On Commutative Clean Rings and pm Rings
preprint version

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Dedicated to Carl Faith and Barbara Osofsky on their birthdays.

Abstract. Rings are assumed to be commutative. Recent work gives some of the tools needed to characterize clean, almost clean, weakly clean and uniquely clean rings by describing their Pierce sheaves. The sheaf descriptions are used to show that weakly clean and almost clean rings which are pm rings are clean. A subclass of clean rings, here called *J-clean rings*, also known as *F-semiperfect rings*, is studied. It includes the uniquely clean rings. There is a mono-functor from commutative rings to J-clean rings which satisfies a universal property. Earlier non-functorial ways of embedding rings in J-clean rings can be derived from the functor. Applications to rings of continuous functions are found throughout.

1. Definitions and preliminaries.

1.1. Introduction. Throughout "ring" will mean a commutative ring with 1 except in the first part of Section 4 where general unitary rings will make an appearance. Various authors have studied *clean rings* and related conditions. The following definition is a composite.

Definition 1.1. *(i) A ring R is called* clean *if each element can be expressed as the sum of a unit and an idempotent. (ii) A ring R is* almost clean *if each element can be expressed as the sum of a non-zero divisor and an idempotent. (iii) A ring R is* weakly clean *if each element can be expressed as the sum or difference of a unit and an idempotent. (iv) a ring R is* uniquely clean *if each $r \in R$ can be written $r = u + e$, u a unit and e an idempotent, in a unique way.*

An informative history of clean rings is found in [**M2**]. (Commutative clean rings coincide with the commutative exchange rings.) The important role of idempotents in the definition and the fact that indecomposable rings of each type have been characterized ([**AA**], [**NZ**]), suggest an approach using the Pierce sheaf. This

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method (details are found below) expresses any ring R as the ring of sections of a sheaf of indecomposable rings over a boolean space (Stone space). One of the aims of this article is to describe clean, weakly clean, almost clean and uniquely clean rings in Pierce sheaf terms. Results previously shown for products or even finite products now have global versions.

The nature of the Pierce sheaf representation of a ring makes it clear that (1) R is clean if and only if each of its stalks is clean ([**BS**, Proposition 1.2]), and (2) R is uniquely clean if and only if each of its stalks is uniquely clean, i.e., local and, modulo its maximal ideal, isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (Corollary 4.3). "Weakly clean" and "almost clean" do not work quite as easily but do have convenient sheaf characterizations (Proposition 2.1 and Theorem 2.4), which clarify the nature of these sorts of rings. Examples and counterexamples are given.

An essential idea in this context is the following:

Definition 1.2. *A ring R is called a* pm ring *if each prime ideal is contained in exactly one maximal ideal.*

A homomorphic image of a pm ring is a pm ring. We recall that any ring of the form $C(X)$, the ring of continuous real valued functions on a (completely regular) topological space X is pm ([**GJ**, 7.15]). Moreover, a clean ring is always a pm ring ([**AC**, Corollary 4]). We will see in Section 3 that the pm ring condition is very powerful when dealing with generalizations of clean rings. In fact a pm ring which is weakly clean or almost clean is clean (Proposition 3.2 and Theorem 3.4). Rings of the form $C(X)$ have the stronger property that the prime ideals containing a given one form a chain: We call rings with this property pm$^+$ rings and give several characterizations (Proposition 3.8) with an application to $C(X)$.

Section 4 concerns a class of clean rings which includes that of the uniquely clean rings. We call these rings "J-clean". One way of describing these rings, even for non-commutative rings, is to say R is J-clean if $R/\mathbf{J}(R)$ is abelian regular and idempotents lift uniquely modulo $\mathbf{J}(R)$. These rings are shown to be precisely the F-semiperfect rings whose idempotents are all central (Theorem 4.1). Once again, the Pierce sheaf point of view is very useful.

