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On ADE quiver models and F-theory compactification

A Belhaj[1], J Rasmussen[2], A Sebbar[1] and M B Sedra[3,4]

[1] Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, ON, K1N 6N5, Canada
[2] Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia
[3] Laboratoire de Physique de la Matière et Rayonnement (LPMR), Morocco Faculté des Sciences, Université Ibn Tofail, Kenitra, Morocco
[4] Virtual African Center for Basic Sciences and Technology (VACBT), Groupement National de Physique des Hautes Energies (GNPHE), Faculté des Sciences, Rabat, Morocco

E-mail: abelhaj@uottawa.ca, J.Rasmussen@ms.unimelb.edu.au, sebbar@mathstat.uottawa.ca and sedra@ictp.it

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Abstract
Based on mirror symmetry, we discuss geometric engineering of $N = 1$ ADE quiver models from F-theory compactifications on elliptic K3 surfaces fibred over certain four-dimensional base spaces. The latter are constructed as intersecting 4-cycles according to ADE Dynkin diagrams, thereby mimicking the construction of Calabi–Yau threefolds used in geometric engineering in type II superstring theory. Matter is incorporated by considering D7-branes wrapping these 4-cycles. Using a geometric procedure referred to as folding, we discuss how the corresponding physics can be converted into a scenario with D5-branes wrapping 2-cycles of ALE spaces.

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1. Introduction

The construction of four-dimensional supersymmetric gauge theories has attracted much attention and has been investigated from various points of view in superstring compactifications, for example. The approach of interest here is the construction of gauge theories from geometric data of superstring backgrounds. Thus, the gauge group and matter content of the resulting models are obtained from the singularities of the K3 fibres and the non-trivial geometry describing the base space of the internal manifolds. In this way, the complete set of physical parameters of the gauge theory is related to the moduli space of the associated manifolds. This programme is called geometric engineering [1–6]. It enables one to represent supersymmetric gauge models by quiver diagrams similar to Dynkin graphs of ordinary, affine or indefinite Lie algebras [2, 4, 5]. One considers the base space to be composed of a collection of intersecting compact cycles, each of which is giving rise to an SU gauge-group factor. To each of these is associated a Dynkin node, and for each pair of groups with matter in bi-fundamental representations, the two corresponding nodes are connected by a line.

Four-dimensional quiver gauge models have been constructed also in M-theory on seven-dimensional manifolds with G_2 holonomy. The compactification manifolds are K3 fibrations over three-dimensional base spaces with ADE geometries. The resulting gauge theories have been discussed in the realm of (p, q) brane webs [7].

The aim of the present paper is to contribute to the programme of geometric engineering by constructing four-dimensional $N = 1$ ADE quiver models from F-theory compactifications. For an earlier work on geometric engineering in F-theory, we refer to [8]. The manifolds of our interest are elliptic K3 surfaces fibred over ADE 4-cycles, where the base is obtained by resolving the ADE hyper-Kähler singularities [9], and are similar to the Calabi–Yau threefolds used in type II geometric engineering [1]. An objective here is to construct explicit models of such geometries leading to gauge theories with bi-fundamental matter in four dimensions. As an illustration, we consider A_r quiver gauge models by introducing 4-cycles in the base which are intersecting according to an A_r Dynkin graph. In particular, we consider in some details the cases of A_1 and A_2, and we find that they are linked to ordinary A_1 and A_2 singularities of the asymptotically locally Euclidean (ALE) spaces. The dual type IIB models involve D7-branes wrapping ADE 4-cycles. Using a geometric procedure referred to as folding, we show that the corresponding physics can be converted into a scenario with D5-branes wrapping 2-cycles of ALE spaces.

The present paper is organized as follows. Section 2 provides a brief review on how geometric engineering may be used to obtain four-dimensional gauge models from superstring theory or M-theory. The basic extension to F-theory is discussed in section 3 where focus is on the construction of ADE fourfolds. Mirror symmetry is employed in section 4 when studying the resulting physics in the presence of D7-branes. The folding procedure linking this to a similar study of D5-branes in superstring theory is also discussed in section 4. Section 5 contains some concluding remarks.

2. Geometric engineering

In this section, we briefly review the main steps in obtaining four-dimensional supersymmetric gauge models either from compactification of type II superstrings on Calabi–Yau threefolds CY^3 [1, 3] or from compactification of M-theory on G_2 manifolds [7]. In either case, the manifold is a K3 fibration over a base space B. The basic tasks are to specify the singularity of the K3 fibration manifolds and to take the limit in which the volume $V(B)$ of the base space is very large so that gravitational effects may be ignored. In particular, one considers the K3 fibres locally as non-compact ALE spaces with ADE singularities. One subsequently examines the compactification in the presence of D2-branes or M2-branes wrapping the vanishing 2-cycles in the ALE spaces. This enables one to make conclusions about the gauge group G and matter content in the four-dimensional model. This analysis can be carried out in two steps as one may perform an initial but partial compactification on the K3 surfaces followed by a further compactification down to four dimensions.

To illustrate this, we consider type IIA superstring theory. Let us study the simplest case of $SU(2)$ gauge theory obtained from an A_1 singularity of the ALE fibre space at the origin. Mathematically, this is described by

$$xy + z^2 = 0 \tag{1}$$

where x, y and z are the complex coordinates. As usual, the singularity can be removed either by deforming the complex structure or by a blow-up procedure. Geometrically, this corresponds to replacing the singular point $(x = y = z = 0)$ by a $\mathbf{CP}^1 \sim \mathbf{S}^2$.

In the case where the A_1 singularity has been resolved, a D2-brane can wrap around the blown-up $\mathbf{CP}^1$ giving rise to a pair of vector particles, $W^{\pm}$. These particles correspond to the two possible orientations of the wrappings. The particles have masses proportional to the volume of the blown-up 2-spheres. $W^{\pm}$ are charged under the $U(1)$ field Z obtained by decomposing the type IIA superstring 3-form $C_{\mu\nu\lambda}$ in terms of the harmonic 2-form on the 2-sphere and a 1-form not in the K3 fibre. In the singular limit where $\mathbf{CP}^1$ shrinks to a point, the three-vector particles become massless and they form an adjoint $SU(2)$ representation. We thus obtain an $N = 2$ $SU(2)$ gauge symmetry in six dimensions.

