# MAT 1339 Vectors Material

# Chapter 1

## Goals

- to understand the difference between scalars and vectors
- to understand the various ways of describing the direction of a vector (and be able to convert between them)
- to be able to add (and subtract) vectors geometrically and understand the properties of addition
- to understand scalar multiplication of vectors and its properties
- to be able to use vector addition to solve some physical problems
- to be able to resolve a vector into perpendicular components

## Introduction to Vectors

The first thing we need to do is distinguish vectors from scalars.

A scalar is a quantity that describes magnitude or size only, like numbers (*eg* $\sqrt{2}$, 1/3, −7), temperature (*eg* 12°C), area (*eg* the area of a rectangle with sides of length 2 cm and 4 cm is 8 $cm^2$), speed (*eg* 80 km/h) or distance (*eg* my friend's house is 600 m away from mine).

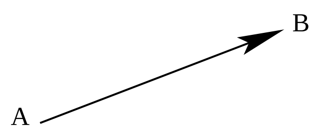
A vector is a quantity that has both magnitude *and* direction, like velocity (*eg* 80 km/h East), displacement (*eg* my friend's house is 600 m North of mine) or force (*eg* 20 N downward).

**Example:**
See if you can distinguish whether the following are scalars or vectors:
(i) my cat's mass is 4.6 kg
(ii) the acceleration of gravity is $g = 9.8$ m/$s^2$ downward
(iii) a boat is sailing at 10 knots westward
(iv) my nephew has 37 video games

(i) mass is a scalar – but be careful, if I had said my cat weighs 45 N, that would be a force and hence a vector (the direction would be downward)
(ii) a vector since there is magnitude and direction

(iii) a vector since there is magnitude and direction
(iv) this is just a number, so a scalar

We can represent a vector in different ways:
(i) in words, like 5 km East
(ii) in a diagram as a geometric vector

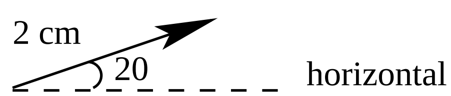



(iii) in a symbolic way as $\vec{v}$ (the arrow above the letter denotes that v is a vector, *not* a scalar)

If we mean a directed line segment from point $A$ (called the starting or initial point or tail) to point $B$ (called the end or terminal point or tip or head) like in the diagram above, we write $\vec{AB}$.

The magnitude or size or length of a vector is written using absolute value bars, so the magnitude of vector $\vec{v}$ is $|\vec{v}|$.

**Example:**
The magnitude of the vector 2 cm at an angle of 20° to the horiziontal is its length, 2 cm.

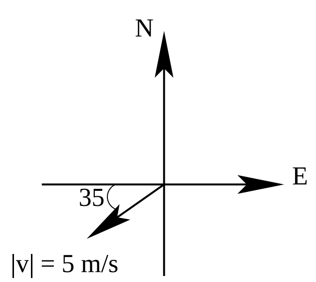



The direction of a vector can be expressed in different ways. In our example above, the direction was specified by an angle, which is measured counterclockwise (ccw) with respect to the horizontal.

Or we could use true or azimuth bearing, which is a three-digit (includes leading zeros if needed) angle measured clockwise (cw) from North. So North is 000°, East is 090°, South is 180° and West is 270°. Our example above is 2 cm 070°.

Or we could use quadrant bearing, which is an angle between 0° and 90° east or west of the north-south line. Our example would be 2 cm N70°E.

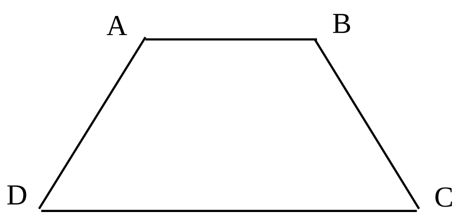
**Example:**



This is 5 m/s at 215° to the horizontal (or 5 m/s at −145° to the horizontal) or 5 m/s at a bearing of 235° or 5 m/s S55°W.
*Can you see where these angles are coming from ?*

Two vectors are said to be parallel if they have the same or opposite directions (but not necessarily the same magnitude)

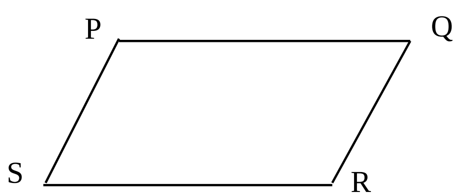
**Example:**
In trapezoid $ABCD$,




$\vec{AB} \parallel \vec{DC}$ (*and* $\vec{AB} \parallel \vec{CD}$).

Two vectors that have the same direction *and* magnitude are said to be equivalent or equal (their actual locations in space do not matter).

**Example:**
In parallelogram $PQRS$,




vectors $\vec{PS}$ and $\vec{QR}$ are equivalent, which we write as $\vec{PS} = \vec{QR}$.
We also have that $\vec{SP} = \vec{RQ}$, $\vec{PQ} = \vec{SR}$ and $\vec{QP} = \vec{RS}$.

Opposite vectors have the same magnitude but opposite direction (again, locations do not matter). The opposite of vector $\vec{v}$ is written as $-\vec{v}$.

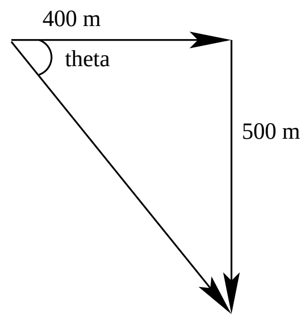
So, in the parallelogram above, $\vec{PS} = -\vec{RQ}$, $\vec{PQ} = -\vec{RS}$, $\vec{QR} = -\vec{SP}$ and $\vec{SR} = -\vec{QP}$.

# Addition and Subtraction of Vectors

If we add two (or more) vectors, we obtain a single vector, often called the resultant, which is the same as the original vectors applied one after another.

Let's see this through an example.

**Example:**
Suppose you walked 400 m East and then 500 m South.

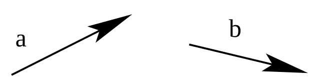



$\tan\theta = \dfrac{500}{400} = \dfrac{5}{4}, \quad \text{so } \theta = \arctan(5/4) \approx 51.3°.$

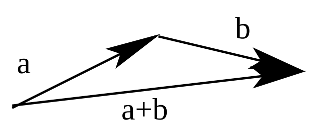
Then, with the help of Pythagoras and trigonometry, we know that this is equivalent to having walked $\sqrt{(400)^2 + (500)^2} \approx 640.3$ m in direction S38.7°E.

This shows us how we can add vectors – by joining them head to tail.

Suppose we have vectors $\vec{a}$ and $\vec{b}$



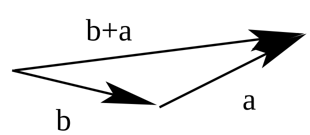

to add them and produce the vector $\vec{a} + \vec{b}$, we move $\vec{b}$ so that its tail touches the head of $\vec{a}$




and then the vector $\vec{a} + \vec{b}$ is the vector that joins the tail of $\vec{a}$ to the tip of $\vec{b}$.

*Can you see how this is exactly what we did in the walking example above?*

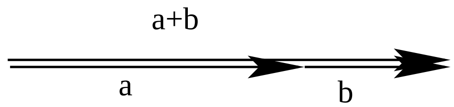
Also, notice this

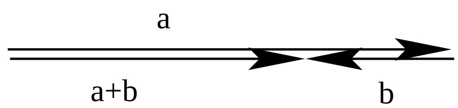



*ie* $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ and so vector addition is commutative.

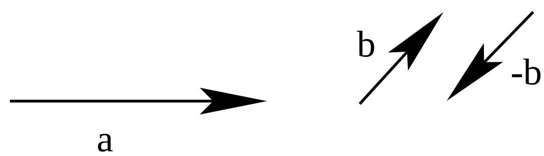
Vector addition is also associative, *ie* $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ and so the order in which we add vectors does not matter (but it will when we subtract vectors below).

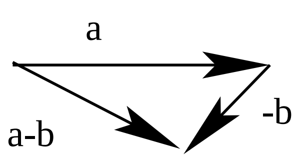
Do we do anything special if the vectors are parallel? No, we do exactly the same thing.



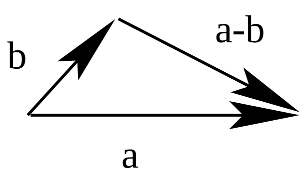




To subtract vectors, we simply recognize that subtraction is adding the negative, which is the opposite when dealing with vectors, so $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.



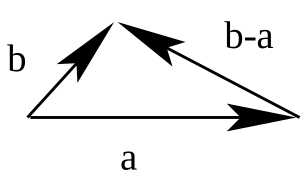




Notice that we could also do this



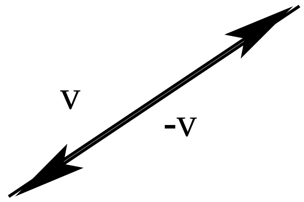

we can find $\vec{a} - \vec{b}$ by joining the vectors tail to tail and joining the head of $\vec{b}$ to the head of $\vec{a}$.

But then




the vector $\vec{b} - \vec{a}$ is the opposite of $\vec{a} - \vec{b}$ and so order in subtraction does matter (just as it does with scalars).

Suppose we take any vector $\vec{v}$ and subtract it from itself,

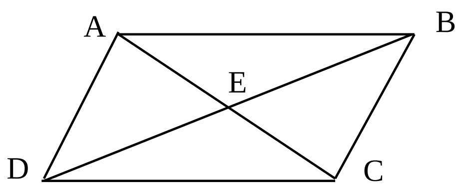



the resultant is $\vec{v} - \vec{v} = \vec{0}$, called the zero vector, which has length or magnitude equal to 0 and points in no specific direction.

The identity property for vector addition should then be clear, $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$.

*Can you see how the properties of vector addition seem to be very similar to those of scalar (number) addition?*

**Example:**
Consider the parallelogram $ABCD$ with diagonals $AC$ and $BD$ that intersect at $E$.




We can write vectors in the figure as sums and differences of others, like $\vec{DB} = \vec{DA} + \vec{AB}$, $\vec{DC} = \vec{DE} + \vec{EC}$, $\vec{AE} = \vec{DE} - \vec{DA}$, and so on...

*Try writing some others this way.*

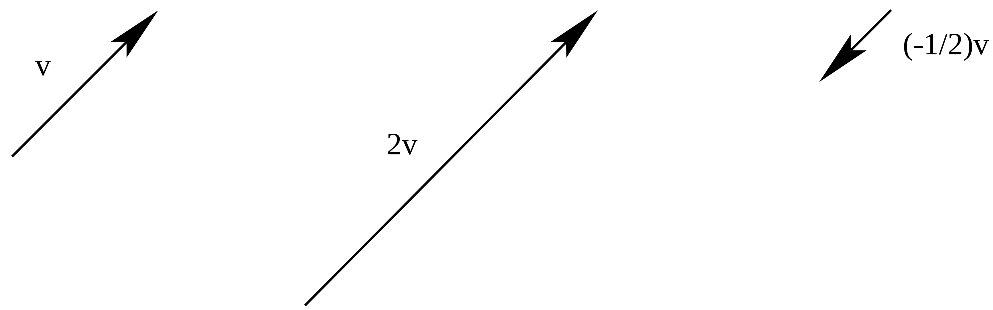
**Example:**
We can simplify expressions like the following:
$((\vec{u} + \vec{v}) - \vec{u}) - \vec{v}$
$= ((\vec{v} + \vec{u}) - \vec{u}) - \vec{v}$ *(commutativity)*
$= ((\vec{v} + \vec{u}) + (-\vec{u})) - \vec{v}$ *(subtraction is adding opposite)*
$= (\vec{v} + (\vec{u} + (-\vec{u})) - \vec{v}$ *(associativity)*
$= (\vec{v} + \vec{0}) - \vec{v}$ *(opposites add to zero vector)*
$= \vec{v} - \vec{v}$ *(identity property)*
$= \vec{v} + (-\vec{v})$
$= \vec{0}$

# Multiplying a Vector by a Scalar

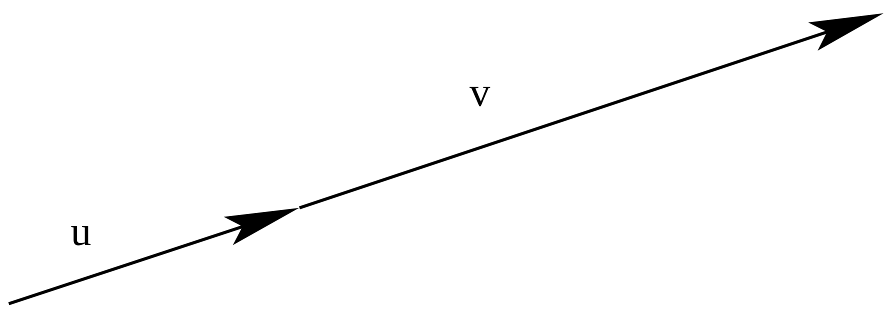
If we take a vector $\vec{v}$ and multiply it by a scalar $k$ (any real number), we are performing scalar multiplication and have produced the scalar multiple $k\vec{v}$. The vector $k\vec{v}$ will be $|k|$ times as long as $\vec{v}$ and it will be parallel to $\vec{v}$.

The magnitude of $k\vec{v}$ is $|k\vec{v}| = |k||\vec{v}|$.
If $k > 0$, $k\vec{v}$ is in the same direction as $\vec{v}$.
If $k < 0$, $k\vec{v}$ is in the opposite direction to $\vec{v}$.
If $|k| > 1$, $k\vec{v}$ is longer than $\vec{v}$.
If $|k| < 1$, $k\vec{v}$ is shorter than $\vec{v}$.

**Example:**




Vectors that are parallel are also said to be collinear because they would lie on a straight line when arranged tail to head. But then they would also be scalar multiples of each other.




*ie* there is some $k \in \mathbb{R}$ such that $\vec{u} = k\vec{v}$.

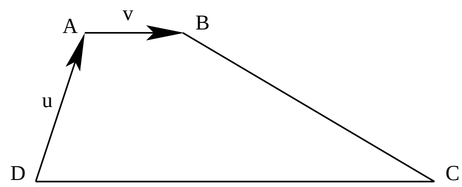
The properties of scalar multiplication are, for any vectors $\vec{u}$ and $\vec{v}$ and scalars $k$ and $c$,
(i) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ *(distributivity)*
(ii) $k(c\vec{v}) = (kc)\vec{v}$ *(associativity)*
(iii) $1\vec{v} = \vec{v}$ *(identity)*

**Example:**
$2(3\vec{u} - \vec{v}) + 4\vec{v}$
$= 2(3)\vec{u} + 2(-\vec{v}) + 4\vec{v}$
$= 6\vec{u} - 2\vec{v} + 4\vec{v}$
$= 6\vec{u} + (-2 + 4)\vec{v}$
$= 6\vec{u} + 2\vec{v}$

A vector of the form $s\vec{u} + t\vec{v}$ (where $s$ and $t$ are scalars) is called a linear combination of vectors $\vec{u}$ and $\vec{v}$.