Reverting again to commutative rings we recall from [**Ca2**] and [**CL**] that there are many ways to embed a ring into a J-clean ring. We develop an important special case of this procedure. We call the categories of commutative rings and of J-clean rings $\mathcal{CR}$ and $\mathcal{JC}$, respectively. A mono-functor $\mathfrak{jc}\colon \mathcal{CR} \to \mathcal{JC}$ is constructed (Construction 4.6). For a ring R, $\mathfrak{jc}(R)$ is defined as a Pierce sheaf over the boolean space Y, where Y is $\operatorname{Spec} R$ with the constructible topology. The category $\mathcal{JC}$ is not a reflective subcategory of $\mathcal{CR}$ but $\mathfrak{jc}$ does have a universal property (Theorem 4.9): If $\phi\colon R \to S$ is a ring homomorphism where S is a J-clean ring then there is a unique $\theta\colon \mathfrak{jc}(R) \to S$ which restricts to ϕ on R.

In [**Ca2**], M. Contessa constructed a canonical embedding of a ring R into a J-clean ring $R^{\mathcal{D}}$. The construction is not functorial but, unlike $\mathfrak{jc}$, is the identity on $\mathcal{JC}$. In [**CL**], the authors describe the space of maximal ideals $\operatorname{Max} R^{\mathcal{D}}$ for some special noetherian domains R. The universal property of $\mathfrak{jc}$ permits a description of $\operatorname{Max} R^{\mathcal{D}}$ in general (Theorem 4.12). As a corollary, $\operatorname{Max} C(X)^{\mathcal{D}}$ is described; it can be identified with the set of prime z-ideals of $C(X)$ given the constructible topology.

1.2. Terminology, notation and preliminaries. The set of *non-zero divisors* (also called *regular elements*) of R will be denoted $\mathfrak{R}(R)$, the set of zero divisors is $\mathfrak{Z}(R)$, the set of units is $\mathfrak{U}(R)$ and the set of idempotents $\mathbf{B}(R)$. A ring R is called *indecomposable* if $\mathbf{B}(R) = \{0,1\}$. When $\mathbf{B}(R)$ is viewed as a boolean algebra, the boolean space $\operatorname{Spec}\mathbf{B}(R) = X(R)$ (or just X) is the base space of the *Pierce sheaf* of R. For each $x \in X$, let Rx be the ideal of R generated by the idempotents in x, then $R_x = R/Rx$ is a *Pierce stalk* of R. If $e \in \mathbf{B}(R)$ then $\operatorname{Supp}(e)$ denotes $\{x \in X(R) \mid e \notin x\}$. See [**P**, Part I] and [**J**, V 2] for detailed descriptions of the sheaf; only a few key properties of it will be quoted here.

The base space of the Pierce sheaf for a ring R is $X(R)$ and the *espace étalé* is $\mathcal{R} = \bigcup_{x\in X(R)} R_x$ which is topologized so that basic open sets in $\mathcal{R}$ are elements of R restricted to sets of $X(R)$ which are both open and closed (such sets are called *clopen*). The ring of global sections of the sheaf is isomorphic to R ([**J**, 2.5 Theorem]). A key fact is that if two sections coincide at some $x \in X(R)$ they coincide over a neighbourhood of x ([**P**, Lemma 4.3]). Moreover, if a statement is true on neighbourhoods in X, it is true on clopen neighbourhoods; this along with the compactness of X, reduces proofs to dealing with a finite number of neighbourhoods. A tool we will use below is the following which is a special case of [**P**, Proposition 3.4].

Lemma 1.3. *Let R be a ring, Z an indeterminate and $f_1, \dots, f_n \in R[Z]$. Suppose that for each $x \in X(R)$ there is $r^{(x)} \in R$ such that $(f_i(r^{(x)}))_x = 0_x$, $i = 1, \dots, n$. Then for some $r \in R$, $f_i(r) = 0$, $i = 1 \dots, n$.*

Throughout, for $r \in R$ and $x \in X(R)$, $r + Rx$ will be denoted r_x, and similarly for subsets of R. As an illustration of the methods of Lemma 1.3, suppose $r \in R$ is such that $r_x \in \mathfrak{U}(R_x)$ for each $x \in X(R)$ then $r \in \mathfrak{U}(R)$. On the other hand, the analogous statement for $\mathfrak{R}(R)$ is not true (and this will be important for us).