A further compactification on the base space B_2 gives pure $SU(2)$ Yang–Mills theory in four dimensions. $N = 2$ models are obtained by taking B_2 as a real 2-sphere. If we instead compactify on Riemann surfaces with 1-cycles, we obtain $N = 4$ gauge models. The extra scalar fields that one can get in such an $N = 4$ system can be identified with the expectation values of Wilson lines on such a surface.

Before turning to geometric engineering in F-theory, we list a couple of comments on extensions of the superstring compactification indicated above:

- The geometric analysis of the $SU(2)$ model can be extended straightforwardly to general simply laced ADE gauge groups. This extension is based on the classification of ADE singularities of ALE spaces.
- Matter is incorporated by considering intersecting 2-cycles with ADE singularities in the base space B_2. In this case, the base is obtained by resolving ADE singularities of ALE spaces [2].
- There are three kinds of models and they are related to the classification of generalized Cartan matrices of Kac–Moody algebras [4, 5].
- Similar comments apply to the case of M-theory on manifolds with G_2 holonomy in which case the D2-branes are replaced by M2-branes [7].

In what follows, we study geometric engineering in F-theory by introducing intersecting geometries in the accompanying compactifications on Calabi–Yau fourfolds. These manifolds are constructed as elliptic K3 surfaces fibred over intersecting 4-cycles according to ADE Dynkin graphs. This geometric engineering may result in $N = 1$ ADE quiver models with bi-fundamental matter in four dimensions, thus extending the result of [10].

3. Geometric engineering in F-theory compactification

In this section, we study geometric engineering of quiver gauge models in F-theory compactification. It is recalled that F-theory defines a non-perturbative vacuum of type IIB superstring theory in which the dilaton and axion fields of the superstring theory are considered dynamical. This introduces an extra complex modulus which is interpreted as the complex parameter of an elliptic curve thereby introducing a non-perturbative vacuum of the type IIB superstring in a *12-dimensional* spacetime [11]. F-theory may also be defined in the context of string dualities, and as we will discuss, F-theory on elliptically fibred Calabi–Yau manifolds may be understood in terms of dual superstring models.

It is also recalled that type IIB superstring theory is a ten-dimensional model of closed strings with chiral $N = 2$ supersymmetry. The bosonic fields of the corresponding low-energy field theory are the graviton $g_{\mu\nu}$, the anti-symmetric tensor $B_{\mu\nu}$ and the dilaton ϕ coming from the NS–NS sector and the axion ξ, and the anti-symmetric tensor fields $\tilde{B}_{\mu\nu}$ and the self-dual (in the physical 8 dimensions) 4-form $D_{\mu\nu\rho\sigma}$ stemming from the R–R sector. There is no non-Abelian gauge field in the massless spectrum of type IIB superstring theory which instead contains Dp-branes with $p = -1, 1, 3, 5, 7$ and 9 on which the gauge fields A_μ live. These extended objects are non-perturbative solutions playing a crucial role in the study of gauge theories in superstring models. It is also noted that type IIB superstring theory has a non-perturbative $PSL(2, \mathbb{Z})$ symmetry with respect to which the fields $g_{\mu\nu}$ and $D_{\mu\nu\rho\sigma}$ are invariant. The complex string coupling $\tau_{\mathrm{IIB}} = \xi + \mathrm{i}e^{-\phi}$ and the doublet $(B_{\mu\nu}, \tilde{B}_{\mu\nu})$ of 2-forms, on the other hand, are believed to transform as

$$\tau_{\mathrm{IIB}} \to \frac{a\tau_{\mathrm{IIB}} + b}{c\tau_{\mathrm{IIB}} + d}, \qquad a, b, c, d \in \mathbb{Z}, \tag{2}$$

and

$$\begin{pmatrix} B_{\mu\nu} \\ \tilde{B}_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_{\mu\nu} \\ \tilde{B}_{\mu\nu} \end{pmatrix}, \tag{3}$$

where the integers a, b, c and d satisfy $ab - cd = 1$.

Following Vafa [11], one may interpret the complex field τ_{IIB} as the complex structure of an extra torus T^2 resulting in the aforementioned 12-dimensional model. From this point of view, type IIB superstring theory may be seen as the compactification of F-theory on T^2. Starting from F-theory, one can similarly look for new superstring models in lower dimensions obtained by compactifications on elliptically fibred Calabi–Yau manifolds. For example, the eight-dimensional F-theory on elliptically fibred K3 is obtained by taking a two-dimensional complex compact manifold given by

$$y^2 = x^3 + f(z)x + g(z), \tag{4}$$

where f and g are polynomials of degree 8 and 12, respectively. One varies the τ torus over the points of a compact space which is taken to be a Riemann sphere $\mathbf{CP}^1$ parametrized by the local coordinate z. In other words, the two-torus complex structure $\tau(z)$ is now a function of z as it varies over the $\mathbf{CP}^1$ base of K3. The above compact manifold generically has 24 singular points corresponding to $\tau(z) \to \infty$. These singularities have a remarkable physical interpretation as each one of the 24 points is associated with the location of a D7-brane in non-perturbative type IIB superstring theory. We assume that the number of D7-branes is arbitrary for non-compact two-dimensional complex space.

We now turn to the study of geometric engineering of quiver gauge models in F-theory. The method we will be using here is similar to the one employed in the engineering of four-dimensional QFT_4 from type II superstring theory or M-theory on G_2 manifolds. Thus, we first build the local fourfolds enabling us to construct four-dimensional quiver gauge models with bi-fundamental matter from compactification of F-theory. The manifolds we will consider are quite simple and involve K3 fibrations specifying the gauge groups in four dimensions. As in the study of string theory, matter may be incorporated by introducing a non-trivial geometry in the base space, and we are naturally led to consider a base space involving a collection of intersecting 4-cycles. Each 4-cycle gives rise to a gauge-group factor while matter is described in terms of bi-fundamental representations of these gauge groups.