*Like we have in the examples above and below.*

**Example:**
In the trapezoid $ABCD$, $\vec{AB} \parallel \vec{DC}$ and $DC = 4AB$.




If we let $\vec{DA} = \vec{u}$ and $\vec{AB} = \vec{v}$, then we can write $\vec{DC} = 4\vec{v}$, $\vec{AC} = \vec{AD} + \vec{DC} = -\vec{u} + 4\vec{v}$ and $\vec{BC} = \vec{BA} + \vec{AD} + \vec{DC} = -\vec{v} + (-\vec{u}) + 4\vec{v} = -\vec{u} + 3\vec{v}$

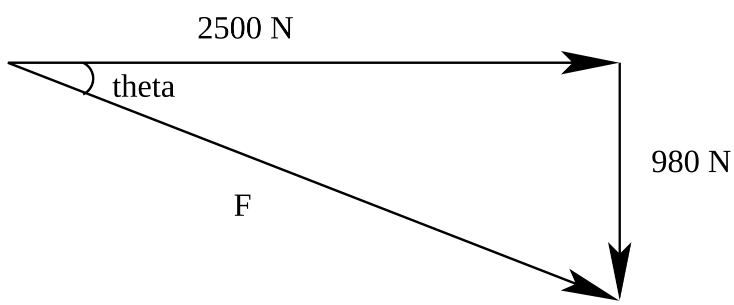
*The idea of linear combinations of vectors is central to linear algebra.*

# Applications of Vector Addition

Two vectors that are perpendicular to each other and add together to give a vector $\vec{v}$ are called <u>rectangular vector components</u> of $\vec{v}$.

**Example:**
A cannonball of mass 100 kg is fired horizontally out of a cannon with a force of 2500 N. Gravity will act vertically (downward) with a force of $(9.8 \text{ m/s}^2)(100 \text{ kg}) = 980 \text{ kgm/s}^2 = 980$ N.

So we have



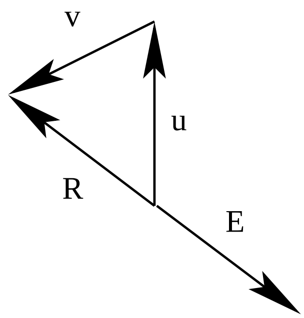

and the horizontal force of 2500 N and the vertical of 980 N are rectangular components of the resultant force $\vec{F}$.

The magnitude of the resultant force is $|\vec{F}| = \sqrt{(2500)^2 + (980)^2} \approx 2685$ N.

The direction is at angle $\theta$ below the horizontal, where $\tan\theta = \dfrac{980}{2500} = \dfrac{98}{250} = \dfrac{49}{125}$,

so $\theta = \arctan(49/125) \approx 21.4°$.

An equilibrant vector $\vec{E}$ is a vector that balances another vector or combination of vectors and hence is equal in magnitude and opposite in direction to the resultant $\vec{R}$.

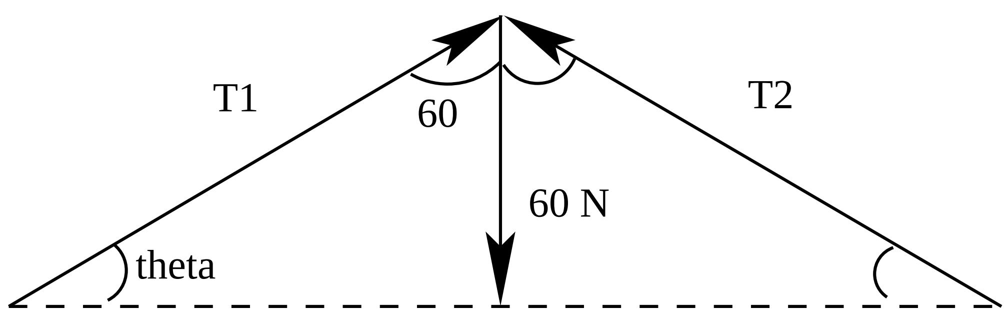



We need to understand this idea to solve tension problems.

Tension is the equilibrant force in a rope or chain that keeps an object in place (or stationary).

**Example:**
A picture that weighs 60 N is hanging from a wire (attached to the picture frame) placed on a hook on the wall such that the hook is in the centre of the wire and the two segments of wire have an angle of 120° between them.

So we have

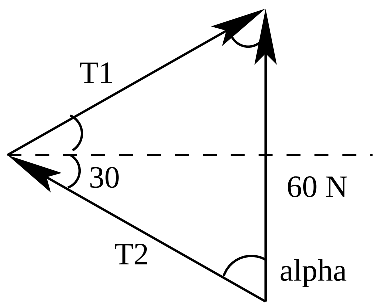



by symmetry, the tensions in the two wire segments will be equal in magnitude, *ie* $|\vec{T_1}| = |\vec{T_2}|$ and the angles they make with the horizontal must also be equal.

In fact, $\theta = 180° - 90° - 60° = 30°$.

The 60 N weight of the picture is the resultant of the system, so we must have that $\vec{T_1} + \vec{T_2} = 60$ N upward.

We can redraw the system as

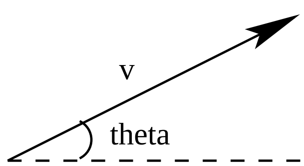



we have an isosceles triangle where $\alpha = 60°$ and hence the triangle must actually be equilateral. Thus $|\vec{T_1}| = |\vec{T_2}| = 60$ N.

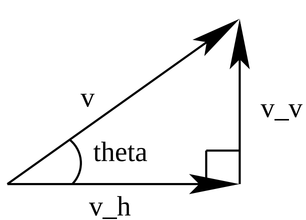
## Resolution of Vectors into Rectangular Components

Any vector can be resolved into rectangular (perpendicular) components, typically horizontal and vertical.

Suppose we have a vector $\vec{v}$ at angle $\theta$ to the horizontal



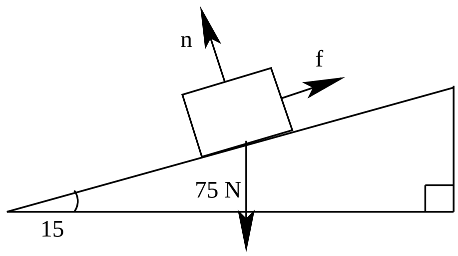

it can be resolved into its horizontal component $\vec{v_h}$ and vertical component $\vec{v_v}$, where $\vec{v} = \vec{v_h} + \vec{v_v}$ $(and\ \vec{v_h} \perp \vec{v_v})$.




then trigonometry tells us that $|\vec{v_h}| = |\vec{v}| \cos\theta$ and $|\vec{v_v}| = |\vec{v}| \sin\theta$.

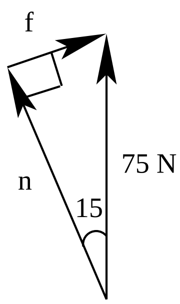
**Example:**
A child pulls a wagon with a force of 40 N at an angle of 25° to the horizontal.
Then the horizontal component has magnitude $|\vec{F_h}| = |\vec{F}| \cos\theta = (40)(\cos(25°)) \approx 36.3$ N and the vertical component has magnitude $|\vec{F_v}| = |\vec{F}| \sin\theta = (40)(\sin(25°)) \approx 16.9$ N.

**Example:**
An object that weighs 75 N is resting on an inclined plane that makes an angle of 15° with the horizontal.




Since the object is at rest, there is a frictional force $\vec{f}$ acting parallel to the ramp and a normal force $\vec{n}$ acting perpendicularly to the ramp that balances the weight (*ie* the force of gravity).

So we have

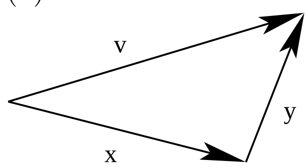



and so $|\vec{f}| = (75)(\sin(15°)) \approx 19.4$ N and $|\vec{n}| = (75)(\cos(15°)) \approx 72.4$ N.

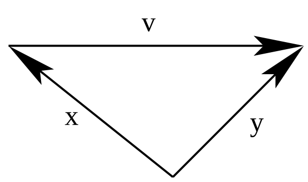
# Practice Problems

**1.** Express the vector $\vec{v}$ as the sum or difference of the other two vectors $\vec{x}$ and $\vec{y}$.

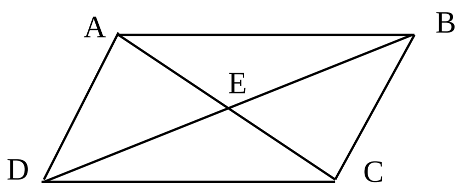
(a)




(b)

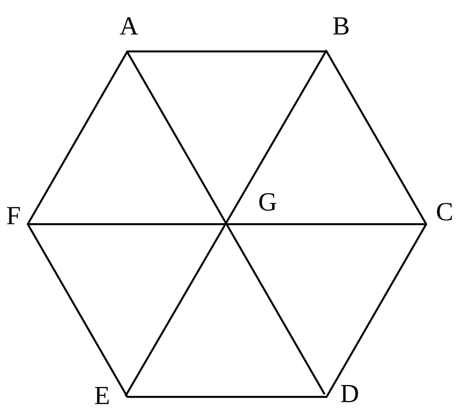



**2.** $ABCD$ is a parallelogram with diagonals $AC$ and $BD$ intersecting at $E$.




Express the following as a single vector:
(a) $\vec{AB} + \vec{BC}$
(b) $\vec{BC} - \vec{BA}$
(c) $\vec{DA} + \vec{AE} - \vec{DC}$

**3.** $ABCDEF$ is a regular hexagon with centre $G$.



Express the following as a single vector:
(a) $\vec{AB} + \vec{BG} + \vec{GF}$
(b) $\vec{AF} - \vec{AG} + \vec{FE}$
(c) $\vec{DG} + \vec{GA} - \vec{DC} - \vec{CB}$

**4.** Let $X$, $Y$, $Z$ and $O$ represent any four points.
(a) Express $\vec{XY}$, $\vec{YZ}$ and $\vec{ZX}$ in terms of $\vec{OX}$, $\vec{OY}$ and $\vec{OZ}$.
(b) Express $\vec{XY} + \vec{YZ}$ in terms of $\vec{OX}$ and $\vec{OY}$ (use answers from (a)).
(c) Show that $\vec{XY} + \vec{YZ} + \vec{ZX} = \vec{0}$.

**5.** Simplify each expression.
(a) $\vec{x} + \vec{y} - 2\vec{x} + 3\vec{y}$
(b) $2(\vec{x} - \vec{y}) + 4(\vec{y} - \vec{x})$
(c) $3(\vec{x} + \vec{y}) - 2(\vec{x} - \vec{y}) - 5\vec{x}$

**6.** An airplane is flying at airspeed 300 km/h on a heading of 270°. There is a wind blowing at 40 km/h from direction 000°. What is the plane's ground velocity?

**7.** An object weighing 250 N is hanging from two chains. The longer one is attached to the ceiling at angle 45° and the shorter one to a wall at an angle of 30° to the wall. Calculate the tensions in the two chains.

**8.** A ferry must cross a river that is 1 km wide to a point directly across. The ferry can travel at 10 km/h relative to the water. There is a current in the river of 7 km/h. What heading should the ferry take?

**9.** Determine the horizontal and vertical components of each force.
(a) 200 N at an angle of 20° ccw from the horizontal
(b) 18 N at angle 15° cw from vertical

**10.** An airplane is moving with velocity 400 km/h in direction N25°W. Calculate the westward displacement of the plane after two hours.

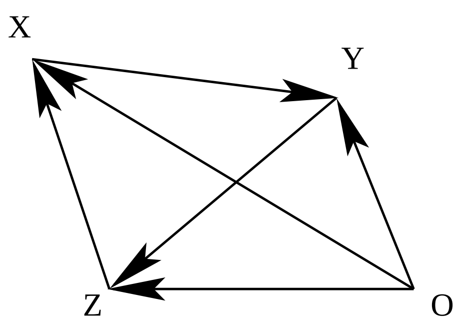
# Practice Problems Solutions

1. (a) $\vec{v} = \vec{x} + \vec{y}$
(b) $\vec{v} = \vec{y} - \vec{x}$

2. (a) $\vec{AC}$
(b) $\vec{AC}$
(c) $\vec{DE} - \vec{DC} = \vec{CE}$

3. (a) $\vec{AF}$
(b) $\vec{GE}$
(c) $\vec{BA}$

4.



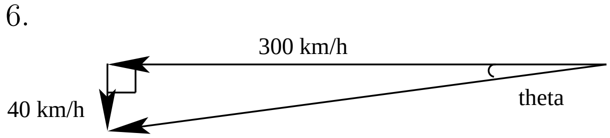

(a) $\vec{XY} = \vec{OY} - \vec{OX}$
$\vec{YZ} = \vec{OZ} - \vec{OY}$
$\vec{ZX} = \vec{OX} - \vec{OZ}$

(b) $\vec{XY} + \vec{YZ} = (\vec{OY} - \vec{OX}) + (\vec{OZ} - \vec{OY}) = \vec{OZ} - \vec{OX}$

(c) $\vec{XY} + \vec{YZ} + \vec{ZX} = (\vec{OZ} - \vec{OX}) + (\vec{OX} - \vec{OZ}) = \vec{0}$

5. (a) $4\vec{y} - \vec{x}$
(b) $2\vec{y} - 2\vec{x}$
(c) $5\vec{y} - 4\vec{x}$

6.



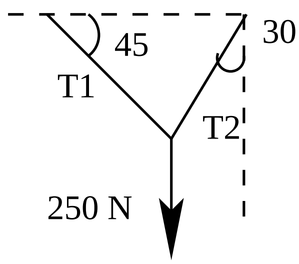
the ground velocity is the resultant

ground speed is $\sqrt{(40)^2 + (300)^2} \approx 303$ km/h

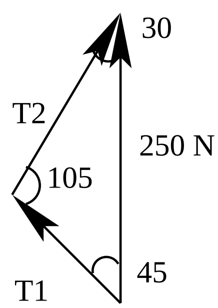
$\tan(\theta) = \dfrac{40}{300} = \dfrac{2}{15}, \quad$ so $\theta \approx 7.6°$

so the ground velocity is 303 km/h S82.8°W or on heading 262.4°

7. the system looks like




which we can redraw as

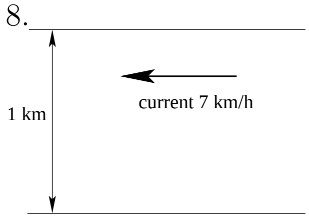


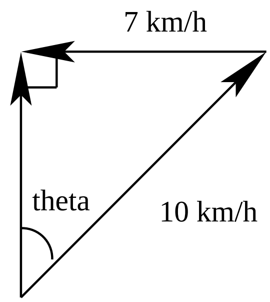

$\vec{T_1} + \vec{T_2} = 250$ N upward

use the sine law to get $\dfrac{|\vec{T_1}|}{\sin(30°)} = \dfrac{250 \text{ N}}{\sin(105°)}, \quad$ so $|\vec{T_1}| \approx 129.4$ N

and $\dfrac{|\vec{T_2}|}{\sin(45°)} = \dfrac{250 \text{ N}}{\sin(105°)}, \quad$ so $|\vec{T_2}| \approx 183$ N

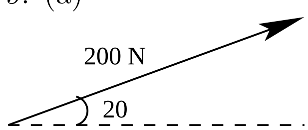
8.