The expression "local ring" will mean a ring with exactly one maximal ideal; no chain condition is implied. If $\mathfrak{p}$ is a prime ideal of R then $R_{\mathfrak{p}}$ denotes the localization at $\mathfrak{p}$. For a ring R, $Q_{\mathrm{cl}}(R)$ refers to the classical ring of quotients (or total ring of fractions) of R; the symbol $q(R)$ is sometimes used for this ring of quotients. The Jacobson radical of R is denoted $\mathbf{J}(R)$. The term "regular ring" will always mean "von Neumann regular ring", i.e., a ring R such that for all $r \in R$ there is $r' \in R$ such that $rr'r = r$. In the context of a ring $C(X)$ of continuous functions on a topological space X, it will always be assumed (as in [**GJ**, Chapter 3]) that the space is *completely regular.*

Before going on to various sorts of rings, we note the following based on [**C**].

Proposition 1.4. *Let R be a clean ring. If $\mathfrak{m} \in \operatorname{Max} R$ and $x = \mathfrak{m} \cap \mathbf{B}(R)$, then the Pierce stalk $R_x = R_{\mathfrak{m}}$.*

Proof. We have that R is a pm ring and then [**C**, Theorem III.1(2)] says that $R \to R_{\mathfrak{m}}$ is a surjection; moreover, [**C**, Theorem I.1(4)] also says that the kernel $O_{\mathfrak{m}}$ of $R \to R_{\mathfrak{m}}$ is generated by idempotents. Hence, $O_{\mathfrak{m}} \subseteq xR$ and, clearly, $x \subseteq O_{\mathfrak{m}}$. $\square$

The following will be used at several points in the article. If S is an extension ring of a ring R, we say S is an *extension by idempotents of* R if R is generated, as a ring, by R and $\mathbf{B}(S)$.

LEMMA 1.5. *Let S be an extension of a ring R by idempotents. Then, the Pierce stalks of S are homomorphic images of the Pierce stalks of R.*

PROOF. Consider the extension of boolean algebras $\mathbf{B}(R) \subseteq \mathbf{B}(S)$. If $z \in \operatorname{Spec}\mathbf{B}(S)$, let $x = z \cap \mathbf{B}(R) \in \operatorname{Spec}\mathbf{B}(R)$. An element $s \in S$ can be expressed in the form $s = \sum_{i=1}^{n} r_i e_i$, where each $r_i \in R$ and $\{e_1, \ldots, e_n\}$ is a complete orthogonal set of idempotents from S. Then, $s_z = (\sum_{e_i \notin z} r_i)_z$. Hence, the composition $R \to S \to S/zS = S_z$ is surjective. Its kernel is $zR = \{r \in R \mid r = re \text{ for some } e \in \mathbf{B}(S)\}$. Since $xR \subseteq zR$, the composition $R \to R_x \to S_z$ is also surjective. $\square$

COROLLARY 1.6. *Let $\mathcal{P}$ be a property of rings such that R has property $\mathcal{P}$ if and only if each Pierce stalk of R has property $\mathcal{P}$ and, moreover, $\mathcal{P}$ is preserved under homomorphic images. If R has property $\mathcal{P}$ and S is an extension of R by idempotents then S has property $\mathcal{P}$.*

2. Pierce sheaf characterizations of clean and related rings.

It has been known for a long time that a ring R is clean if and only if each of its Pierce stalks is clean if and only if each of its Pierce stalks is local; see, for example [**BS**, Proposition 1.2] (which only requires that the idempotents of R be central). Since a homomorphic image of a local ring is local, Corollary 1.6 applies to show that an extension of a clean ring by idempotents is clean.

We will see that weakly clean rings and uniquely clean rings have convenient characterizations in terms of their Pierce stalks but that the characterization of almost clean rings will involve more than just the stalks. The descriptions of uniquely clean rings and related rings will appear in Section 4.