3.1. Construction of fourfolds

The manifolds that we propose to use can be described as hyper-Kähler quotients as they are of the form

$$X_8: \qquad C^2 \text{ fibered over } V^2. \tag{5}$$

Here, V^2 is a two-dimensional toric variety specified below, while C^2 is the fibre which can be converted into a local elliptic K3 surface by an orbifold action [12]. Let us briefly discuss this construction. We consider the limit

$$C^2 \to \mathbb{H} \times \mathbb{C} \tag{6}$$

where $\mathbb{H}$ is the complex upper half-plane,

$$\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}. \tag{7}$$

We also introduce Γ_M as the principal subgroup of $SL(2, \mathbb{Z})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_M \subset SL(2, \mathbb{Z}), \qquad a = d = 1 \bmod M, \quad b = c = 1 \bmod M. \tag{8}$$

Γ acts freely on the upper half-plane $\mathbb{H}$. In this way, the semi-direct product $\Gamma \times \mathbb{Z}^2$ acts freely on $\mathbb{H} \times \mathbb{C}$ as

$$(\gamma; m, n) \cdot (\tau, z) = \left(\gamma\tau, \frac{z + m\tau + n}{c\tau + d}\right) \tag{9}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_M, \qquad (m, n) \in \mathbb{Z}^2, \qquad \tau \in \mathbb{H}, \quad z \in \mathbb{C}. \tag{10}$$

It follows that the quotient surface $(\mathbb{H} \times \mathbb{C})/(\Gamma \times \mathbb{Z}^2)$ is an elliptic fibration over the curve $\mathbb{H}/\Gamma$.

An advantage of working with an (real) eight-dimensional geometry like (5) is that one may study an intersecting structure in the base V^2 using physical arguments. This is done in part by mimicking the similar analysis of ADE singularities in terms of $N = 2$ toric sigma models. First, we let the manifold appear in the construction of a two-dimensional field theory with $N = 4$ supersymmetry. This model has $r + 2$ hypermultiplets which are charged under r Abelian vector multiplets $U(1)^r$ with charges Q_i^a. The geometry given by (5) solves the D-flat constraint equations of this $N = 4$ field theory:

$$\sum_{i=1}^{r+2} Q_i^a \phi_\alpha^i \bar{\phi}_i^\beta = \vec{\xi}^a \cdot \vec{\sigma}_\alpha^\beta, \qquad a = 1, \ldots, r. \tag{11}$$

Here, the fields ϕ_α denote $(r + 2)$-component complex doublets of hypermultiplets, $\vec{\xi}^a$ are the three-vector coupling parameters, while $\vec{\sigma}$ are the traceless 2×2 Pauli matrices. In a description that makes manifest only half the supersymmetry of the gauge theory, that is $N = 2$, the D-flatness conditions read

$$\sum_{i=1}^{r+2} Q_i^a \left(\left|\phi_1^i\right|^2 - \left|\phi_2^i\right|^2\right) = \xi_{\mathbb{R}}^a, \tag{12}$$

$$\sum_{i=1}^{r+2} Q_i^a \phi_1^i \bar{\phi}_2^i = \xi_{\mathbb{C}}^a. \tag{13}$$

By a set of rotations, we may set the complex components $\xi_{\mathbb{C}}^a$ of $\vec{\xi}^a$ equal to zero. Likewise, we may set the real components $\xi_{\mathbb{R}}^a$ to positive or negative values. It is noted that, in the resulting description, one may locally view any set $\left\{\phi_{\alpha_i}^i, i = 1, \ldots, r + 2\right\}$ as the toric coordinates of the two-dimensional base V^2. The remaining fields thus describe the fibre C^2.

In this way, the intersection matrix of the 4-cycles defining the base can be identified with the charge matrix of an $N = 4$ gauge theory in two dimensions. As we will see, this matrix can be related to a Cartan matrix of an ADE Lie algebra. Geometric engineering in F-theory compactification thereby enables us to describe four-dimensional ADE quiver gauge models with bi-fundamental matter.

3.2. ADE fourfolds

As already indicated, we will consider fourfolds whose base spaces are constructed as 4-cycles intersecting according to ADE Dynkin diagrams. We refer to these manifolds as ADE fourfolds. They constitute a very natural class of manifolds in this context. It is pointed out, though, that we in principle could consider more complicated geometries. We will restrict ourselves to the ADE fourfolds as they allow us to extract the corresponding physics in a straightforward manner.

For simplicity, we will here focus on the case of A_r. In two-dimensional $N = 4$ field theory, the matrix Q_i^a can then be identified with a simple extension of minus the Cartan matrix of the A_r Lie algebras. The charge matrix thus reads

$$Q_i^a = \delta_i^a - 2\delta_i^{a+1} + \delta_i^{a+2}, \qquad a = 1, \ldots, r, \qquad i = 1, \ldots, r+2. \tag{14}$$

Inserting this matrix into the D-flatness conditions (12)–(13) with $\xi_{\mathbb{C}}^a = 0$ yields the conditions

$$\left(\left|\phi_1^a\right|^2 - 2\left|\phi_1^{a+1}\right|^2 + \left|\phi_1^{a+2}\right|^2\right) - \left(\left|\phi_2^a\right|^2 - 2\left|\phi_2^{a+1}\right|^2 + \left|\phi_2^{a+2}\right|^2\right) = \xi_{\mathbb{R}}^a, \tag{15}$$

$$\phi_1^a\bar{\phi}_2^a - 2\phi_1^{a+1}\bar{\phi}_2^{a+1} + \phi_1^{a+2}\bar{\phi}_2^{a+2} = 0. \tag{16}$$

With $\xi_{\mathbb{R}}^a = \mu^a > 0$ for a odd while $\xi_{\mathbb{R}}^a = -\mu^a < 0$ for a even (that is, $\mu^a > 0$ for all a) and with the field redefinitions

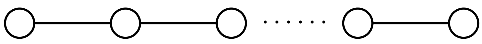
$$(\varphi^a, \psi^a) = \begin{cases} \left(\phi_1^a, \phi_2^a\right) & \text{if } a \text{ odd} \\ \left(\bar{\phi}_2^a, -\bar{\phi}_1^a\right) & \text{if } a \text{ even,} \end{cases} \tag{17}$$

the D-flatness conditions reduce to

$$(|\varphi^a|^2 + 2|\varphi^{a+1}|^2 + |\varphi^{a+2}|^2) - (|\psi^a|^2 + 2|\psi^{a+1}|^2 + |\psi^{a+2}|^2) = \mu^a, \tag{18}$$

$$\varphi^a\bar{\psi}^a + 2\varphi^{a+1}\bar{\psi}^{a+1} + \varphi^{a+2}\bar{\psi}^{a+2} = 0. \tag{19}$$