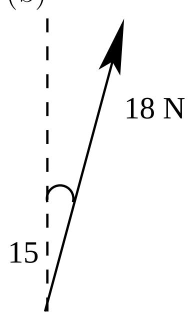

the ferry should head in a direction that makes an angle of $\sin(\theta) = \dfrac{7}{10}$,
so $\theta = \arcsin(7/10) \approx 44.4°$

9. (a)



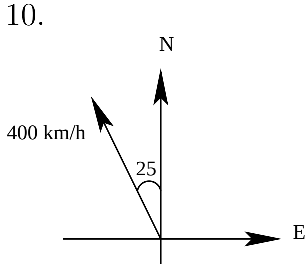

$|\vec{F}_h| = (200\ \text{N})\cos(20°) \approx 188\ \text{N}$
$|\vec{F}_v| = (200\ \text{N})\sin(20°) \approx 68.4\ \text{N}$

(b)




$|\vec{F}_h| = (18\ \text{N})\sin(15°) \approx 4.7\ \text{N}$
$|\vec{F}_v| = (18\ \text{N})\cos(15°) \approx 17.4\ \text{N}$

10.



the westward velocity component is $|\vec{v_w}| = (400 \text{ km/h}) \sin(25°) \approx 169$ km/h

so after two hours, the plane has travelled 338 km westward

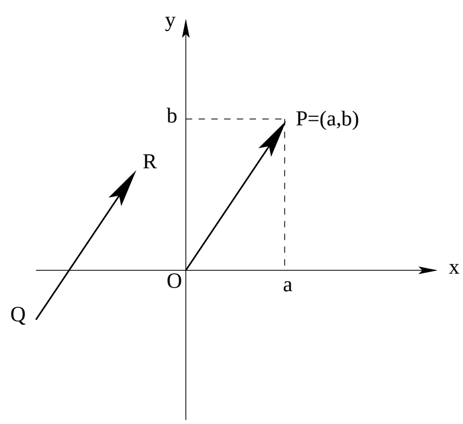
# Chapter 2

## Goals

- to understand how to represent vectors in two- and three-dimensional space in Cartesian form and perform the operations of addition(subtraction) and scalar multiplication on them
- to understand both the bracket $[a, b, c]$ and $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ representations of Cartesian vectors and be able to use both notations
- to be able to perform the dot product of two vectors and understand its properties
- to be able to use the dot product in applications, such as calculating work done or vector projections
- to be able to perform the cross product of two vectors and understand its properties
- to be able to use the cross product in applications, such as calculating torque

## Cartesian Vectors

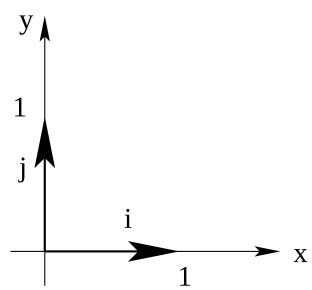
Consider any vector $\vec{v}$ in the plane – its initial and terminal points, $Q$ and $R$, can be defined using Cartesian coordinates. If we translate $\vec{v}$ so that its tail is at the origin $O$, then its head will be at some point $P$, with coordinates $(a, b)$. This representation of $\vec{v}$ (*ie* $\vec{OP}$) is called the <u>position vector</u> $[a, b]$.



And any vector with the same magnitude and direction can be represented as $[a, b]$.

If we resolve $\vec{v} = [a, b]$ into horizontal and vertical vector components, we'll have a vector of length $a$ along the $x$-axis and one of length $b$ along the $y$-axis (which will add up to $\vec{v}$).

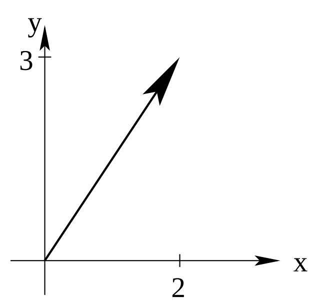
We define the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ to be vectors of length 1 (hence the use of the word unit) that point in the positive $x$ and $y$ directions, respectively.




So $\hat{\imath}$ has position vector $[1, 0]$ and it's $[0, 1]$ for $\hat{\jmath}$ and so we can write that $\hat{\imath} = [1, 0]$ and $\hat{\jmath} = [0, 1]$.

Then the horizontal and vertical components of $\vec{v}$ are $\vec{v_h} = a\hat{\imath} = a[1, 0] = [a, 0]$ and $\vec{v_v} = b\hat{\jmath} = b[0, 1] = [0, b]$. And thus $\vec{v} = \vec{v_h} + \vec{v_v} = [a, 0] + [0, b] = [a, b]$ or $\vec{v} = a\hat{\imath} + b\hat{\jmath}$.
And we also see that $|\vec{v}| = \sqrt{a^2 + b^2}$ (*by Pythagoras*).

**Example:**




The vector $[2, 3]$ has horizontal component of length 2, vertical component of length 3 and length or magnitude $\sqrt{(2)^2 + (3)^2} = \sqrt{13}$. It can be written as $[2, 3] = [2, 0] + [0, 3] = 2[1, 0] + 3[0, 1] = 2\hat{\imath} + 3\hat{\jmath}$.

Suppose we add vectors $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$:
$\vec{u} + \vec{v} = [u_1, u_2] + [v_1, v_2]$
$= [u_1, 0] + [0, u_2] + [v_1, 0] + [0, v_2]$
$= u_1\hat{\imath} + u_2\hat{\jmath} + v_1\hat{\imath} + v_2\hat{\jmath}$
$= (u_1 + v_1)\hat{\imath} + (u_2 + v_2)\hat{\jmath}$
$= (u_1 + v_1)[1, 0] + (u_2 + v_2)[0, 1]$
$= [u_1 + v_1, u_2 + v_2]$

*Once we understand how we add vectors in this manner, we do not have to write out all these details.*

And scalar multiplication is $k\vec{v} = k[v_1, v_2] = k(v_1\hat{\imath} + v_2\hat{\jmath}) = kv_1\hat{\imath} + kv_2\hat{\jmath} = [kv_1, kv_2]$.

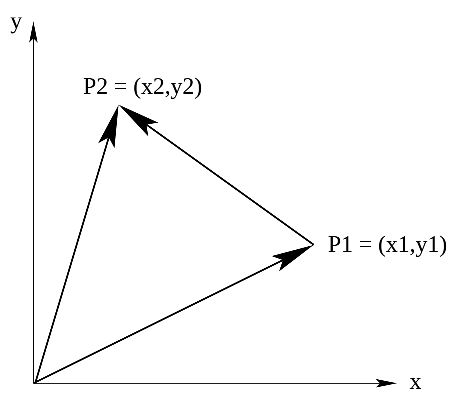
So the opposite of $\vec{v}$ is $-\vec{v} = [-v_1, -v_2]$.

**Example:**
Suppose $\vec{u} = [2, 7]$ and $\vec{v} = [1, 3]$, then
(i) $\vec{u} + \vec{v} = [2 + 1, 7 + 3] = [3, 10] = 3\hat{\imath} + 10\hat{\jmath}$
(ii) $\vec{u} - \vec{v} = [2 - 1, 7 - 3] = [1, 4] = \hat{\imath} + 4\hat{\jmath}$
(iii) $-3\vec{v} = [-3(1), -3(3)] = [-3, -9] = -3\hat{\imath} - 9\hat{\jmath}$
(iv) $2\vec{u} + 4\vec{v} = [4, 14] + [4, 12] = [8, 26] = 8\hat{\imath} + 26\hat{\jmath}$

Given any vector $\vec{v} = [v_1, v_2] = v_1\hat{\imath} + v_2\hat{\jmath}$, we can always find a unit vector in the same direction by dividing by the magnitude, *ie* $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{v_1\hat{\imath} + v_2\hat{\jmath}}{\sqrt{v_1^2 + v_2^2}}$ and one in the opposite direction by multiplying by $-1$, *ie* $-\hat{v}$.

**Example:**
If $\vec{v} = [-2, 3]$, then $\hat{v} = \dfrac{[-2, 3]}{\sqrt{(-2)^2 + (3)^2}} = \frac{1}{\sqrt{13}}[-2, 3] = \frac{1}{\sqrt{13}}(-2\hat{\imath} + 3\hat{\jmath})$.

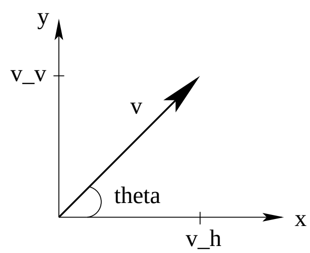
Suppose we have two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$,




then the vector that joins $P_1$ to $P_2$ is $\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = [x_2, y_2] - [x_1, y_1] = [x_2 - x_1, y_2 - y_1]$ and $|\vec{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (*which is the distance between the points*).

**Example:**
The vector that joins point $A = (-1, 0)$ to point $B = (5, 2)$ is $\vec{AB} = [5 - (-1), 2 - 0] = [6, 2]$ and its length is $\sqrt{(6)^2 + (2)^2} = \sqrt{40} = 2\sqrt{10}$.

We have also seen that the horizontal and vertical vector components are expressible as $|\vec{v_h}| = |\vec{v}|\cos\theta$ and $|\vec{v_v}| = |\vec{v}|\sin\theta$, where $\theta$ is the angle (measured counterclockwise) the vector makes with the positive $x$-axis.



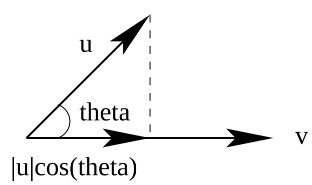

$\vec{v} = [v_1, v_2] = [|\vec{v_h}|, |\vec{v_v}|] = [|\vec{v}|\cos\theta, |\vec{v}|\sin\theta] = |\vec{v}|\cos\theta\hat{\imath} + |\vec{v}|\sin\theta\hat{\jmath} = |\vec{v}|(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$

*Can you then see that* $\hat{v} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ *?*

**Example:**
A force of 150 N at 15° to the horizontal is then
$\vec{F} = [|\vec{F}|\cos\theta, |\vec{F}|\sin\theta] = [(150)\cos(15°), (150)\sin(15°)] \approx [144.9, 38.8]$ N $= 144.9\hat{\imath}+38.8\hat{\jmath}$ N.

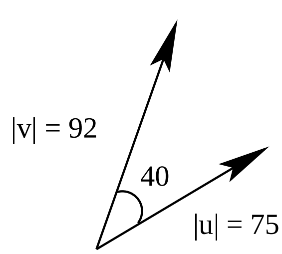
# The Dot Product

The dot product of two vectors $\vec{u}$ and $\vec{v}$ is defined to be $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ (read as $\vec{u}$ dot $\vec{v}$), where $\theta$ is the acute $(0 \le \theta \le 180°)$ angle between the vectors $\vec{u}$ and $\vec{v}$ when arranged tail to tail.




The dot product will clearly be commutative, *ie* $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$. Also, the dot product produces a scalar (not vector) result and represents a measurement of the projection of one vector onto the other, or of the tendency of the vectors to point in the same direction.

**Example:**

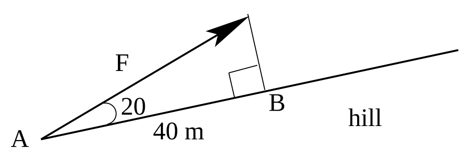



$\vec{u}\cdot\vec{v} = (75)(92)\cos(40°) \approx 5286$

We can notice some of the properties of the dot product immediately:
(i) if $\vec{u}$ or $\vec{v}$ is $\vec{0}$, then $\vec{u}\cdot\vec{v}=0$
(ii) also if $\theta=90°$, then $\vec{u}\cdot\vec{v}=0$ (*vectors are orthogonal*)
(iii) if $0<\theta<90°$, then $\vec{u}\cdot\vec{v}>0$ (*vectors point in same general direction*)
(iv) but if $90°<\theta<180°$, then $\vec{u}\cdot\vec{v}<0$ (*vectors point in generally opposite directions*)
(v) if $\theta=0$, then $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|$ (and so $\vec{v}\cdot\vec{v}=|\vec{v}|^2$)
(vi) if $\theta=180°$, then $\vec{u}\cdot\vec{v}=-|\vec{u}||\vec{v}|$
(vii) $\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{\imath}=0$

The dot product is used to calculate work, which is the product of the magnitude of displacement of an object with the magnitude of the force applied in the direction of motion.

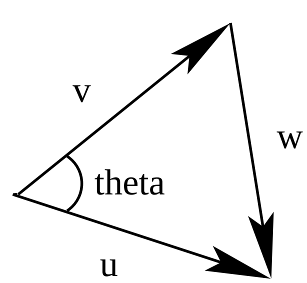
**Example:**
A child pulls a sled 40 m up a hill with a force of 125 N at an angle of 20° to the surface of the hill.




The work done is
$W=$ (displacement)(horizontal component of force)
$=|\vec{AB}||\vec{F}|\cos(20°)$
$=\vec{AB}\cdot\vec{F}$
$=(40\text{ m})(125\text{ N})\cos(20°)$
$\approx 4698$ J.

Other properties of the dot product are:
(i) if $\vec{u}\neq\vec{0}$ and $\vec{v}\neq\vec{0}$, then $\vec{u}\cdot\vec{v}=0$ if and only if $\vec{u}\perp\vec{v}$
(ii) for any scalar $k$, $(k\vec{u})\cdot\vec{v}=k(\vec{u}\cdot\vec{v})=\vec{u}\cdot(k\vec{v})$ (*associative*)
(iii) $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ and $(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$ (*distributive*)

There is a nice way to calculate $\vec{u}\cdot\vec{v}$ using the Cartesian representations of the vectors.




By the cosine law, $|\vec{w}|^2=|\vec{u}|^2+|\vec{v}|^2-2|\vec{u}||\vec{v}|\cos\theta=|\vec{u}|^2+|\vec{v}|^2-2\vec{u}\cdot\vec{v}$.
But $\vec{w}=\vec{u}-\vec{v}$, so we have
$\vec{u}\cdot\vec{v}=\frac{1}{2}(|\vec{u}|^2+|\vec{v}|^2-|\vec{u}-\vec{v}|^2)$

$= \frac{1}{2}(u_1^2 + u_2^2 + v_1^2 + v_2^2 - ((u_1 - v_1)^2 + (u_2 - v_2)^2))$
$= \frac{1}{2}(u_1^2 + u_2^2 + v_1^2 + v_2^2 - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2)$
$= u_1v_1 + u_2v_2$

*ie* $\vec{u} \cdot \vec{v} = [u_1, u_2] \cdot [v_1, v_2] = u_1v_1 + u_2v_2$

**Example:**
(i) $[-2, 4] \cdot [3, 2] = (-2)(3) + (4)(2) = -6 + 8 = 2$
(ii) Suppose $\vec{u} = [0, 2]$ and $\vec{v} = [7, 3]$,
then $\vec{u} \cdot \vec{v} = (0)(7) + (2)(3) = 6$
and $(3\vec{u} - \vec{v}) \cdot (\vec{u} + 5\vec{v}) = [-7, 3] \cdot [35, 17] = (-7)(35) + (3)(17) = -245 + 51 = -194$.

# Applications of the Dot Product

Let's look at what the dot product can do for us. We have already seen that the work done by a force $\vec{F}$ in direction of a displacement $\vec{s} = \vec{AB}$ is $W = \vec{F} \cdot \vec{s}$.

**Example:**
A force of 30 N is acting in the direction of the vector $\vec{v} = [3, 2]$ and is exerted on an object moving from point $(0, 2)$ to point $(2, 7)$ (distances measured in metres). What is the work done ?

A unit vector in the direction of $\vec{v}$ is $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{[3, 2]}{\sqrt{(3)^2 + (2)^2}} = \frac{1}{\sqrt{13}}[3, 2]$.

So the force is $\vec{F} = (30 \text{ N})\hat{v} = \frac{30}{\sqrt{13}}[3, 2] = \frac{30}{\sqrt{13}}(3\hat{\imath} + 2\hat{\jmath})$ N.

