], which is that when α has mean dimension zero, the crossed product is \mathcal{Z} -stable. \square

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GEORGE A. ELLIOTT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO,
TORONTO, ONTARIO, CANADA, M5S 2E4

E-mail address: `elliott@math.toronto.edu`

ZHUANG NIU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE,
WY, 82071, USA.

E-mail address: `zniu@uwyo.edu`

LUIS SANTIAGO AND AARON TIKUISIS, INSTITUTE OF MATHEMATICS, UNIVER-
SITY OF ABERDEEN, ABERDEEN, UNITED KINGDOM, AB24 3UE.

E-mail address: `lmoreno@abdn.ac.uk`

E-mail address: `a.tikuisis@abdn.ac.uk`