A simple examination of these equations reveals that the base, being described by φ^a, consists of r intersecting $\mathbf{WP}^2_{1,2,1}$ according to the A_r Dynkin diagram

$\mathbf{A_r}$:  (20)

As an illustration, let us consider the A_1 geometry. Its description involves three hypermultiplets and one charge vector $Q = (1, -2, 1)$. The base V^2 is extracted by setting $\psi^1 = \psi^2 = \psi^3 = 0$ in (18) which then reduces to $|\varphi^1|^2 + 2|\varphi^2|^2 + |\varphi^3|^2 = \mu$ describing $\mathbf{WP}^2_{1,2,1}$ with area proportional to μ. Thus, the total geometry is a C^2 fibration over $\mathbf{WP}^2_{1,2,1}$. As already indicated, for generic values of r, V^2 can be identified with a system of intersecting $\mathbf{WP}^2_{1,2,1}$. It follows that $\mathbf{WP}^2_{1,2,1}$ plays the role of $\mathbf{CP}^1$ in the standard description of an A-series ALE space whose base can be identified with a system of intersecting $\mathbf{CP}^1$ according to the A_r Dynkin diagram (20).

4. ADE quiver gauge models in F-theory compactification

Having constructed a class of ADE fourfolds, we now discuss the physics resulting from compactifying F-theory on these manifolds. Our analysis here is based on a dual type IIB superstring description where the compactification on Calabi–Yau threefolds is considered in the presence of D7-branes wrapping 4-cycles and filling the four-dimensional Minkowski spacetime. As in the string theory, we initially compactify on the K3 fibre. This results in an eight-dimensional supersymmetric gauge theory which can be identified with a gauge model living in the worldvolume of the D7-branes. A subsequent compactification of F-theory to four dimensions is then equivalent to wrapping the D7-branes on V^2 in the type IIB superstring compactification. In this scenario, V^2 is taken to be embedded in the three-dimensional Calabi–Yau compactification space.

The present task is therefore to look for the type IIB geometries dual to the F-theory on ADE fourfolds. Since V^2 is a compact toric manifold, it is natural to expect that the type IIB geometry may be a toric variety as well. At first sight, such a manifold appears quite complicated. As we will see, however, if we restrict ourselves to the physics coming from the D7-branes, the dual type IIB geometry may be described as a toric Calabi–Yau manifold with V^2 as base space. We will assume that it corresponds to a line bundle over V^2. It is also noted that the toric property allows us to describe the corresponding D7-brane physics in terms of the toric data of the Calabi–Yau threefold.

We are interested in fourfolds with ADE intersecting 4-cycles in the base, and we will discuss how F-theory on these ADE fourfolds can be interpreted in terms of type IIB superstrings on Calabi–Yau threefolds in the presence of D5-branes wrapping ADE intersecting 2-cycles of ALE spaces. This connection is based on a geometric procedure called folding. For type IIB toric geometry, this procedure has been used to geometrically engineer non-simply laced quiver gauge theories [3]. In brief, one identifies the toric vertices of the Calabi–Yau threefold which are permuted under the folding action Γ (which is an outer automorphism of the associated toric graph not to be confused with the group Γ appearing in (8)). This imposes certain constraints on the toric data depending on the precise action of Γ. It turns out that such actions become very simple using local mirror transformations and that there are two possible representations [3]. The one we are interested in here has the property of resulting in geometries with one dimension less than the 'natural' one. This dimensional reduction follows straightforwardly from the toric data of the resulting geometry. From the string theory point of view, however, this missing complex direction will resurface. This will be addressed below.

4.1. A_1 quiver model

F-theory on an A_1 fourfold is expected to be equivalent to type IIB superstring theory on $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$ in the presence of D7-branes having a worldvolume $\mathbb{R}^4 \times \mathbf{WP}^2_{1,1,2}$. This would give $N = 1$ pure Yang–Mills theory in four dimensions, and the Yang–Mills coupling constant g_{YM} is related to the volume of $\mathbf{WP}^2_{1,1,2}$ as

$$\mathrm{Vol}\big(\mathbf{WP}^2_{1,1,2}\big) = 1\big/g^2_{YM}. \tag{21}$$

4.2. Models with bi-fundamental matter

Here, we discuss how to incorporate bi-fundamental matter in geometric engineering of F-theory. Such matter can be introduced by replacing the single $\mathbf{WP}^2_{1,1,2}$ considered above by an intersecting geometry according to ADE toric Dynkin graphs. As we will see, these complicated gauge theories can be reduced to well-known models based on ordinary ADE singularities of ALE spaces involving $\mathbf{CP}^1$ complex curves. In this analogy, the D7-branes wrapping the intersecting $\mathbf{WP}^2_{1,1,2}$ constituents are replaced by D5-branes wrapping the intersecting $\mathbf{CP}^1$ constituents of the deformed ALE spaces. This connection is based on the geometric procedure of folding which we alluded to above. To illustrate the mechanism, we initially consider the model A_1.

We consider $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$ which is a toric manifold. It can be described by an $N = 2$ gauged linear sigma model with four chiral superfields and $U(1)$ gauge symmetry with respect to which the superfields have charges

$$q = (1, 1, 2, -4) \tag{22}$$

satisfying the constraint

$$\sum_{i=1}^{4} q_i = 0. \tag{23}$$

This condition implies that the space is a local Calabi–Yau manifold. In toric geometry, it can be represented by

$$\sum_{i=1}^{4} q_i v_i = 0, \tag{24}$$

and a particular toric vertex realization of this manifold is given by

$$v_1 = (-2, 1, 1), \qquad v_2 = (2, 1, 1), \qquad v_3 = (0, -1, 1), \qquad v_4 = (0, 0, 1). \tag{25}$$

4.2.1. Mirrors of $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$. As usual, the mirror of $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$ is obtained by solving the following constraint equations [13–15]:

$$\sum_{i=1}^{4} a_i y_i = 0, \qquad \prod_{i=1}^{4} y_i^{q_i} = 1, \tag{26}$$

where $\{a_i\}$ are the complex parameters defining the complex structure of the mirror geometry. However, only one of these parameters is physical and it describes the mirror of the size of $\mathbf{WP}^2_{1,1,2}$. For simplicity, one can also fix this parameter to 1.