Clean rings have also appeared in [**Ca1**] under the name *topologically boolean rings*, i.e., those rings R where the mapping $\operatorname{Max} R \to X(R)$ given by $\mathfrak{m} \mapsto \mathfrak{m} \cap \mathbf{B}(R)$ is one-to-one and, hence, a homeomorphism ([**Ca1**, Definition 3.6] and see also [**Ca2**, §3]). It is easy to see that this is the same as saying that the Pierce stalks are all local. The name given to rings of global sections of local rings over boolean spaces in [**OS**] is *lokal Boolesch*. Lemma 3.2 in [**OS**] says that R is "lokal Boolesch" exactly when for each $\mathfrak{m} \in \operatorname{Max} R$, the canonical map $R \to R_{\mathfrak{m}}$ is onto with kernel xR, $x = \mathfrak{m} \cap \mathbf{B}(R)$; this, again, is the same as saying that each Pierce stalk is local.

2.1. Weakly clean rings. In [**AA**, Question 1.11] the authors ask whether a weakly clean ring T which is not clean splits into $T \cong R \times S$, where R is indecomposable weakly clean but not clean and S is clean. The answer is "no". Let $A = \mathbb{Z}_{(p)}$ and $B = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$, where p and q are distinct odd primes, and let T be the ring of sequences from A which are eventually constant and in B. The ring B is weakly clean but not clean by [**AC**, Example 17]. Since B is a homomorphic image of T, T is not clean but is easily seen to be weakly clean. However, no direct factor of T is indecomposable and weakly clean but not clean. In fact, the following is true.

PROPOSITION 2.1. *Let R be a ring. Then R is weakly clean if and only if all its Pierce stalks are weakly clean and all but at most one of them are clean.*

PROOF. We note that for $x \neq y$ in X, $Rx + Ry = R$. Hence, $R/(Rx \cap Ry) \cong R_x \times R_y$, and, also, that R is clean if and only if every element is the *difference* of a unit and an idempotent.

If R is weakly clean so are all the Pierce stalks by [**AA**, Lemma 1.2], since the stalks are homomorphic images of R. Moreover, if R_x are R_y are distinct stalks, $R_x \times R_y$ is a homomorphic image of R and so must be weakly clean and at most one of the factors is weakly clean but not clean by [**AA**, Theorem 1.7].

In the other direction, if all the stalks are clean (i.e., indecomposable and clean) then, as already mentioned, R is clean. If, on the other hand, one stalk is weakly clean but not clean, say R_z, then, for any $r \in R$, there are two possibilities. If r_z is the sum of a unit and an idempotent from R_z, then r_x is the sum of a unit and an idempotent in R_x, for all $x \in X$. It follows that r is the sum of a unit and an idempotent from R (an application of Lemma 1.3). If r_z can only be expressed as a unit minus an idempotent in R_z, then, each r_x has such an expression and then so does r. □

2.2. Almost clean rings. The property "almost clean" does not work quite as smoothly via Pierce sheaves. The notation used above is kept. Recall that a local ring is (almost) clean and any domain is almost clean. In the first example which is not almost clean all the stalks are almost clean and all except one are local. Examples 2.9 contain a ring where all the stalks except one are fields and the remaining one is $\mathbb{Z}$.

Example 2.2. *There is a ring which is not almost clean but whose Pierce stalks are almost clean; in fact, all of the stalks but one are clean.*

Proof. Fix a field K and for odd n let $R_n = K[X]_{(X)}/(X^{n+1})$ and for even n let $R_n = K[X]_{(X-1)}/((X-1)^{n+1})$. The example will be the subring R of $\prod_{n\geq 1} R_n$ defined as the set of sequences of the form $(r_1, r_2, \ldots)$ such that for some $m \geq 1$ and for some $f \in K[X]$, $r_i = \overline{f}$ (meaning the image of f in R_i) for all $i \geq m$.