The displacement is $\vec{s} = (2 - 0)\hat{\imath} + (7 - 2)\hat{\jmath} = 2\hat{\imath} + 5\hat{\jmath}$ m.

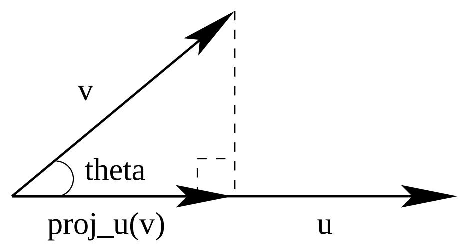
So the work done is
$W = \vec{F} \cdot \vec{s} = \frac{30}{\sqrt{13}}(3\hat{\imath} + 2\hat{\jmath}) \cdot (2\hat{\imath} + 5\hat{\jmath}) \text{ J} = \frac{30}{\sqrt{13}}((3)(2) + (2)(5)) \text{ J} = \frac{480}{\sqrt{13}} \text{ J} \approx 133$ J.

Since the dot product is $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$, we have that $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$ and so $\theta = \arccos\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$ and hence we can find the angle between the vectors.

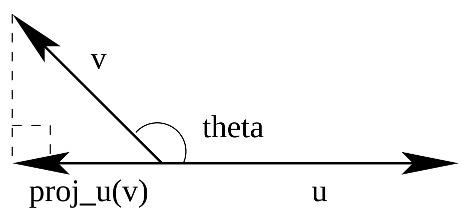
**Example:**
The angle between the vectors $\vec{u} = [6, -2]$ and $\vec{v} = [-1, 4]$ is
$\theta = \arccos\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$
$= \arccos\left(\dfrac{[6, -2] \cdot [-1, 4]}{\sqrt{(6)^2 + (-2)^2}\sqrt{(-1)^2 + (4)^2}}\right)$
$= \arccos\left(\dfrac{(6)(-1) + (-2)(4)}{\sqrt{40}\sqrt{17}}\right)$

$= \arccos\left(\frac{-14}{\sqrt{40}\sqrt{17}}\right) \approx 122.5°.$

The shadow of one vector $\vec{v}$ onto another $\vec{u}$ is called the projection of $\vec{v}$ on $\vec{u}$ and is written $proj_{\vec{u}}\vec{v}$.




If the angle between the vectors, $\theta$, is less than 90°, then $proj_{\vec{u}}\vec{v}$ is the component of $\vec{v}$ in the direction of $\vec{u}$ and thus
$proj_{\vec{u}}\vec{v} = |\vec{v}|\cos\theta\,\hat{u} = |\vec{v}|\cos\theta\left(\frac{\vec{u}}{|\vec{u}|}\right) = \left(\frac{|\vec{v}|}{|\vec{u}|}\cos\theta\right)\vec{u}.$



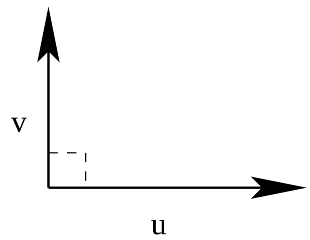

If $90° < \theta < 180°$, $proj_{\vec{u}}\vec{v}$ is in the direction opposite to $\vec{u}$ and thus
$proj_{\vec{u}}\vec{v} = |\vec{v}|\cos(180° - \theta)(-\hat{u}) = |\vec{v}|(-\cos\theta)\left(\frac{-\vec{u}}{|\vec{u}|}\right) = \left(\frac{|\vec{v}|}{|\vec{u}|}\cos\theta\right)\vec{u}$

(*the same formula as above*).

And so $proj_{\vec{u}}\vec{v} = \left(\frac{|\vec{v}|}{|\vec{u}|}\cos\theta\right)\vec{u} = \left(\frac{|\vec{v}||\vec{u}|\cos\theta}{|\vec{u}||\vec{u}|}\right)\vec{u} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}.$

**Example:**
The projection of $\vec{v} = 2\hat{\imath} - 4\hat{\jmath}$ on $\vec{u} = \hat{\imath} + 3\hat{\jmath}$ is
$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$
$= \left(\frac{(2\hat{\imath} - 4\hat{\jmath})\cdot(\hat{\imath} + 3\hat{\jmath})}{(\hat{\imath} + 3\hat{\jmath})\cdot(\hat{\imath} + 3\hat{\jmath})}\right)(\hat{\imath} + 3\hat{\jmath})$
$= \left(\frac{(2)(1) + (-4)(3)}{(1)(1) + (3)(3)}\right)(\hat{\imath} + 3\hat{\jmath})$
$= \left(\frac{-10}{10}\right)(\hat{\imath} + 3\hat{\jmath})$
$= (-1)(\hat{\imath} + 3\hat{\jmath})$
$= -\hat{\imath} - 3\hat{\jmath}$
$= -\vec{u}.$

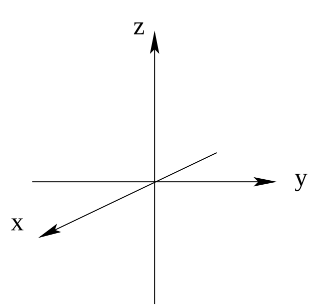
Notice that if $\theta = 90°$, $proj_{\vec{u}}\vec{v} = \vec{0}$ as there is no shadow of $\vec{v}$ on $\vec{u}$ or, equivalently, there is no component of $\vec{v}$ in the direction of $\vec{u}$.




We can use the dot product to calculate other quantities. For example, suppose the Candy Store sold 40 bags of jelly beans and 25 giant jawbreakers one week. Then we could represent these sales with the vector $[40, 25]$. If a bag of jelly beans sells for \$1.25 and a giant jawbreaker for 50¢, we could represent these prices with the vector $[1.25, 0.50]$. The dot product of these vectors $[40, 25] \cdot [1.25, 0.50] = (40)(1.25) + (25)(0.50) = \$62.50$ would be the revenue from these sales.

## Vectors in Three-Space

The three-dimensional Cartesian coordinate system is

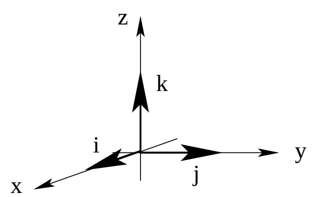



this system is called right-handed since, if we were to curl the fingers of our right hand from the $x$-axis towards the $y$-axis, our thumb would point in the direction of the $z$-axis.

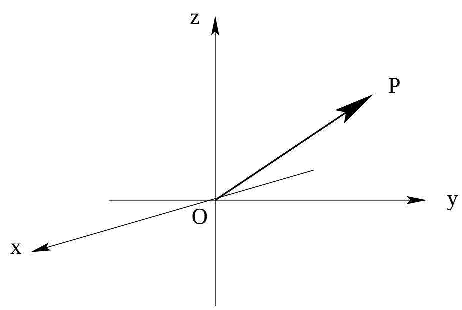
Points are specified by ordered triples $P = (x, y, z)$. The axes divide space into 8 octants, the one where all three coordinates are positive is called the first octant (but there is no agreement on how to label the remaining seven). The origin $O = (0, 0, 0)$ is the point where the three axes intersect.

**Example:**
The point $(2, 3, -1)$ is 2 units in front, 3 units to the right and 1 unit below the origin.

The unit vectors in 3-space are $\hat{\imath} = [1, 0, 0]$, $\hat{\jmath} = [0, 1, 0]$ and $\hat{k} = [0, 0, 1]$.




The position vector of the point $P = (x, y, z)$ is $\vec{v} = \vec{OP}$ (from the origin $O$ to the point $P$).
$\vec{OP} = [x, y, z] = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.




The magnitude (or length) of the vector would be $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$ (*from Pythagoras*).

**Example:**
The vector $\vec{v} = [2, 4, -1] = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ has magnitude $|\vec{v}| = \sqrt{(2)^2 + (4)^2 + (-1)^2} = \sqrt{21}$.

A scalar multiple of vector $\vec{v} = [v_1, v_2, v_3]$ is $k\vec{v} = [kv_1, kv_2, kv_3]$, where $k \in \mathbb{R}$ and which is a vector collinear with $\vec{v}$.

**Example:**
For what value of $c$ is $\vec{u} = [2, c, 1]$ collinear with $\vec{v} = [4, 6, 2]$ ?
For $\vec{u}$ to be collinear with $\vec{v}$, it must be a scalar multiple of it, *ie* $\vec{u} = k\vec{v}$
or $[2, c, 1] = k[4, 6, 2] = [4k, 6k, 2k]$ or $2 = 4k$, $c = 6k$ and $1 = 2k$.
So clearly $k = 1/2$ and thus $c = 3$.

For two vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$ and $\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2, u_3 - v_3]$.

And the vector from point $P_1 = (x_1, y_1, z_1)$ to point $P_2 = (x_2, y_2, z_2)$ is
$\vec{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$.

**Example:**
Suppose $\vec{u} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{v} = \hat{\imath} + 3\hat{\jmath} - 4\hat{k}$, then
(i) $\vec{u} + \vec{v} = (2 + 1)\hat{\imath} + (-1 + 3)\hat{\jmath} + (3 - 4)\hat{k} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$
(ii) $\vec{v} - \vec{u} = (1 - 2)\hat{\imath} + (3 - (-1))\hat{\jmath} + (-4 - 3)\hat{k} = -\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$
(iii) $2\vec{u} - 3\vec{v} = 2(2\hat{\imath} - \hat{\jmath} + 3\hat{k}) - 3(\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = (4\hat{\imath} - 2\hat{\jmath} + 6\hat{k}) - (3\hat{\imath} + 9\hat{\jmath} - 12\hat{k}) = \hat{\imath} - 11\hat{\jmath} + 18\hat{k}$
(iv) A unit vector in the direction of $\vec{u}$ is $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{2\hat{\imath} - \hat{\jmath} + 3\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (3)^2}} = \frac{1}{\sqrt{14}}(2\hat{\imath} - \hat{\jmath} + 3\hat{k})$.

The dot product of $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$ is $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = u_1v_1 + u_2v_2 + u_3v_3$.

**Example:**
(i) If $\vec{u} = [-1, 5, 0]$ and $\vec{v} = [2, 6, -1]$, then $\vec{u} \cdot \vec{v} = (-1)(2) + (5)(6) + (0)(-1) = 28$.
(ii) What is the angle bewteen $\vec{u} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{v} = 4\hat{\imath} + 7\hat{\jmath} + \hat{k}$ ?

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$
$$= \arccos\left(\frac{(1)(4) + (-2)(7) + (3)(1)}{\sqrt{(1)^2 + (-2)^2 + (3)^2}\sqrt{(4)^2 + (7)^2 + (1)^2}}\right)$$
$$= \arccos\left(\frac{-7}{\sqrt{14}\sqrt{66}}\right) \approx 103.3°.$$

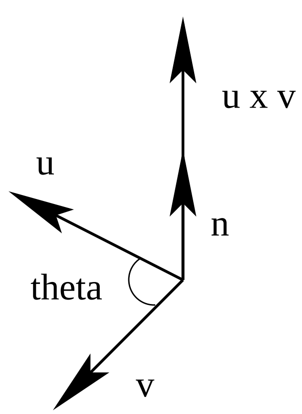
*Are you comfortable with the two notations for vectors ?*

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$. So if we want to find a vector orthogonal (perpendicular) to a given one, we use the dot product to help us.

**Example:**
Find a vector orthogonal to $\vec{p} = [2, -3, 5]$.
Let $\vec{v} = [x, y, z]$ be a vector orthogonal to $\vec{p}$, then $\vec{p} \cdot \vec{v} = 0$, which means that $[2, -3, 5] \cdot [x, y, z] = 0$ or $2x - 3y + 5z = 0$. So there are infinitely many vectors orthogonal to $\vec{p}$ (*all of which will lie in the same plane*) and we can pick any solution to the equation, say $\vec{v} = [1, -1, -1]$.

# The Cross Product

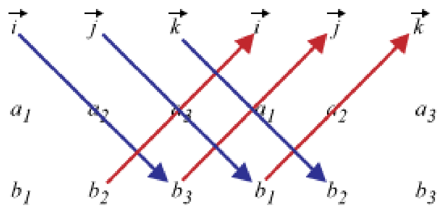
Now, we'll define a vector operation that will produce a vector result. The cross product of two vectors $\vec{u}$ and $\vec{v}$ is defined to be $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta\, \hat{n}$ (which is read as $\vec{u}$ cross $\vec{v}$) where $\theta$ is the acute angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ such that $\vec{u}$, $\vec{v}$ and $\hat{n}$ form a right-handed system (curling the fingers of the right hand from $\vec{u}$ towards $\vec{v}$ gives the thumb pointing in the direction of $\hat{n}$).



Immediately, we have that $\vec{v} \times \vec{u} = |\vec{v}||\vec{u}| \sin\theta\, (-\hat{n}) = -\vec{u} \times \vec{v}$ and hence the cross product is *not* commutative.

If $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, we calculate the components of $\vec{u} \times \vec{v}$ this way
$\vec{u} \times \vec{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$.

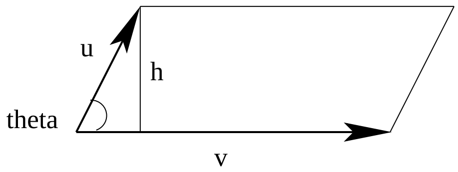
There is a visual way to remember how to do the calculation:




*Can you see how to use it?*

**Example:**
If $\vec{u} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{v} = \hat{\imath} + 3\hat{\jmath} + 4\hat{k}$, then
$\vec{u} \times \vec{v} = ((-1)(4) - (3)(3))\hat{\imath} + ((3)(1) - (2)(4))\hat{\jmath} + ((2)(3) - (-1)(1))\hat{k} = -13\hat{\imath} - 5\hat{\jmath} + 7\hat{k}$,
also $(\vec{u} \times \vec{v}) \cdot \vec{u} = [-13, -5, 7] \cdot [2, -1, 3] = (-13)(2) + (-5)(-1) + (7)(3) = 0$,
and $(\vec{u} \times \vec{v}) \cdot \vec{v} = [-13, -5, 7] \cdot [1, 3, 4] = (-13)(1) + (-5)(3) + (7)(4) = 0$
and so $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$ (*as it should be*).

The magnitude of the cross product $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ is equal to the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.




The area is $A = |\vec{v}|h = |\vec{v}||\vec{u}| \sin\theta$.

So the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$ in our example above is
$\sqrt{(-13)^2 + (-5)^2 + (7)^2} = \sqrt{243} \approx 15.6$.

Suppose that $\vec{u} \times \vec{v} = \vec{0}$, what would have to be true?
Either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u}$ and $\vec{v}$ are parallel (*ie* $\theta = 0$) and there is $k \in \mathbb{R}$ such that $\vec{u} = k\vec{v}$.

Other properties of the cross product are:
(i) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ (*distributive*)
(ii) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ (*distributive*)
(iii) $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$ (*associative*)

We can use the cross product to find the angle between vectors (*though using the dot product would be easier*) as $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ means that $\sin\theta = \dfrac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}$ and so $\theta = \arcsin\left(\dfrac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}\right)$.

For our example, $|\vec{u}| = \sqrt{(2)^2+(-1)^2+(3)^2} = \sqrt{14}$ and $|\vec{v}| = \sqrt{(1)^2+(3)^2+(4)^2} = \sqrt{26}$,

so $\theta = \arcsin\left(\dfrac{15.6}{\sqrt{14}\sqrt{26}}\right) \approx 54.9°$.