It follows from the Calabi–Yau condition (23) that the mirror manifold (26) is invariant under the projective transformation $y_i \to \lambda y_i$. The projectively invariant solution to the constraint equations (23) thus defines a $4 - 1 - 1 - 1 = 1$ dimensional toric manifold given by a holomorphic hypersurface in $\mathbb{C}^2$,

$$F(x, y) = 0. \tag{27}$$

To recover the dimension of the original manifold, that is, a complex three-dimensional local Calabi–Yau manifold, we may use the ad hoc trick of introducing by hand the two extra holomorphic variables u and v combined in the quadratic form uv, thereby modifying the previous equation to

$$F(x, y) = uv. \tag{28}$$

In what follows, though, we will ignore this quadratic term in the mirror geometry.

We return now to the examination of (26). It turns out that there are many ways of solving the mirror system (26). A nice way is to introduce two complex variables $U_i = (U_1, U_2)$, specified later on, and a system of two-dimensional vectors of integer entries:

$\{n_i, i = 1, \ldots, 4\}$. A simple examination reveals that (26) can be solved using the following parametrization:

$$y_i = \prod_{j=1}^{2} U_j^{n_i^j}, \qquad i = 1, 2, 3, 4 \tag{29}$$

subject to

$$\sum_{i=1}^{4} q_i n_i^1 = 0, \qquad \sum_{i=1}^{4} q_i n_i^2 = 0. \tag{30}$$

In this way, the corresponding mirror geometry reads

$$\sum_{i=1}^{4} \prod_{j=1}^{2} U_j^{n_i^j} = 0. \tag{31}$$

Setting $v_i = (n_i, 1)$, we get the following mirror geometry:

$$1 + U_1^{-2} U_2 + U_1^2 U_2 + U_2^{-1} = 0. \tag{32}$$

This can be described by a homogeneous polynomial in a weighted projective space. Indeed, we consider $\mathbf{WP}^2_{\lambda_1,\lambda_2,\lambda_3}(x_1, x_2, x_3)$ and introduce the following general gauge invariants:

$$U_1 = \frac{x_1^{\lambda_3/g_1}}{x_3^{\lambda_1/g_1}}, \qquad U_2 = \frac{x_2^{\lambda_3/g_2}}{x_3^{\lambda_2/g_2}} \tag{33}$$

where

$$g_1 = gcd(\lambda_1, \lambda_3), \qquad g_2 = gcd(\lambda_2, \lambda_3). \tag{34}$$

The division by these common divisors is in order to keep the ratios in (33) minimal. The geometry (32) may then be written as

$$0 = x_1^{2\lambda_3/g_1} x_2^{\lambda_3/g_2} x_3^{2\lambda_1/g_1+\lambda_2/g_2} + x_2^{2\lambda_3/g_2} x_3^{4\lambda_1/g_1} + x_1^{4\lambda_3/g_1} x_2^{2\lambda_3/g_2} + x_1^{2\lambda_3/g_1} x_3^{2\lambda_1/g_1+2\lambda_2/g_2} \tag{35}$$

where we have multiplied by $x_1^{2\lambda_3/g_1} x_2^{\lambda_3/g_2} x_3^{2\lambda_1/g_1+\lambda_2/g_2}$. Multiplying by an additional factor of $x_2^{2\lambda_3/g_2}$ leads to

$$\begin{aligned} 0 = {} & x_1^{2\lambda_3/g_1} x_2^{3\lambda_3/g_2} x_3^{2\lambda_1/g_1+\lambda_2/g_2} + x_2^{4\lambda_3/g_2} x_3^{4\lambda_1/g_1} + x_1^{4\lambda_3/g_1} x_2^{4\lambda_3/g_2} \\ & + x_1^{2\lambda_3/g_1} x_2^{2\lambda_3/g_2} x_3^{2\lambda_1/g_1+2\lambda_2/g_2}. \end{aligned} \tag{36}$$

Elliptic solution

Now we introduce the parameters (or coordinates)

$$z_1 = x_2^{\lambda_3/g_2} x_3^{\lambda_1/g_1}, \qquad z_2 = x_1^{\lambda_3/g_1} x_2^{\lambda_3/g_2}, \qquad z_3 = x_1^{\lambda_3/g_1} x_2^{\lambda_3/g_2} x_3^{\lambda_1/g_1+\lambda_2/g_2} \tag{37}$$

with weights

$$\begin{aligned} (\mu_1, \mu_2, \mu_3) &= ((\lambda_1/g_1 + \lambda_2/g_2)\lambda_3, (\lambda_1/g_1 + \lambda_2/g_2)\lambda_3, 2(\lambda_1/g_1 + \lambda_2/g_2)\lambda_3) \\ &= (\lambda_1/g_1 + \lambda_2/g_2)\lambda_3 \times (1, 1, 2). \end{aligned} \tag{38}$$

In terms of these parameters, (36) reads

$$0 = z_1 z_2 z_3 + z_1^4 + z_2^4 + z_3^2. \tag{39}$$

This corresponds to a homogeneous polynomial of degree 4 in

$$\mathbf{WP}^2_{(\lambda_1/g_2+\lambda_2/g_2)\lambda_3,(\lambda_1/g_2+\lambda_2/g_2)\lambda_3,2(\lambda_1/g_2+\lambda_2/g_2)\lambda_3} = \mathbf{WP}^2_{1,1,2}. \tag{40}$$

The mirror geometry given by (39) is seen to describe an elliptic curve (since the degree satisfies $d = \mu_1 + \mu_2 + \mu_3$ in $\mathbf{WP}^2_{\mu_1,\mu_2,\mu_3}$).