Notice that if $r \in R$ is "eventually" $\overline{f}$ and $s \in R$ is "eventually" $\overline{g}$ then $r = s$ implies $f = g$. Indeed, for all $i \geq m$, for some m, $\overline{f-g} = \bar{0}$ showing that, for some $s_k \in K[X]$ not divisible by X, $(f-g)s_k$ is divisible by each X^k, for large enough odd k. Hence, $f = g$. As a result, the function $R \to K[X]$ defined by $r = (r_1, \ldots, r_{m-1}, \overline{f}, \overline{f}, \ldots) \mapsto f$ is a ring surjection with kernel $\bigoplus_{n\geq 1} R_n$. The stalks are the rings R_n along with one more, $R_\infty = K[X]$.

Now consider the element $r = (\bar{X}, \bar{X}, \ldots)$. If R were almost clean there would exist a non-zero divisor s and an idempotent e with $r = s + e$. There are two cases. In the first e is "eventually" 0. Then, for all $i \geq m$, for some $m \geq 1$, $\bar{X} = s_i$ is a non-zero divisor. This is impossible since for i odd, $\bar{X}$ is nilpotent. In the second case, e is "eventually" 1. Then for all $i \geq m$, some $m \geq 1$, $\bar{X} = s_i + 1$ and $\overline{X-1}$ is a non-zero divisor. This, too, is impossible since for even i, $\overline{X-1}$ is nilpotent. □

Constructions of rings R where $X(R) = \mathbb{N} \cup \{\infty\}$, as in Example 2.2, occur in many places in the literature.

The above example suggests a characterization of almost clean rings in Pierce sheaf terms.

As shown in [**AA**, Example 2.9], homomorphic images of almost clean rings need not be almost clean. However, they are when the kernel is generated by idempotents.

Proposition 2.3. *Let R be an almost clean ring and I a proper ideal of R generated by idempotents. Then $\bar{R} = R/I$ is almost clean. In particular, the Pierce stalks of an almost clean ring are almost clean.*

Proof. It is easy to see that $I = \{r \in R \mid r = re \text{ for some } e = e^2 \in I\}$ (e.g., [**NR**, Lemma 1.2]). If $s \in \mathfrak{R}(R)$ and $(s+I)(t+I) = I$ then $st = ste$, for some $e = e^2 \in I$. Since $st(1-e) = 0$, $t = te \in I$. Hence, $s + I \in \mathfrak{R}(R/I)$. The result now follows. $\square$

The following result generalizes [**AA**, Theorem 2.5] which dealt only with finite products of indecomposable rings. A condition, which will be called the *Non-zero Divisor Condition* (NZDC), will simplify statements. In the special case of [**AA**, Theorem 2.5], it automatically holds.

(NZDC): *For all $r \in R$ and $x \in X = \operatorname{Spec} \mathbf{B}(R)$, there is a neigbourhood N of x such that for all $y \in N$, $r_y \in \mathfrak{R}(R_y)$, or there is a neighbourhood N of x such that for all $y \in N$, $r_y - 1_y \in \mathfrak{R}(R_y)$.*

It is clear that the (NZDC) implies that the stalks of R are almost clean.

Theorem 2.4. *Let R be a ring and $X = \operatorname{Spec} \mathbf{B}(R)$. Then the following are equivalent.*

(1) R is almost clean.

(2) R satisfies the (NZDC).

(3) For each $r \in R$ and $x \in X$ there is $e \in \mathbf{B}(R) \backslash x$ such that $re + (1-e) \in \mathfrak{R}(R)$ or $(r-1)e + (1-e) \in \mathfrak{R}(R)$.

Proof. Assume (1). If $r \in R$, it can be written $r = s + e$, $s \in \mathfrak{R}(R)$ and $e \in \mathbf{B}(R)$. When $x \in \operatorname{Supp}(1-e)$, $r_x = s_x$, and when $x \in \operatorname{Supp}(e)$, $r_x - 1_x = s_x$. Since $\operatorname{Supp}(e)$ and $\operatorname{Supp}(1-e)$ are complementary clopen subsets, (NZDC) follows.