Since $\hat{\imath} = [1,0,0]$ and $\hat{\jmath} = [0,1,0]$,
$\hat{\imath}\times\hat{\jmath} = [(0)(0)-(0)(1),(0)(0)-(1)(0),(1)(1)-(0)(0)]$
$= [0,0,1]$
$= \hat{k}$.

*Can you see this visually from the coordinate system?*

# Applications of the Dot and Cross Products

The formulas for work and projection are the same in 3-space as they are in 2-space.

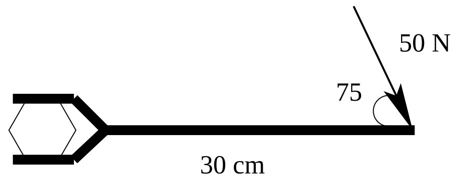
**Example:**
A force $\vec{F} = [20,100,75]$ N acts on an object with displacement $\vec{d} = [2,4,5]$ m, so the work done is $W = \vec{F}\cdot\vec{d} = [20,100,75]\cdot[2,4,5] = (20)(2)+(100)(4)+(75)(5) = 815$ J.
Gravity acts in direction $-\hat{k}$ (*ie* downward), so the work done against gravity is
$[0,0,75]\cdot[0,0,5] = 375$ J.

**Example:**
The projection of $\vec{v} = [2,-1,3]$ on $\vec{u} = [3,2,7]$ is

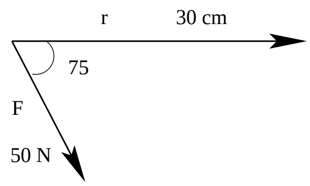
$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{[2,-1,3]\cdot[3,2,7]}{[3,2,7]\cdot[3,2,7]}\right)[3,2,7]$$
$$= \left(\frac{(2)(3)+(-1)(2)+(3)(7)}{(3)^2+(2)^2+(7)^2}\right)[3,2,7]$$
$$= \tfrac{25}{62}[3,2,7]$$
$$= \tfrac{75}{62}\hat{\imath} + \tfrac{50}{62}\hat{\jmath} + \tfrac{175}{62}\hat{k}.$$

Torque is a measure of the force $\vec{F}$ acting on an object that causes it to rotate and is given by $\vec{\tau} = \vec{r}\times\vec{F}$, where $\vec{r}$ is the torque arm, the vector from the pivot point to the point where the force $\vec{F}$ acts.

**Example:**
A wrench of length 30 cm is used to tighten a bolt. A force of 50 N is applied in a clockwise direction at an angle of 75° to the handle.



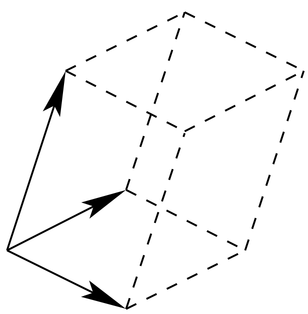

or




so the torque is $\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}| \sin\theta\, \hat{n} = (0.3 \text{ m})(50 \text{ N}) \sin(75°)\, \hat{n} \approx 14.5$ Nm inward (the bolt is being tightened).

*We use Nm as the units of the vector torque and not J which is used for the scalar energy.*

If we have the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, we can define the scalar triple product to be $\vec{w} \cdot \vec{u} \times \vec{v}$ (*the cross product has to be done first in order for the product to make sense*). This represents the volume of the parallelopiped formed by the vectors. We use absolute value bars if the result is negative.



**Example:**
The volume of the parallelopiped formed by $\vec{u} = [1, 1, 1]$, $\vec{v} = [2, 2, 4]$ and $\vec{w} = [3, 1, 6]$ is
$\vec{w} \cdot \vec{u} \times \vec{v} = [3, 1, 6] \cdot [1, 1, 1] \times [2, 2, 4]$
$= [3, 1, 6] \cdot [(1)(4) - (1)(2), (1)(2) - (1)(4), (1)(2) - (1)(2)]$
$= [3, 1, 6] \cdot [2, -2, 0]$
$= (3)(2) + (1)(-2) + (6)(0)$
$= 4$

# Practice Problems

**1.** Express each vector in the $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ form.
(a) $[2, 0, 7]$
(b) $[-1, \sqrt{2}, 3]$
(c) $[2, -1, 6]$
(d) $[0, -5, 0]$

**2.** Express each vector in the $[a, b, c]$ form.
(a) $6\hat{\imath} - \hat{k}$
(b) $\sqrt{2}\hat{\jmath} + 2\hat{k}$
(c) $\hat{\imath} + \hat{\jmath} + \hat{k}$
(d) $2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$

**3.** Find a unit vector in the direction of each given vector.
(a) $[2, 7, -1]$
(b) $[0, 3, 0]$
(c) $\hat{\imath} + \hat{\jmath} + \hat{k}$
(d) $4\hat{\imath} - \hat{\jmath} + 2\hat{k}$

**4.** If $\vec{u} = [2, 1, -1], \vec{v} = [1, -2, 3]$ and $\vec{w} = [-3, 0, 2]$, find the following.
(a) $2\vec{u} - \vec{v} + 3\vec{w}$
(b) $3(\vec{u} - 2\vec{w}) + 4(\vec{v} + \vec{w})$
(c) $\vec{u} \cdot (3\vec{w})$
(d) $\vec{v} \cdot \vec{w}$
(e) $\vec{u} \times (\vec{v} + \vec{w})$
(f) $\vec{v} \times \vec{u} + \vec{w}$
(g) $\vec{v} \cdot \vec{w} \times (2\vec{u})$
(h) $2(\vec{u} \cdot \vec{v}) + 6\vec{w} \cdot \vec{v} \times \vec{u}$

**5.** If $\vec{u} = 3\hat{\imath} - 2\hat{k}$ and $\vec{v} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, find both $proj_{\vec{u}}\vec{v}$ and $proj_{\vec{v}}\vec{u}$. What is the angle between the vectors?

**6.** Use the cross product to find the angle bewteen the vectors in question 5.

**7.** A force $\vec{F} = 4\hat{\imath} + 6\hat{\jmath} + \hat{k}$ N is applied to move an object from point $(1, 2, 0)$ to point $(7, 11, -2)$ (measured in metres). What is the work done?

**8.** 40 N is applied (clockwise) at an angle of 75° to a 18 cm long wrench. Find the torque.

**9.** Find the volume of the parallelopiped formed by the vectors $\vec{u} = [1, 1, 0], \vec{v} = [-2, 0, 1]$ and $\vec{w} = [1, 1, 4]$.

**10.** Find a vector that is perpendicular to both $\vec{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{v} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

# Practice Problems Solutions

1. (a) $2\hat{\imath} + 7\hat{k}$
(b) $-\hat{\imath} + \sqrt{2}\hat{\jmath} + 3\hat{k}$
(c) $2\hat{\imath} - \hat{\jmath} + 6\hat{k}$
(d) $-5\hat{\jmath}$

2. (a) $[6, 0, -1]$
(b) $[0, \sqrt{2}, 2]$
(c) $[1, 1, 1]$
(d) $[2, -4, -3]$

3. (a) $|[2, 7, -1]| = \sqrt{(2)^2 + (7)^2 + (-1)^2} = 3\sqrt{6}, \quad \text{so } \hat{v} = \frac{1}{3\sqrt{6}}[2, 7, -1]$

(b) $|[0, 3, 0]| = \sqrt{(0)^2 + (3)^2 + (0)^2} = 3, \quad \text{so } \hat{v} = [0, 1, 0]$

(c) $|\hat{\imath} + \hat{\jmath} + \hat{k}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}, \quad \text{so } \hat{v} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$

(d) $|4\hat{\imath} - \hat{\jmath} + 2\hat{k}| = \sqrt{(4)^2 + (-1)^2 + (2)^2} = \sqrt{21}, \quad \text{so } \hat{v} = \frac{1}{\sqrt{21}}(4\hat{\imath} - \hat{\jmath} + 2\hat{k})$

4. (a) $2\vec{u} - \vec{v} + 3\vec{w} = [4, 2, -2] - [1, -2, 3] + [-9, 0, 6] = [-6, 4, 1]$

(b) $3(\vec{u} - 2\vec{w}) + 4(\vec{v} + \vec{w}) = 3[8, 1, -5] + 4[-2, -2, 5] = [16, -5, 5]$

(c) $\vec{u} \cdot (3\vec{w}) = [2, 1, -1] \cdot [-9, 0, 6] = (2)(-9) + (1)(0) + (-1)(6) = -24$

(d) $\vec{v} \cdot \vec{w} = (1)(-3) + (-2)(0) + (3)(2) = 3$

(e) $\vec{u} \times (\vec{v} + \vec{w}) = [2, 1, -1] \times [-2, -2, 5]$
$= [(1)(5) - (-1)(-2), (-1)(-2) - (2)(5), (2)(-2) - (-1)(-2)] = [3, -8, -2]$

(f) $\vec{v} \times \vec{u} + \vec{w} = [(-2)(-1) - (3)(1), (3)(2) - (1)(-1), (1)(1) - (-2)(2)] + [-3, 0, 2]$
$= [-1, 7, 5] + [-3, 0, 2] = [-4, 7, 7]$

(g) $\vec{v} \cdot \vec{w} \times (2\vec{u}) = [1, -2, 3] \cdot [(0)(-2) - (2)(2), (2)(4) - (-3)(-2), (-3)(2) - (0)(4)]$
$= [1, -2, 3] \cdot [-4, 2, -6] = (1)(-4) + (-2)(2) + (3)(-6) = -26$

(h) $2(\vec{u} \cdot \vec{v}) + 6\vec{w} \cdot \vec{v} \times \vec{u} = 2((2)(1) + (-2)(1) + (3)(-1)) + 6[-3, 0, 2] \cdot [-1, 7, 5]$
$= -6 + 6((-3)(-1) + (0)(7) + (2)(5)) = -6 + 78 = 72$

5. $$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \left(\frac{(\hat{\imath} + 2\hat{\jmath} + \hat{k}) \cdot (3\hat{\imath} - 2\hat{k})}{(3\hat{\imath} - 2\hat{k}) \cdot (3\hat{\imath} - 2\hat{k})}\right)(3\hat{\imath} - 2\hat{k})$$

$$= \left(\frac{(1)(3)+(2)(0)+(1)(-2)}{(3)(3)+(0)(0)+(-2)(-2)}\right)(3\hat{\imath}-2\hat{k}) = \tfrac{1}{13}(3\hat{\imath}-2\hat{k})$$

$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{1}{(\hat{\imath}+2\hat{\jmath}+\hat{k})\cdot(\hat{\imath}+2\hat{\jmath}+\hat{k})}\right)(\hat{\imath}+2\hat{\jmath}+\hat{k})$$
$$= \left(\frac{1}{(1)(1)+(2)(2)+(1)(1)}\right)(\hat{\imath}+2\hat{\jmath}+\hat{k}) = \tfrac{1}{6}(\hat{\imath}+2\hat{\jmath}+\hat{k})$$

since $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$,

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{1}{\sqrt{13}\sqrt{6}}\right) \approx 83.5°$$

6. $\vec{u}\times\vec{v} = [(0)(1)-(-2)(2),(-2)(1)-(3)(1),(3)(2)-(0)(1)] = [4,-5,6]$
so $|\vec{u}\times\vec{v}| = \sqrt{(4)^2+(-5)^2+(6)^2} \approx 8.77$

since $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$,

$$\theta = \arcsin\left(\frac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}\right) = \arcsin\left(\frac{8.77}{\sqrt{13}\sqrt{6}}\right) \approx 83.2°$$
(the small difference in the angle in the two calculations is caused by round-off error)

7. the displacement is $\vec{d} = (7-1)\hat{\imath}+(11-2)\hat{\jmath}+(-2-0)\hat{k} = 6\hat{\imath}+9\hat{\jmath}-2\hat{k}$

so the work is $W = \vec{F}\cdot\vec{d} = (4\hat{\imath}+6\hat{\jmath}+\hat{k})\cdot(6\hat{\imath}+9\hat{\jmath}-2\hat{k}) = (4)(6)+(6)(9)+(1)(-2) = 76$ J

8. $\vec{\tau} = \vec{r}\times\vec{F} = |\vec{r}||\vec{F}|\sin\theta\hat{n} = (0.18)(40)\sin(75°)$ inward $\approx 6.95$ Nm inward

9. $\vec{u}\times\vec{v} = [(1)(1)-(0)(0),(0)(-2)-(1)(1),(1)(0)-(1)(-2)] = [1,-1,2]$
then $\vec{w}\cdot\vec{u}\times\vec{v} = [1,1,4]\cdot[1,-1,2] = (1)(1)+(1)(-1)+(4)(2) = 8$,
so the parallelopiped has a volume of 8 cubic units

10. one way: $\vec{u}\times\vec{v} = [(1)(3)-(1)(-1),(1)(2)-(1)(3),(1)(-1)-(1)(2)] = [4,-1,-3]$
another way: let $\vec{w} = [x,y,z]$, then we need $\vec{u}\cdot\vec{w} = 0$ and $\vec{v}\cdot\vec{w} = 0$,
so we must have $x+y+z = 0$ and $2x-y+3z = 0$, but there are infinitely many solutions,
so we could take any one of them
(all of them are scalar multiples of $[4,-1,-3]$)

# Chapter 3

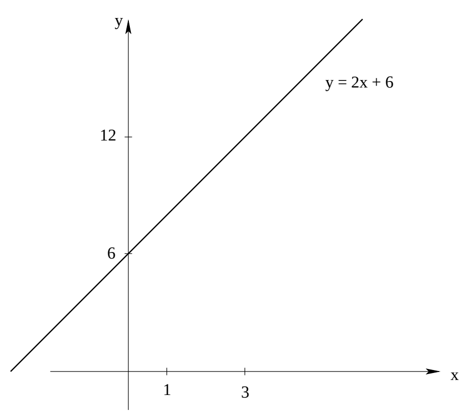
## Goals

- to be able to recognize the various forms (scalar, parameter, vector) of the equations for lines in two- and three-space and planes (three-space) and be able to convert between them
- to understand the properties of lines and planes, including parallelism and perpendicularity and be able to solve distance problems
- to understand the physical properties for the intersections of lines in two- and three-space, lines and planes (three-space) and planes (three-space) and see the connection with solutions of linear systems of equations
- to understand how to perform elimination on the augmented matrix representing a linear system to find the solution

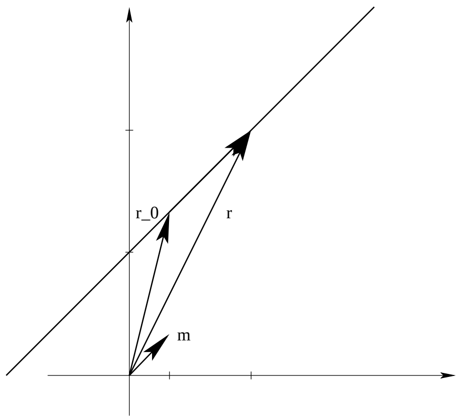
## Equations of Lines in Two– and Three–Space

In two-space, a line can be given in slope-intercept form $y = mx+b$ or with the scalar equation $Ax + By + C = 0$.

**Example:**



The line $y = 2x + 6$ has slope $m = 2$ and $y$-intercept $b = 6$. Its scalar equation form would be $2x - y + 6 = 0$.