Non-elliptic solution

A simple example is given by the following parametrization:

$$U_1 = \frac{x_1}{x_2}, \qquad U_2 = \frac{x_1 x_2}{x_3^2}, \tag{41}$$

differing in form from (33). This results in a polynomial constraint of the form

$$x_1 x_2 x_3^2 + x_2^4 + x_1^4 + x_3^4 = 0 \tag{42}$$

which corresponds to a homogeneous polynomial in $\mathbf{WP}^2_{1,1,1} = \mathbf{CP}^2$. It is ensured by construction that this satisfies the mirror constraint which here reads

$$\left(x_1 x_2 x_3^2\right)^4 = x_2^4 \times x_1^4 \times \left(x_3^4\right)^2. \tag{43}$$

The algebraic curve (42) is not elliptic.

Another non-elliptic curve follows from (33) when based on $\mathbf{WP}^2_{1,2,3}$, for example, as the mirror is described by

$$x_1^3 x_2^3 x_3 + x_1^{12} + x_2^6 + x_3^4 = 0 \tag{44}$$

with associated mirror constraint given by

$$\left(x_1^3 x_2^3 x_3\right)^4 = x_1^{12} \times \left(x_2^6\right)^2 \times x_3^4. \tag{45}$$

4.2.2. Folding procedure. Having described the toric geometry of type IIB superstring theory in the presence of D7-branes and its mirror version, the next step is to show how to convert the corresponding model into the well-known gauge theories living on the worldvolume of wrapped D5-brane. This can be done with the help of an outer automorphism group action Γ of the toric graphs of the Calabi–Yau threefolds in the dual type IIB geometry. In this approach, ADE toric graphs of ALE spaces may be obtained, from type IIB geometries dual to F-theory on ADE fourfolds, by identifying vertices which are permuted by Γ. Once this action has been specified, one should solve the corresponding toric constraint equations. It turns out that these can be derived easily from the equation of the algebraic curve appearing in the mirror of toric Calabi–Yau threefolds.

To understand how this *a priori* surprising connection can exist, we first consider the A_1 fourfolds in F-theory compactification corresponding to $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$. Indeed, let us consider the $\mathbb{Z}_2$ subgroup of $PSL(2,\mathbb{Z})$ acting as

$$v_1 \longleftrightarrow v_2, \tag{46}$$

that is, the vertices v_1 and v_2 are in the same orbit of this action as they transform as a doublet. Note that (24) is invariant under this transformation. The folding procedure now amounts to identifying these two vertices. In the mirror version, this folding action is equivalent to identifying the corresponding monomials as

$$U_1^{n_1^1} U_2^{n_1^2} = U_1^{n_2^1} U_2^{n_2^2}. \tag{47}$$

Using (25) and (33), this merely reduces to the following simple constraint

$$x_1^{\lambda_3} = x_3^{\lambda_1} \tag{48}$$

in the weighted space. Implementing (48) into (36), we get

$$0 = x_2^{2\lambda_3/g_2}\left(x_2^{\lambda_3/g_2} x_3^{\lambda_2/g_2} + 2x_2^{2\lambda_3/g_2} + x_3^{2\lambda_2/g_2}\right). \tag{49}$$

In terms of the new coordinates $z_1 = x_2^{\lambda_3/g_2}$ and $z_2 = x_3^{\lambda_2/g_2}$, this corresponds to

$$0 = z_1 z_2 + z_1^2 + z_2^2. \tag{50}$$

This describes the deformation of an A_1 singularity of K3 and is easily seen to result as the mirror of $\mathcal{O}_{\mathbf{CP}^1}(-2)$ following from

$$y_1 + y_2 + y_3 = 0, \tag{51}$$

$$\prod_{i=1}^{3} y_i^{q_i} = 1, \tag{52}$$

where

$$q = (1, 1, -2), \tag{53}$$

cf (26).

We note that the Mori vector (53) can be obtained from the Mori vector given in (22). The above vertex identification leads to the following reduction in the toric data of the Calabi–Yau threefolds:

$$(1, 1, 2, -4) \to 2(1, 1, -2) \equiv (1, 1, -2). \tag{54}$$

The appearance of this vector is not surprising. It comes originally from the A_1 geometry on which the F-theory is compactified. The key observation here is that our procedure may be regarded as the following topological change:

$$\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4) \to \mathcal{O}_{\mathbf{CP}^1}(-2) \times \mathbb{C}. \tag{55}$$

This topological change admits an interpretation in terms of D-branes. On the left-hand side geometry, F-theory on an A_1 fourfold can be interpreted as type IIB superstring theory with D7-branes having worldvolume $\mathbb{R}^4 \times \mathbf{WP}^2_{1,1,2}$ where the factor $\mathbf{WP}^2_{1,1,2}$ is the compact space of $\mathcal{O}_{\mathbf{WP}^2_{1,1,2}}(-4)$. After the folding procedure, this can be re-interpreted in terms of D5-branes with worldvolume $\mathbb{R}^4 \times \mathbf{CP}^1$, where the factor $\mathbf{CP}^1$ is the compact part of $\mathcal{O}_{\mathbf{CP}^1}(-2)$. Moreover, we believe that the type IIB geometry becomes a *trivial* fibration of A_1 ALE spaces over the complex plane. In this way, the D7-branes reduce to D5-branes wrapping the complex curve $\mathbf{CP}^1$ in the blown-up A_1 singularity given by (1).

It has been seen that this quiver gauge theory has $N = 2$ supersymmetry, whereas the original gauge model living in the D7-brane worldvolume has $N = 1$ supersymmetry. This requires that the above $N = 2$ supersymmetry should be broken down to $N = 1$. This reduction may be administered by the addition of a tree-level superpotential of the form

$$W_{\text{tree}} = \frac{1}{\mathrm{i}} \sum_j g_j tr_j \phi^j. \tag{56}$$

A similar mechanism has been studied in the context of large-N duality in [16], see also [17]. The introduction of this superpotential modifies the geometry and leads to a *non-trivial* fibration of A_1 ALE spaces over the complex plane. This modified fibration may be described by

$$xy + z^2 + W'(t)^2 = 0 \tag{57}$$

where t is the coordinate in the base, that is, in the complex plane. It is noted that this new geometry may undergo a geometric transition where the vanishing $\mathbf{S}^2$ is replaced by an $\mathbf{S}^3$ while the D5-branes are replaced by 3-form fluxes.

4.2.3. On mirror geometry. Here, we would like to extend the above results to higher order Dynkin geometries. To start, it is recalled that a general complex p-dimensional toric variety can be described by

$$V^p = \frac{\mathbb{C}^{p+n} \setminus S}{\mathbb{C}^{\star n}} \tag{58}$$

where the $n\mathbb{C}^\star$ actions are given by

$$\mathbb{C}^{\star n} : z_i \mapsto \lambda^{q_i^a} z_i, \qquad i = 1, \dots, p+n; \qquad a = 1, \dots, n. \tag{59}$$

Requiring this to correspond to a Calabi–Yau manifold imposes the conditions

$$\sum_{i=1}^{p+n} q_i^a = 0, \qquad a = 1, \dots, n. \tag{60}$$

A toric vertex realization reads

$$\sum_{i=1}^{p+n} q_i^a v_i = 0, \qquad a = 1, \dots, n \tag{61}$$

where the vertices v_i are of dimension p. In the case of a toric Calabi–Yau manifold, we may choose the vertices as

$$v_i = (m_i, 1) \tag{62}$$

as they implement the Calabi–Yau conditions in the sense that

$$0 = \sum_{i=1}^{p+n} q_i^a v_i = \left(\sum_{i=1}^{p+n} q_i^a m_i, \sum_{i=1}^{p+n} q_i^a \right) = \left(\sum_{i=1}^{p+n} q_i^a m_i, 0 \right). \tag{63}$$

The jth coordinate of m_i is written as m_{ij} where $j = 1, \dots, p-1$. The mirror manifold is given as a solution to

$$\begin{aligned} &\sum_{i=1}^{p+n} b_i y_i = 0 \\ &\prod_{i=1}^{p+n} y_i^{q_i^a} = 1, \qquad a = 1, \dots, n. \end{aligned} \tag{64}$$

To solve these constraints, one may introduce $p-1$ gauge invariants U_j and write

$$y_i = \prod_{j=1}^{p-1} U_j^{m_{ij}}. \tag{65}$$

This automatically solves the constraint equation in (64) since

$$\prod_{i=1}^{p+n} \left(\prod_{j=1}^{p-1} U_j^{m_{ij}} \right)^{q_i^a} = \prod_{j=1}^{p-1} U_j^{\sum_{i=1}^{p+n} m_{ij} q_i^a} = 1 \tag{66}$$

due to (63). The mirror manifold is then given by

$$0 = \sum_{i=1}^{p+n} b_i y_i = \sum_{i=1}^{p+n} b_i \prod_{j=1}^{p-1} U_j^{m_{ij}}. \tag{67}$$

Assuming that this may be described by a homogeneous polynomial in a weighted projective space, we introduce $\mathbf{WP}^{p-1}_{\lambda_1,\ldots,\lambda_p}(x_1,\ldots,x_p)$ and write

$$U_1 = \frac{x_1^{\lambda_p/g_1}}{x_p^{\lambda_1/g_1}}, \qquad U_2 = \frac{x_2^{\lambda_p/g_2}}{x_p^{\lambda_2/g_2}}, \qquad \cdots, \qquad U_{p-1} = \frac{x_{p-1}^{\lambda_p/g_{p-1}}}{x_p^{\lambda_{p-1}/g_{p-1}}} \tag{68}$$

where

$$g_j = gcd(\lambda_j, \lambda_p), \qquad j = 1, \ldots, p-1. \tag{69}$$

The mirror manifold may then be described by

$$0 = \sum_{i=1}^{p+n} b_i \left(\frac{x_1^{\lambda_p/g_1}}{x_p^{\lambda_1/g_1}}\right)^{m_{i1}} \times \cdots \times \left(\frac{x_{p-1}^{\lambda_p/g_{p-1}}}{x_p^{\lambda_{p-1}/g_{p-1}}}\right)^{m_{ip-1}}. \tag{70}$$

In what follows, we will initially be interested in the case $p = 3$.

4.2.4. A_2 quiver model. Here, we specialize to the case of A_2 quiver models in F-theory compactification. The dual type IIB geometry is given by a $U(1)^2$ linear sigma model with five chiral fields with charges

$$q = \begin{pmatrix} 1 & 2 & 1 & 0 & -4 \\ 0 & 1 & 2 & 1 & -4 \end{pmatrix} \tag{71}$$

as Mori matrix. A simple vertex realization is given by $v_i = (m_i, 1)$ where

$$m_1 = (2, 1), \qquad m_2 = (-1, 0), \qquad m_3 = (0, -1), \qquad m_4 = (1, 2), \qquad m_5 = (0, 0). \tag{72}$$

We will see that this geometry reduces to an ALE space with A_2 singularity. As before, we will base our analysis on mirror symmetry. Indeed, the associated mirror manifold (70) is thus defined by

$$\begin{aligned} 0 = b_1 \left(\frac{x_1^{\lambda_3/g_1}}{x_3^{\lambda_1/g_1}}\right)^2 \left(\frac{x_2^{\lambda_3/g_2}}{x_3^{\lambda_2/g_2}}\right) + b_2 \left(\frac{x_1^{\lambda_3/g_1}}{x_3^{\lambda_1/g_1}}\right)^{-1} + b_3 \left(\frac{x_2^{\lambda_3/g_2}}{x_3^{\lambda_2/g_2}}\right)^{-1} \\ + b_4 \left(\frac{x_1^{\lambda_3/g_1}}{x_3^{\lambda_1/g_1}}\right) \left(\frac{x_2^{\lambda_3/g_2}}{x_3^{\lambda_2/g_2}}\right)^2 + b_5. \end{aligned} \tag{73}$$

For simplicity, we set $b_1 = b_2 = b_3 = b_4 = b_5 = 1$ in the following. Multiplying (73) by $x_1^{\lambda_3/g_1} x_2^{\lambda_3/g_2} x_3^{4\lambda_1/g_1+4\lambda_2/g_2}$ leads to

$$\begin{aligned} 0 = x_1^{3\lambda_3/g_1} x_2^{2\lambda_3/g_2} x_3^{2\lambda_1/g_1+3\lambda_2/g_2} + x_2^{\lambda_3/g_2} x_3^{5\lambda_1/g_1+4\lambda_2/g_2} + x_1^{\lambda_3/g_1} x_3^{4\lambda_1/g_1+5\lambda_2/g_2} \\ + x_1^{2\lambda_3/g_1} x_2^{3\lambda_3/g_2} x_3^{3\lambda_1/g_1+2\lambda_2/g_2} + x_1^{\lambda_3/g_1} x_2^{\lambda_3/g_2} x_3^{4\lambda_1/g_1+4\lambda_2/g_2}. \end{aligned} \tag{74}$$