We can represent this line with a vector equation. A direction vector parallel to the line is $\vec{m} = [1, 2]$. A position vector that has its tip on the line would be $\vec{r_0} = [1, 8]$ (since the point $(1, 8)$ is on the line). Another position vector $\vec{r} = [3, 12]$ also touches the line (since $(3, 12)$ is on the line). Then, if we let $\vec{s}$ be the vector joining $(1, 8)$ to $(3, 12)$, we would have that $\vec{r} = \vec{r_0} + \vec{s}$. But since $\vec{s}$ is parallel to $\vec{m}$, we have that $\vec{r} = \vec{r_0} + t\vec{m}$. In this particular case, $t = 2$ – *can you see that?* If we were to let $t$ vary, we could get any other point on the line, so we have that the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbb{R}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$. *Notice that this is not unique – we can use any point $(x_0, y_0)$ on the line and any direction vector $\vec{m}$ parallel to it.*

If we separate the components, we have the parametric equations of the line, $x = x_0 + tm_1$, $y = y_0 + tm_2$, $t \in \mathbb{R}$.

So for our example above, $x = 1 + t$ and $y = 8 + 2t$.

**Example:**
Is the point $(-1, 4)$ on the line? How about the point $(2, 9)$?
If $(-1, 4)$ is on the line, then we must have that $-1 = 1 + t$ and $4 = 8 + 2t$ for the same value of $t$. So, yes, $(-1, 4)$ is on the line since $t = -2$ satisfies both equations.
For the other point, $2 = 1 + t$ requires $t = 1$ but $9 = 8 + 2t$ requires that $t = 1/2$ (*ie* not the same value) so the point $(2, 9)$ is not on the line.

**Example:**
Find the vector equation of the line passing through the points $P_1 = (-1, 2)$ and $P_2 = (2, -4)$.
We need a direction vector, so we use $\vec{m} = \vec{P_1P_2} = [3, -6]$. We can use either point, so take $\vec{r_0} = [-1, 2]$ and thus $\vec{r} = \vec{r_0} + t\vec{m}$ is $[x, y] = [-1, 2] + t[3, -6]$ (and then $P_2$ corresponds to $t = 1$ ).

Two lines $L_1$ and $L_2$ are parallel if they have the same slope and are coincident if they are the same line. For lines in vector equation form, they will be parallel if the direction vectors are parallel.

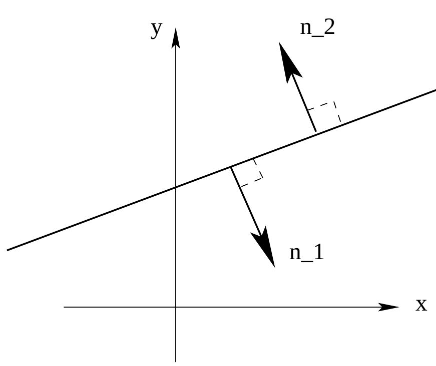
So the lines in our two examples above are not parallel, whereas the lines
$L_1$ $\vec{r} = (2+3t)\hat{\imath} + (1-2t)\hat{\jmath}$ and $L_2$ $\vec{r} = (6-6t)\hat{\imath} + (4+4t)\hat{\jmath}$ are parallel.
(*The notation has changed – can you see what you need to see to see that the lines are parallel?*)

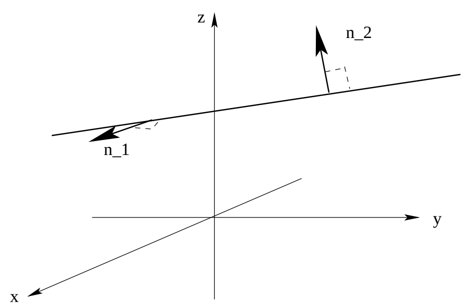
Let's go back to our example $y = 2x + 6$. We know that $\vec{m} = [1, 2]$ is a direction vector for the line, so any vector $\vec{n}$ that is perpendicular to $\vec{m}$ (*ie* $\vec{m} \cdot \vec{n} = 0$ ) will be normal or perpendicular to the line, like $\vec{n} = [2, -1]$. But notice the scalar equation $2x - y + 6 = 0$. In other words, given the scalar equation $Ax + By + C = 0$ for a line, the vector $\vec{n} = [A, B]$ is normal to it.

What do we do in three-space?
First of all, a scalar equation of three variables $Ax + By + Cz + D = 0$ describes a plane in three-space, not a line, so we do not have a scalar equation, nor do we have a slope-intercept form. But we do have a vector equation and parametric equations. The line passing through $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{m} = [m_1, m_2, m_3]$ is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbb{R}$ or $[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3]$ or $x = x_0 + tm_1$, $y = y_0 + tm_2$ and $z = z_0 + tm_3$ for $t \in \mathbb{R}$.

**Example:**
The line passing through $P_1 = (1, 0, 5)$ and $P_2 = (4, 2, 1)$ has $\vec{m} = [3, 2, -4]$, so
$\vec{r} = [4, 2, 1] + t[3, 2, -4]$ (and then $P_1$ corresponds to $t = -1$ ).
Does $(10, 5, -7)$ lie on the line?
$4 + 3t = 10 \implies t = 2$
$2 + 2t = 5 \implies t = 3/2$
$1 - 4t = -7 \implies t = 2$ so no, it does not.

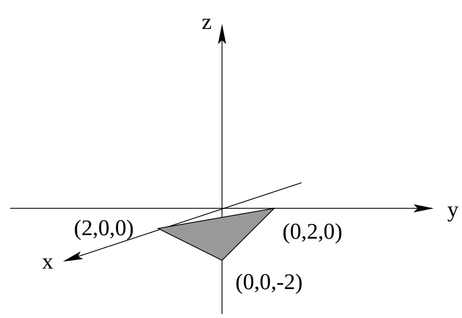
All of the vectors normal to a line in two-space are parallel, but this is not the case in three-space.






# Equations of Planes

**Example:**
Consider the scalar equation $x+y-z-2=0$. if we set two of the coordinates equal to 0, we can see where the graph of the object would cross the axes. If $y=z=0$, $x+(0)-(0)-2=0 \implies x=2$, so the $x$-intercept is 2. If $x=z=0$, $(0)+y-(0)-2=0 \implies y=2$, so the $y$-intercept is 2. If $x=y=0$, $(0)+(0)-z-2=0 \implies z=-2$, so the $z$-intercept is $-2$.

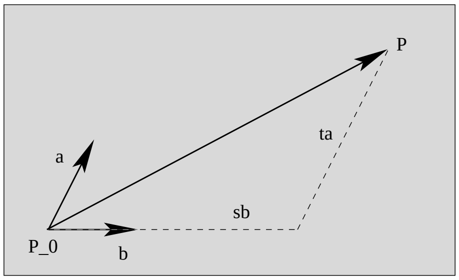



This object is certainly not a line. The graph of this equation is a plane, a flat, two-dimensional surface in three-space that extends off to infinity in every direction.

Other points on the plane are $P=(2,2,2)$, $Q=(0,7,5)$, $R=(6,3,7)$ and $S=(4,0,2)$.
*There are infinitely many points on the plane – can you see that these points satisfy the equation?*
However, the point $(10,5,7)$ is not on the plane as $(10)+(5)-(7)-2 \neq 0$ (*ie the point does not satisfy the equation*).

Using the points on the plane, we can find direction vectors that are parallel to the plane (and not to each other). For example, $\vec{PQ}=[-2,5,3]$ and $\vec{RS}=[-2,-3,-5]$.

Suppose we have two non-parallel direction vectors $\vec{a}$ and $\vec{b}$ on a plane.




Then, the vector from any point $P_0 = (x_0, y_0, z_0)$ to any other point $P = (x, y, z)$ on the plane can be represented by a linear combination of the direction vectors, *ie* $\vec{P_0P} = t\vec{a} + s\vec{b}$, where $t, s \in \mathbb{R}$. But since $\vec{P_0P} = [x, y, z] - [x_0, y_0, z_0]$, we can rearrange the vector equation to write $[x, y, z] = [x_0, y_0, z_0] + t\vec{a} + s\vec{b} = [x_0, y_0, z_0] + t[a_1, a_2, a_3] + s[b_1, b_2, b_3]$ or $\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}$ and we have the vector equation of a plane.

So for our example above, we can take $P_0 = (2, 2, 2)$, $\vec{a} = [-2, 5, 3]$ and $\vec{b} = [-2, -3, -5]$, so $[x, y, z] = [2, 2, 2] + t[-2, 5, 3] + s[-2, -3, -5]$. Notice that if $t = -1/4$ and $s = 1/4$, we have $[x, y, z] = [2, 0, 0]$, the position vector of the $x$-intercept.

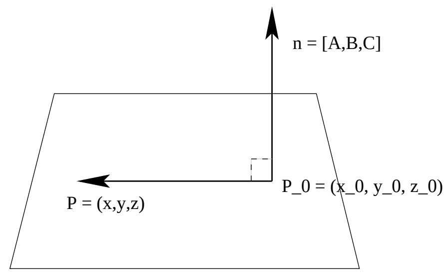
The parametric equations of a plane would be $x = x_0 + ta_1 + sb_1$, $y = y_0 + ta_2 + sb_2$ and $z = z_0 + ta_3 + sb_3$ where $t, s \in \mathbb{R}$.

So for our example, $x = 2 - 2t - 2s$, $y = 2 + 5t - 3s$ and $z = 2 + 3t - 5s$. Picking values for $t$ and $s$ will give us other points on the plane, like if $t = 1$ and $s = -2$, we'll have $(4, 13, 15)$. *Can you see that this does satisfy the original scalar equation?*

**Example:**
Let's find the equation of the plane that contains the line $[x, y, z] = [2, 1, 4] + t[2, 3, 4]$ and is parallel to the line $[x, y, z] = [7, 4, 2] + s[-1, 0, 6]$.
Since the first line lies on the plane, we can use $(2, 1, 4)$ as the point on the plane and $[2, 3, 4]$ as a direction vector. Since the plane is parallel to the second line and $[-1, 0, 6]$ is not parallel to $[2, 3, 4]$ (*not a scalar multiple of it*), we can use $[-1, 0, 6]$ as the second direction vector. A vector equation for the plane would be $[x, y, z] = [2, 1, 4] + p[2, 3, 4] + q[-1, 0, 6]$ for $p, q \in \mathbb{R}$. Or, in parametric form, $x = 2 + 2p - q$, $y = 1 + 3p$ and $z = 4 + 4p + 6q$.

# Properties of Planes

Suppose we know a point $P_0 = (x_0, y_0, z_0)$ on a plane and that the vector $\vec{n} = [A, B, C] = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$ is normal (perpendicular or orthogonal) to the plane.



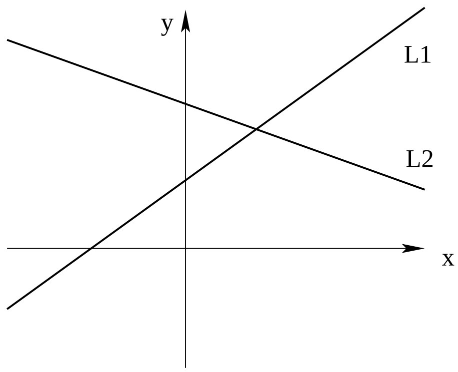

If we take any point $P = (x, y, z)$ on the plane, then the vector $\vec{P_0P} = [x - x_0, y - y_0, z - z_0]$ is parallel to the plane and perpendicular to $\vec{n}$. Thus $\vec{P_0P} \cdot \vec{n} = 0$.
So $[x - x_0, y - y_0, z - z_0] \cdot [A, B, C] = 0$
or $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
or $Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$
or $Ax + By + Cz + D = 0$ where $D = -Ax_0 - By_0 - Cz_0$ is a constant. This is the scalar equation of the plane.

**Example:**
The plane containing the point $P_0 = (4, -2, 3)$ and having normal vector $\vec{n} = [1, -2, 1]$ has scalar equation $(1)x+(-2)y+(1)z+(-(1)(4)-(-2)(-2)-(1)(3)) = 0$ or $x-2y+z-11 = 0$, which is also written as $x - 2y + z = 11$.

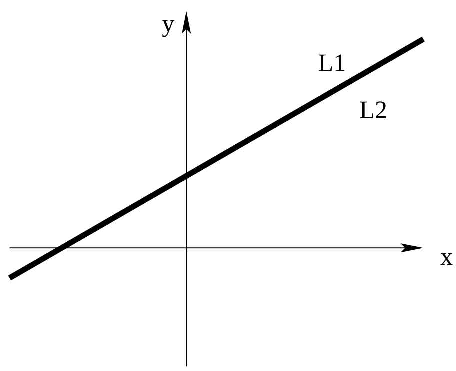
**Example:**
Find the scalar equation of the plane containing the points $P = (2, 1, 4)$, $Q = (4, 0, 3)$ and $R = (3, 4, -2)$.
The vectors $\vec{PQ} = [2, -1, -1]$ and $\vec{PR} = [1, 3, -6]$ are vectors on the plane. Then $\vec{PQ} \times \vec{PR}$ will be normal to the plane, so
$\vec{n} = [2, -1, -1] \times [1, 3, -6]$
$= [(-1)(-6) - (-1)(3), (-1)(1) - (2)(-6), (2)(3) - (-1)(1)]$
$= [9, 11, 7]$.
The scalar equation has the form $9x+11y+7z+D = 0$. We can plug any of the three points in to find $D$. Let's use $Q = (4, 0, -3)$, then $9(4) + 11(0) + 7(3) + D = 0 \implies D = -57$ and the scalar eqaution of the plane is $9x + 11y + 7z = 57$.

# Intersections of Lines in Two– and Three–Space

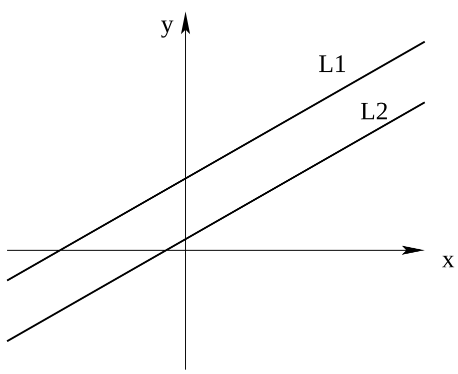
For two lines $L_1$ and $L_2$ in two-space, there are three possibilities.




(i) The lines intersect at a single point.




(ii) The lines are coincident (and hence intersect at infinitely many points).




(iii) The lines are parallel and distinct (and hence do not intersect).

There is a direct connection to linear systems of two equations in two unknowns.

(a) Consider
$2x + 3y = 4$
$3x + 4y = 9$

we can rewrite the system as an augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 3 & 4 & 9 \end{array}\right]$$

where the first row represents the first equation and the second row represents the second equation. The first column represents $x$ and the second column represents $y$. The vertical bar represents the equalities and the last column represents the constants on the right-hand sides of the equations. This is simply a shorthand notation for the system of equations where we keep track of the coefficients. We perform elementary row operations like multiplying or dividing a row by a nonzero scalar constant, interchanging rows or adding or subtracting multiples of one row from another. The goal is to change the matrix into its reduced row echelon form (RREF), where the first nonzero entry in each row is a 1, called a leading 1 or pivot. Each leading 1 is strictly to the right of any leading 1 in a row above it and all other elements in a column that contains a leading 1 are zero (so both above and below the leading 1). Any zero rows must appear at the bottom of the matrix. When we have reduced row echelon form, we will be able to read the solution of the linear system right out of the matrix. The procedure to reduce a matrix to this form is called Gauss-Jordan elimination.