Let us introduce the coordinates

$$z_1 = x_1^{\lambda_3/g_1} x_3^{\lambda_2/g_2}, \qquad z_2 = x_2^{\lambda_3/g_2} x_3^{\lambda_1/g_1}, \qquad z_3 = x_3^{\lambda_1/g_1+\lambda_2/g_2} \tag{75}$$

with weights

$$(\mu_1, \mu_2, \mu_3) = (\lambda_1/g_1 + \lambda_2/g_2)\lambda_3 \times (1, 1, 1). \tag{76}$$

In terms of these, (74) reads

$$0 = z_1^3 z_2^2 + z_2 z_3^4 + z_1 z_3^4 + z_1^2 z_2^3 + z_1 z_2 z_3^3. \tag{77}$$

This corresponds to a homogeneous polynomial of degree 5 in $\mathbf{WP}^2_{1,1,1}$. The folding action of interest here identifies the vertices v_2 and v_5,

$$v_2 \longleftrightarrow v_5. \tag{78}$$

In the mirror geometry, this action simply corresponds to

$$z_2 = z_3. \tag{79}$$

Having determined the toric constraint, we can now derive the equation defining the mirror geometry. A simple computation yields

$$0 = z_2^5 + z_1 z_2^4 + z_1^2 z_2^3 + z_1^3 z_2^2 = \sum_{i=1}^{4} z_1^{i-1} z_2^{6-i}. \tag{80}$$

In terms of the new coordinates

$$y_i = z_1^{i-1} z_2^{6-i}, \tag{81}$$

we have

$$\begin{aligned} &\sum_{i=1}^{4} y_i = 0, \\ &\prod_{i=1}^{4} y_i^{q_i^a} = 1, \qquad a = 1, 2 \end{aligned} \tag{82}$$

where

$$q^1 = (1, -2, 1, 0), \qquad q^2 = (0, 1, -2, 1). \tag{83}$$

This coincides with the mirror constraint equations of an A_2 singularity of K3. This may alternatively be expressed through

$$y_1 y_3 = y_2^2, \qquad y_2 y_4 = y_3^2. \tag{84}$$

4.2.5. More on the folding procedure. The folding procedure identifying v_k and v_l imposes the identification of y_k with y_l in the mirror geometry. In terms of the gauge invariants (65), this implies that

$$\prod_{j=1}^{p-1} U_j^{m_{kj}} = \prod_{j=1}^{p-1} U_j^{m_{lj}}. \tag{85}$$

Assuming the association of the weighted projective space $\mathbf{WP}^{p-1}_{\lambda_1,\ldots,\lambda_p}(x_1, \ldots, x_p)$, cf (68), the identification (85) may be expressed as

$$\prod_{j=1}^{p-1} \left(\frac{x_j^{\lambda_p/g_j}}{x_p^{\lambda_j/g_j}} \right)^{m_{kj}-m_{lj}} = 1. \tag{86}$$

To simplify our considerations, let us assume that the toric vertices of the original model were chosen such that

$$m_1 = (1, 0, \ldots, 0), \qquad m_{n+p} = (0, \ldots, 0). \tag{87}$$

In this case, (86) reduces to

$$1 = U_1 = \frac{x_1^{\lambda_p/g_1}}{x_p^{\lambda_1/g_1}}. \tag{88}$$

The new geometry obtained by the folding action identifying v_1 and v_{n+p} now follows from (70). It reads

$$0 = \sum_{i=1}^{p+n} b_i \left(\frac{x_2^{\lambda_p/g_2}}{x_p^{\lambda_2/g_2}} \right)^{m_{i2}} \times \cdots \times \left(\frac{x_{p-1}^{\lambda_p/g_{p-1}}}{x_p^{\lambda_{p-1}/g_{p-1}}} \right)^{m_{ip-1}} \tag{89}$$

and describes (up to a simple rewriting) a polynomial equation in the weighted projective space $\mathbf{WP}^{p-2}_{\lambda_2,\ldots,\lambda_p}(x_2, \ldots, x_p)$.

We see that x_1 and m_{i1} no longer enter the game. This suggests that the original geometry, whose mirror we just found by folding, is described effectively by $n + p - 1$ vertices. Since n is unaltered by the folding procedure, it follows that the original toric variety is of dimension $p-1$, cf (58). Thus, interpreting the new geometry (89) as the mirror of a $(p-1)$-dimensional toric variety, we should require that the set

$$\{\hat{v}_i = (m_{i,2}, \ldots, m_{i,p-1}, 1);\ i = 1, \ldots, n + p - 1\} \tag{90}$$

defines a toric vertex realization of a Calabi–Yau manifold whose charge matrix $\hat{q}$ satisfies

$$\begin{aligned} &\sum_{i=1}^{p+n-1} \hat{q}_i^a = 0, \qquad a = 1, \ldots, n, \\ &\sum_{i=1}^{p+n-1} \hat{q}_i^a \hat{v}_i = 0, \qquad a = 1, \ldots, n. \end{aligned} \tag{91}$$

It would be interesting to pursue this general link further.

5. Conclusion

We have studied geometric engineering of $N = 1$ ADE quiver models. In particular, we have considered a class of such models obtained by the compactification of F-theory on manifolds defined as elliptic K3 surfaces fibred over certain ADE 4-cycles. The latter are constructed by resolving the ADE hyper-Kähler singularities. Our main focus has been on A_r quiver models resulting when the base space of the compactification fourfold of F-theory is built from intersecting 4-cycles according to A_r Dynkin graphs. The dual type IIB superstring theory involves D7-branes on such cycles embedded in toric Calabi–Yau threefolds. We have analysed in some details the cases of A_1 and A_2 and found that they are linked to the A_1 and A_2 geometries of ALE spaces. Our approach involves a particular geometric procedure referred to as folding. Using this, we have discussed how the physics of F-theory on ADE fourfolds in the presence of D7-branes wrapping 4-cycles can be related to a scenario with D5-branes wrapping 2-cycles of ALE spaces.

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