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 3 & 4 & 9 \end{array}\right] R_1/2 \text{ (create leading 1 in first row)}$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 3 & 4 & 9 \end{array}\right] R_2 - 3R_1 \text{ (clear column under leading 1)}$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & -1/2 & 3 \end{array}\right] R_2 \times -2 \text{ (make leading 1 in second row)}$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 1 & -6 \end{array}\right] R_1 - (3/2)R_2 \text{ (clear column above leading 1)}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -6 \end{array}\right] \text{this is RREF, the solution is } x = 11 \text{ and } y = -6.$$

*These lines intersect at a single point, which is a unique solution.*

(b) Consider
$2x + 3y = 4$
$6x + 9y = 12$

as a matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 6 & 9 & 12 \end{array}\right] R_1/2$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 6 & 9 & 12 \end{array}\right] R_2 - 6R_1$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 0 & 0 \end{array}\right] \text{(RREF)}$$

we have a row of zeros and the solution is $x+(3/2)y = 2$, which means that there are infinitely many solutions (*these lines are coincident*). Let $y = t$ be a parameter, then $x + (3/2)t = 2$ or $x = 2 - (3/2)t$. So the parametric equations of the line of solutions are $x = 2 - (3/2)t$ and $y = t$ and the vector equation would be $[x, y] = [2, 0] + t[-3/2, 1]$.

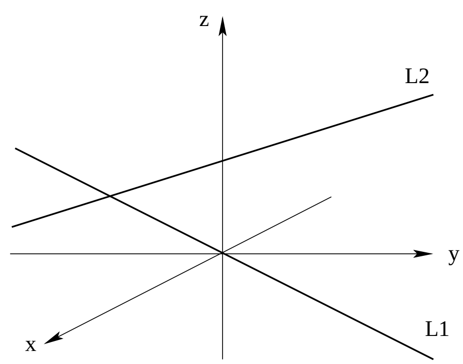
(c) Consider
$2x + 3y = 4$
$6x + 9y = 16$

as a matrix

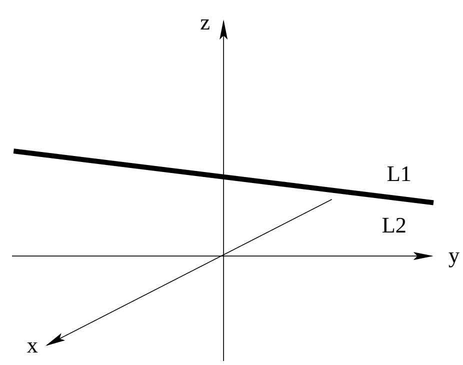
$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 6 & 9 & 16 \end{array}\right] R_1/2$$
$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 6 & 9 & 16 \end{array}\right] R_2 - 6R_1$$
$$\left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 0 & 4 \end{array}\right]$$

this system is <u>inconsistent</u>. The last row says $0x + 0y = 4$, which is impossible, so there is no solution here (*these lines are parallel*).

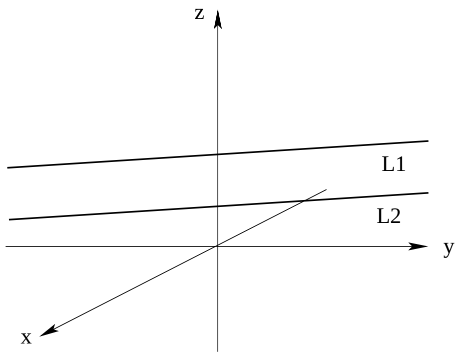
If we have two lines $L_1$ and $L_2$ in three-space, there are four possibilities.



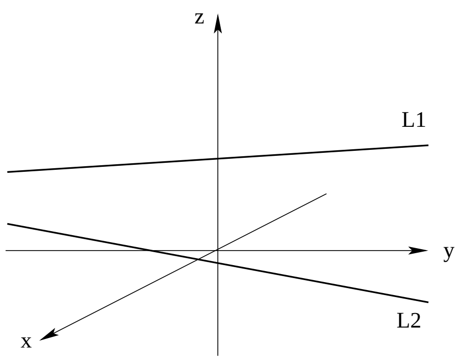

(i) The lines intersect at a single point (unique solution).



(ii) The lines are coincident (there are infinitely many solutions).




(iii) The lines are parallel and distinct (no solution).



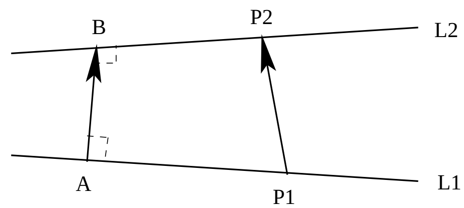

(iv) The lines are <u>skew</u> – they are distinct, not parallel and do not intersect (no solution).

**Example:**
The lines $L_1$ $[x, y, z] = [2, 3, 4] + t[1, 1, 3]$ and $L_2$ $[x, y, z] = [1, 0, -1] + s[1, 1, 3]$ are parallel (same direction vector). Are they coincident or distinct ?
If they are coincident, every point on $L_1$ would be on $L_2$, so all we have to do is check if $(2, 3, 4)$ is on $L_2$. The parametric equations for $L_2$ are $x = 1 + s$, $y = s$ and $z = -1 + 3s$. If we substitute the point $(2, 3, 4)$ in, $2 = 1 + s \implies s = 1$
$3 = s \implies s = 3$
and $4 = -1 + 3s \implies s = 5/3$ and so the point is not on $L_2$ and the lines are distinct.

**Example:**
Find the point of intersection of $L_1$ $\vec{r} = [2, 1, 5] + t[3, 1, -2]$ and $L_2$ $\vec{r} = [0, 3, 1] + s[5, -1, 2]$
*These lines are not parallel because their direction vectors are not parallel.*
We write the lines in parametric form $x = 2 + 3t$, $y = 1 + t$ and $z = 5 - 2t$; and $x = 5s$, $y = 3 - s$ and $z = 1 + 2s$. We look for a point of intersection by equating the coordinates $2 + 3t = 5s$, $1 + t = 3 - s$ and $5 - 2t = 1 + 2s$. This system has a unique solution $s = t = 1$. Plugging these values for the parameters into the equations for the lines yields the point of intersection $(5, 2, 3)$ (*which is a unique solution*).

**Example:**
Are the lines $L_1$ $[x, y, z] = [2, 1, 5] + t[3, 1, -2]$ and $L_2$ $[x, y, z] = [1, 3, 2] + s[5, -1, 2]$ skew?
The lines are not parallel, so we check for a point of intersection.
$2 + 3t = 1 + 5s \implies 3t - 5s = -1$
$1 + t = 3 - s \implies t + s = 2$
and $5-2t = 2+2s \implies 2t+2s = 3$, which contradicts the previous equation, so this system is inconsistent and there is no point of intersection. $\therefore$ the lines are skew.

The shortest distance between two skew lines $L_1$ and $L_2$ would be the length of the common perpendicular $|\vec{AB}|$.



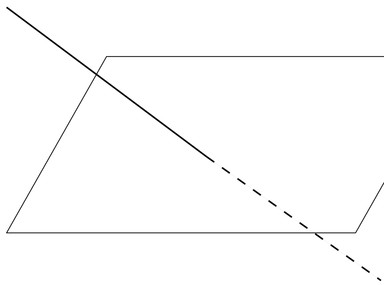

Suppose we have points $P_1$ and $P_2$ on the lines, then if $\vec{n}$ is any vector in the direction of $\vec{AB}$, we would have that $proj_{\vec{n}}\vec{P_1P_2} = \vec{AB}$. But $|\vec{AB}|$ is the length we want, so $|\vec{AB}| = |proj_{\vec{n}}\vec{P_1P_2}| = \dfrac{|\vec{P_1P_2} \cdot \vec{n}|}{|\vec{n}|}$. We can find a normal vector $\vec{n}$ that is perpendicular to both lines by taking the cross product of the direction vectors of the lines, *ie* $\vec{n} = \vec{m_1} \times \vec{m_2}$.

**Example:**
For our skew lines above, $\vec{m_1} = [3, 1, -2]$ and $\vec{m_2} = [5, -1, 2]$.
So $\vec{n} = \vec{m_1} \times \vec{m_2}$
$= [3, 1, -2] \times [5, -1, 2]$
$= [(1)(2) - (-2)(-1), (-2)(5) - (3)(2), (3)(-1) - (1)(5)]$
$= [0, -16, -8]$.
We can divide by $-8$ and use $\vec{n} = [0, 2, 1]$ for convenience.

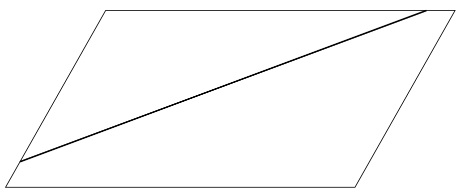
Use the points on the lines $P_1 = (2, 1, 5)$ and $P_2 = (1, 3, 2)$ to get $\vec{P_1P_2} = [-1, 2, -3]$.
So $|\vec{AB}| = \dfrac{|\vec{P_1P_2} \cdot \vec{n}|}{|\vec{n}|}$
$= \dfrac{|[-1, 2, -3] \cdot [0, 2, 1]|}{\sqrt{(0)^2 + (2)^2 + (1)^2}}$
$= \dfrac{|(-1)(0) + (2)(2) + (-3)(1)|}{\sqrt{5}}$
$= 1/\sqrt{5} \approx 0.45$ (*so the lines are close*).

# Intersections of Lines and Planes

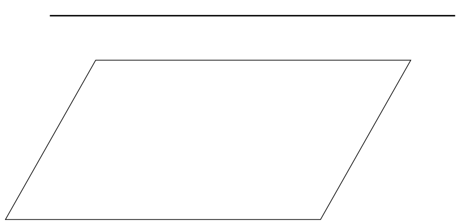
If we have a line and a plane in three-space, there are three possibilities.



(i) The line intersects the plane at a single point (unique solution).



(ii) The line lies on the plane (so they intersect at infinitely many points and there are infinitely many solutions).



(iii) The line is parallel to the plane (but not on it), so the line does not intersect the plane (no solution).

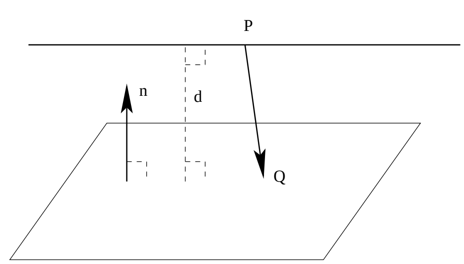
**Example:**
Dose the line $x = 2 + t$, $y = 6 - t$ and $z = 3 + 2t$ intersect the plane $2x + 3y + 5z = 55$ ? If yes, at what point ?
We plug the parametric equations of the line into the scalar equation of the plane to see if there is a solution.
$2(2 + t) + 3(6 - t) + 5(3 + 2t) = 55$
$4 + 2t + 18 - 3t + 15 + 10t = 55$
$9t = 18 \implies t = 2$
Since there is a single solution for $t$, the line and plane intersect at the single point $(4, 4, 7)$.

**Example:**
Is the line $\vec{r} = [2, 3, 1] + t[1, 5, -2]$ parallel to the plane $3x + y + 4z = 13$ ? If yes, is the line

in the plane or not ?
The direction vector of the line is $\vec{m} = [1, 5, -2] = \hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and a normal vector of the plane is $\vec{n} = [3, 1, 4] = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$.
Let's check $\vec{m} \cdot \vec{n} = [1, 5, -2] \cdot [3, 1, 4] = (1)(3) + (5)(1) + (-2)(4) = 0$.
Since $\vec{m} \cdot \vec{n} = 0$, $\vec{m} \perp \vec{n}$ and hence the line is parallel to the plane.
Let's check for points of intersection.
$3(2 + t) + (3 + 5t) + 4(1 - 2t) = 13$
$6 + 3t + 3 + 5t + 4 - 8t = 13$
$13 + 0t = 13$
$0t = 0$
Since all values of t satisfy this equation, there are infinitely many points of intersection and hence the line does lie on the plane.

The line would be parallel to the plane $3x + y + 4z = 10$ as well, but it would not lie on it.
$3(2 + t) + (3 + 5t) + 4(1 - 2t) = 10$
$13 + 0t = 10$
$0t = -3$
Which has no solution.

If a line is parallel to a plane, but not on it, how far apart are they ? Suppose we have a point $P$ on the line and a point $Q$ on the plane.




If $\vec{n}$ is a normal vector to the plane, then the projection of $\vec{PQ}$ onto $\vec{n}$ would be the perpendicular distance $d$ from $P$ to the plane. So we have $d = \dfrac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$ (*which is similar to what we did for the skew lines*).

**Example:**
How far is the line $\vec{r} = [2, 3, 1] + t[1, 5, -2]$ from the plane $3x + y + 4z = 10$ ?

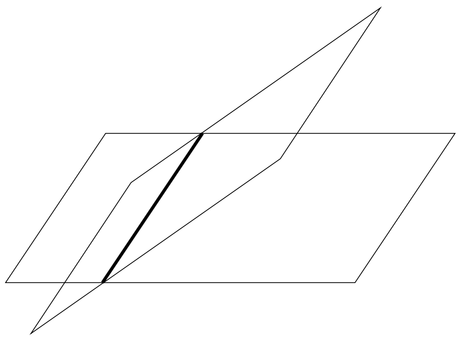
Take $P = (2, 3, 1)$ and $Q = (2, 0, 1)$, then $\vec{PQ} = [0, -3, 0]$ and we know $\vec{n} = [3, 1, 4]$.

So $d = \dfrac{|[0, -3, 0] \cdot [3, 1, 4]|}{\sqrt{(3)^2 + (1)^2 + (4)^2}}$
$= \dfrac{|(0)(3) + (-3)(1) + (0)(4)|}{\sqrt{26}}$

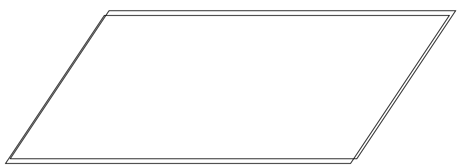
$= 3/\sqrt{26} \approx 0.59$ (*so pretty close*).

## Intersections of Planes

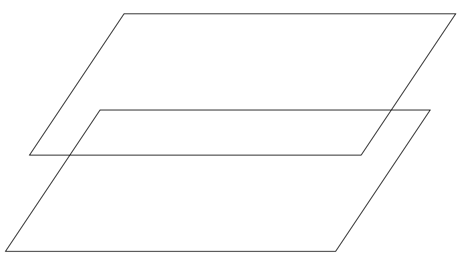
If we have two planes in three-space, there are three possibilities.



(i) They intersect in a line (infinitely many solutions with one parameter).



(ii) They are coincident (infinitely many solutions with two parameters).



(iii) They are parallel and distinct (no solution).

A system of two equations in three unknowns is either inconsistent (has no solution) or has infinitely many solutions – a unique solution is not possible (two planes cannot intersect at a single point).

**Example:**
Consider the planes $2x + 2y + z = 7$ and $x + y + z = 6$.
The normals for the planes are not parallel, so the planes are not parallel and hence they must intersect.
Let's solve for the line of intersection. We'll rewrite the system as a matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 2 & 1 & 7 \end{array}\right] R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & -1 & -5 \end{array}\right] R_2 \times -1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right] R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right] \text{(RREF)}$$

The solution is $x + y = 1$, $z = 5$.
Let $y = t$, then $x = 1 - t$, $y = t$ and $z = 5$ are the parametric equations of the line.
The vector equation is $[x, y, z] = [1, 0, 5] + t[-1, 1, 0]$.

**Example:**
Consider the planes $x - 2y + 3z = 10$ and $2x - 4y + 6z = 12$.
The normals are parallel, so the planes are parallel. But are they coincident or distinct ?

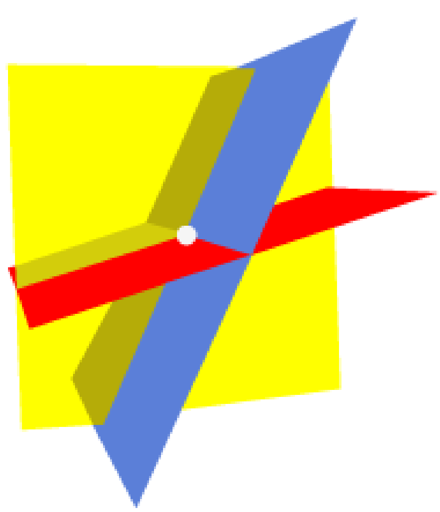
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 2 & -4 & 6 & 12 \end{array}\right] R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

This is an inconsistent system, so there is no solution and the planes are distinct.

*The planes $x - 2y + 3z = 10$ and $2x - 4y + 6z = 20$ are coincident – can you see that ?*

For three planes, there are six possibilities. There are three where the linear system is consistent (has solutions).



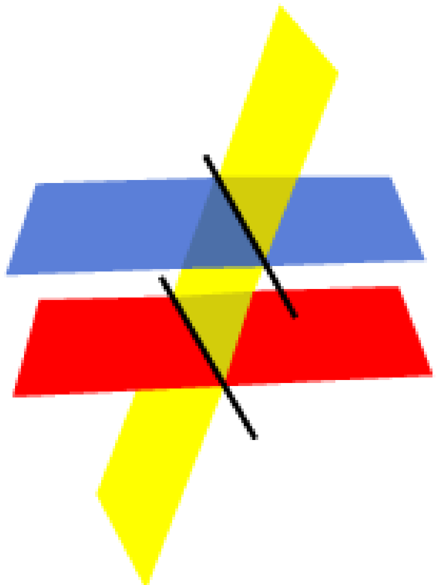
(i) The planes intersect at a single point (unique solution).

(ii) The planes intersect in a line (infinitely many solutions with one parameter).

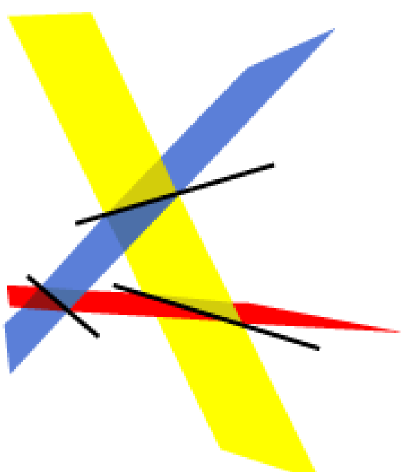
(iii) The planes are coincident (infinitely many solutions with two parameters).

And there are three where the system is <u>inconsistent</u> (no solutions).

(iv) The planes are parallel (and at least two are distinct).



(v) Two of the planes are parallel (and distinct), but the third is not parallel.



(vi) The planes intersect in pairs.

The normals in the situations would have to be:
(i) not parallel, nor coplanar
(ii) coplanar, but not parallel
(iii) parallel
(iv) parallel
(v) two are parallel
(vi) coplanar, but not parallel.

**Example:**
Consider the planes $x + 2y + 3z = 10$, $2x + 3y - z = 4$ and $3x + 4y + 5z = 12$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & 3 & -1 & 4 \\ 3 & 4 & 5 & 12 \end{array}\right] R_2 - 2R_1, R_3 - 3R_1$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -1 & -7 & -16 \\ 0 & -2 & -4 & -18 \end{array}\right] R_2 \times -1$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 7 & 16 \\ 0 & -2 & -4 & -18 \end{array}\right] R_3 + 2R_2$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 7 & 16 \\ 0 & 0 & 10 & 14 \end{array}\right] R_3/10$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 7 & 16 \\ 0 & 0 & 1 & 7/5 \end{array}\right] R_1 - 3R_3, R_2 - 7R_3$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 29/5 \\ 0 & 1 & 0 & 31/5 \\ 0 & 0 & 1 & 7/5 \end{array}\right] R_1 - 2R_2$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -33/5 \\ 0 & 1 & 0 & 31/5 \\ 0 & 0 & 1 & 7/5 \end{array}\right] \text{(RREF)}$$

These planes intersect at a single point $(-33/5, 31/5, 7/5)$.

**Example:**
Consider the planes $x + y - 2z = 4$, $2x + 2y - 4z = 6$ and $3x + 5y + 2z = 10$.

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 2 & 2 & -4 & 6 \\ 3 & 5 & 2 & 10 \end{array}\right] R_2 - 2R_1, R_3 - 3R_1$$
$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & 8 & -2 \end{array}\right]$$

This is an inconsistent system, so there is no solution.
*Can you see that two of the planes are parallel but distinct?*

**Example:**
Consider the planes $x - 5y + 2z = 10$, $x + 7y - 2z = -6$ and $8x + 5y + z = 20$.

$$\left[\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 1 & 7 & -2 & -6 \\ 8 & 5 & 1 & 20 \end{array}\right] R_2 - R_1, R_3 - 8R_1$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & 10 \\ 0 & 12 & -4 & -16 \\ 0 & 45 & -15 & -60 \end{array}\right] R_2/12$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & 10 \\ 0 & 1 & -1/3 & -4/3 \\ 0 & 45 & -15 & -60 \end{array}\right] R_3 - 45R_2$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & 10 \\ 0 & 1 & -1/3 & -4/3 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 + 5R_2$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1/3 & 10/3 \\ 0 & 1 & -1/3 & -4/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{(RREF)}$$

So we have $x + (1/3)z = 10/3$ and $y - (1/3)z = -4/3$.
Let $z = t$ be the parameter, then we have $x = 10/3 - (1/3)t$, $y = -4/3 + (1/3)t$ and $z = t$ or $\vec{r} = [10/3, -4/3, 0] + t[-1/3, 1/3, 1]$ and the planes intersect in a line.

# Practice Problems

**1.** Given the line in vector or parametric form, convert it to the other.
(a) $x = 2 + t$, $y = 7 - 5t$, $z = 2t$
(b) $[x, y, z] = [3, 4, -2] + t[2, -1, 0]$

**2.** Write the vector and parametric forms of the line parallel to $2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ passing through the point $(4, 1, -3)$. Does the line pass through the points $(16, -17, 3)$ and $(5, 2, -2)$?

**3.** Write all three forms of the equation of the plane passing through the point $(-2, 1, 3)$ with normal vector $\vec{n} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

**4.** Write all three forms of the equation of the plane containing the points $(1, 1, 4)$, $(-2, -1, 0)$ and $(3, 2, -5)$. Is the point $(9, 5, 3)$ on the plane?

**5.** Determine if the lines intersect and write the solutions if they do.
(a) $\vec{r} = [2, 3, -1] + t[2, -1, 2]$ and $\vec{r} = [1, 4, 6] + s[4, -2, 4]$
(b) $x = 2 + t$, $y = -3 - 2t$, $z = 4 - t$ and $x = 3 + s$, $y = 1 + s$, $z = 17 + 6s$

**6.** Show that the lines $[x, y, z] = [1, 0, -1] + s[2, 3, -4]$ and $[x, y, z] = [8, 1, 3] + t[4, -5, 1]$ are skew and find the distance between them.

**7.** Does the line intersect the plane? If yes, give the solution.
(a) $x = 2 + t$, $y = 1 - 2t$, $z = 3 - t$ and $4x + y + 2z = 10$
(b) $x = 1 + 2t$, $y = -2 + 3t$, $z = 4 - t$ and $x + y + 5z = 19$
(c) $\vec{r} = [2, 0, -4] + t[1, 5, -1]$ and $x + 2y + z = 6$

**8.** What is the distance between the point $(3, 2, -1)$ and the plane $2x + 3y + z = 8$?

**9.** Find the line of intersection of the planes $x + 2y + z = 12$ and $2x - y + 3z = 4$.

**10.** How do the planes intersect?
(a) $x + 2y + z = 4$, $2x - y + z = 7$ and $3x + 2y + 2z = 8$
(b) $x + 2y + z = 4$, $2x + 4y + 2z = 8$ and $3x + y + 3z = 6$
(c) $x + 2y + z = 4$, $5x + 6y - z = -2$ and $2x + 4y + 2z = 3$

# Practice Problems Solutions

1. (a) $[x, y, z] = [2, 7, 0] + t[1, -5, 2]$
(b) $x = 3 + 2t$, $y = 4 - t$, $z = -2$

2. the line is $\vec{r} = [4, 1, -3] + t[2, -3, 1]$ or $x = 4 + 2t$, $y = 1 - 3t$, $z = -3 + t$
$16 = 4 + 2t \implies t = 6$
$-17 = 1 - 3t \implies t = 6$
$3 = -3 + t \implies t = 6$, so yes, $(16, -17, 3)$ is on the line
$5 = 4 + 2t \implies t = 1/2$
$2 = 1 - 3t \implies t = -1/3$
$-2 = -3 + t \implies t = 1$, so no, $(5, 2, -2)$ is not on the line

3. since the normal is $\vec{n} = [1, -2, 4]$, we have $A = 1$, $B = -2$ and $C = 4$ and the scalar equation is $x - 2y + 4z + D = 0$

plug in the point $(-2) - 2(1) + 4(3) + D = 0 \implies D = -8$ and so the plane is $x - 2y + 4z = 8$

next, find two other points on the plane – let's use $(0, 0, 2)$ and $(8, 0, 0)$

then we have direction vectors $[-2, 1, 1]$ and $[-10, 1, 3]$

so the vector equation is $[x, y, z] = [-2, 1, 3] + s[-2, 1, 1] + t[-10, 1, 3]$
and the parametric equations are $x = -2 - 2s - 10t$, $y = 1 + s + t$, $z = 3 + s + 3t$

4. we have direction vectors $\vec{a} = [-3, -2, -4]$ and $\vec{b} = [2, 1, -9]$
so the vector equation is $\vec{r} = [1, 1, 4] + t[-3, -2, -4] + s[2, 1, -9]$
and the parametric equations $x = 1 - 3t + 2s$, $y = 1 - 2t + s$, $z = 4 - 4t - 9s$

a normal to the plane is $\vec{n} = \vec{a} \times \vec{b} = [-3, -2, -4] \times [2, 1, -9] = [22, -35, 1]$
so the scalar equation has the form $22x - 35y + z + D = 0$

plug in the point $(1, 1, 4)$ to see $22(1) - 35(1) + (4) + D = 0 \implies D = 9$
so the plane is $22x - 35y + z = -9$

plug $(9, 5, 3)$ into the equation $22(9) - 35(5) + (3) = 26 \neq -9$, so no, $(9, 5, 3)$ is not on the plane

5. (a) the lines are parallel, so they are either distinct or coincident
is the point $(2, 3, -1)$ on the second line ?
$2 = 1 + 4s \implies s = 1/4$
$3 = 4 - 2s \implies s = 1/2$
$-1 = 6 + 4s \implies s = -7/4$, so no, that point is not on the second line, so the lines are

distinct and there is no intersection

(b) the lines are not parallel, so they are either skew or intersect at a point
equate the coordinates $2+t=3+s \implies t-s=1$
$-3-2t=1+s \implies -2t-s=4$
$4-t=17+6s \implies -t-6s=13$
the solution $s=-2$, $t=-1$ satisfies all three, so there is a (unique) point of intersection $(1,-1,5)$

6. the lines are not parallel, so if they do not intersect at a single point, they must be skew

write the parametric forms and equate the coordinates to get
$1+2s=8+4t \implies 2s-4t=7$ (#1.)
$3s=1-5t \implies 3s+5t=1$ (#2.)
$-1-4s=3+t \implies -4s-t=4$ (#3.)

but then $2\times$(#1.) + (#3.) gives that $-9t=18 \implies t=-2 \implies s=-1/2$, but that does not satisfy (#2.), so there is no point of intersection

we have $P_1=(1,0,-1)$ and $P_2=(8,1,3)$ so $\vec{P_1P_2}=[7,1,4]$

the normal is $\vec{n}=[2,3,-4]\times[4,-5,1]=[-17,-18,-22]$

so the distance is $d=\dfrac{|\vec{P_1P_2}\cdot\vec{n}|}{|\vec{n}|}=\dfrac{|[7,1,4]\cdot[-17,-18,-22]|}{\sqrt{(-17)^2+(-18)^2+(-22)^2}}=225/\sqrt{1097}\approx 6.79$

7. (a) plug the line into the plane
$4(2+t)+(1-2t)+2(3-t)=8+4t+1-2t+6-2t=15\neq 10$
the line and plane are parallel, but the line does not lie on the plane
*can you see that from the equations?*

(b) $(1+2t)+(-2+3t)+5(4-t)=1+2t-2+3t+20-5t=19$
the line lies on the plane, so every point on the line is a solution

(c) $(2+t)+2(5t)+(-4-t)=2+t+10t-4-t=-2+10t=6$ if $t=4/5$
so there is a point of intersection $(14/5,4,-24/5)$

8. we can take the point $(0,0,8)$ on the plane, so then $\vec{PQ}=[-3,-2,9]$

so $d=\dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}=\dfrac{|[-3,-2,9]\cdot[2,3,1]|}{\sqrt{(2)^2+(3)^2+(1)^2}}=3/\sqrt{14}\approx 0.80$

9. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 2 & -1 & 3 & 4 \end{array}\right] R_2 - 2R_1$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & -5 & 1 & -20 \end{array}\right] R_2/-5$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 1 & -1/5 & 4 \end{array}\right] R_1 - 2R_2$

$\left[\begin{array}{ccc|c} 1 & 0 & 7/5 & 4 \\ 0 & 1 & -1/5 & 4 \end{array}\right]$

let $z = t$, then we have $x = 4 - (7/5)t$, $y = 4 + (1/5)t$, $z = t$

10. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 7 \\ 3 & 2 & 2 & 8 \end{array}\right] R_2 - 2R_1, R_3 - 3R_1$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & -1 & -1 \\ 0 & -4 & -1 & -4 \end{array}\right] R_2/-5$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1/5 & 1/5 \\ 0 & -4 & -1 & -4 \end{array}\right] R_3 + 4R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1/5 & 1/5 \\ 0 & 0 & -1/5 & -16/5 \end{array}\right] R_3 \times -5$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1/5 & 1/5 \\ 0 & 0 & 1 & 16 \end{array}\right] R_1 - R_3, R_2 - (1/5)R_3$

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 16 \end{array}\right] R_1 - 2R_2$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 16 \end{array}\right]$

so there is a single point of intersection $(-6, -3, 16)$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 8 \\ 3 & 1 & 3 & 6 \end{array}\right] R_2 - 2R_1, R_3 - 3R_1$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -6 \end{array}\right] R_2 \leftrightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2/-5$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 6/5 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/5 \\ 0 & 1 & 0 & 6/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $z = t$, then we have $x = 8/5 - t$, $y = 6/5$, $z = t$
and the planes intersect in a line
*two of the planes are coincident – can you see that ?*

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 5 & 6 & -1 & -2 \\ 2 & 4 & 2 & 3 \end{array}\right] R_2 - 5R_1, R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -4 & -6 & -22 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

this system is inconsistent, so there is no solution
*two of the planes are parallel and distinct – can you see that ?*