We now assume (2). This direction uses the usual technique of building elements in a Pierce sheaf. Fix $r \in R$. For each $x \in X$, because R_x is almost clean, there is a clopen neighbourhood N_x such that for all $y \in N_x$ we have r_y is a non-zero divisor in R_y or for all $y \in N_x$ we have $r_y - 1_y$ is a non-zero divisor in R_y. The compactness of X and the fact the covering $X = \bigcup_{x \in X} N_x$ consists of clopen sets, show that there is a partition of X into disjoint clopen sets $N_1, \ldots, N_k, M_1, \ldots, M_l$ such that for each $i = 1, \ldots, k$ and all $y \in N_i$, r_y is a non-zero divisor in R_y and for each $j = 1, \ldots, l$ and all $y \in M_j$, $r_y - 1_y$ is a non-zero divisor in R_y. Let e be the idempotent whose support is $M_1 \cup \cdots \cup M_l$ and $s \in R$ the element such that $s_y = r_y$ if $y \in N_1 \cup \cdots \cup N_k$ and $s_y = r_y - 1_y$ if $y \in M_1 \cup \cdots \cup M_l$; s is a non-zero divisor in R. We get an expression $r = s + e$, as required.

The equivalence of (2) and (3) is straightforward. $\square$

The (NZDC) is weaker than the statement: for $r \in R$ and $x \in X$, if $r_x \in \mathfrak{R}(R_x)$ then r is regular on a neighbourhood of x. Another way of putting the stronger condition is: for each $x \in X$, $\mathfrak{R}(R)_x = \mathfrak{R}(R_x)$. The next example shows that the stronger condition cannot replace the (NZDC) in Theorem 2.4.

Example 2.5. *There is a ring R which satisfies the (NZDC) but not the stronger condition that $\mathfrak{R}(R_x) = \mathfrak{R}(R)_x$, for all $x \in X(R)$.*

Proof. Consider an example R like that of Example 2.2 except that the local rings $K[X]_{(X)}/(X)^n$ and $K[X]_{(X-2)}/(X-2)^n$, with K a field of characteristic 0, are used. Once again the stalk R_∞ is $K[X]$. If $f \in K[X]$ is neither divisible by X nor by $X - 2$ then any element of R eventually f is a non-zero divisor on a neighbourhood of ∞. However, if $f(0) = 0$ then $f(2)$ is an even integer. Then

$f(X)-1$ is divisible neither by X nor by $X-2$. Moreover, if $f(2)=0$ but $f(0)\neq 0$ then $f(0)$ is an even integer and $f(X)-1$ is divisible neither by X nor by $X-2$. Hence, the (NZDC) has been verified; however, an element which is a non-zero divisor at ∞ need not be a non-zero divisor on a neighbourhood of ∞. □

Indecomposable almost clean rings are characterized in [**AA**, Theorem 2.3].

EXAMPLE 2.6. *If A is an indecomposable almost clean ring and X is a boolean space, then the ring R of sections of the simple sheaf (X, A) is almost clean.*

PROOF. Recall that the *simple sheaf* ([**P**, Definition 11.2]) has *espace étalé* $X\times A$, where A has the discrete topology. The stalks are copies of A and if, for $r\in R$, $x\in X$, $r_x\in\mathfrak{R}(A)$ then r is a non-zero divisor on a neighbourhood of x. Hence, the stronger version of (NZDC) holds. □

Example 2.6 yields special cases of the following theorem. It was shown in [**BR**, Example 1.10] that if $S=R[\mathbf{B}(S)]$ where R is almost clean (i.e., S is an extension of R by idempotents) then S is not necessarily almost clean. We now look at special kind of extension by idempotents.

Given a ring R and an extension of boolean algebras $\mathbf{B}(R)\subseteq B$, there is a method of constructing a generic extension ring R_B of R so that $\mathbf{B}(R_B)=B$ and if $R\subseteq S$ is any ring extension with $\mathbf{B}(S)\cong B$ then there is a unique ring homomorphism $R_B\to S$ whose image is the subring of S generated by R and $\mathbf{B}(S)$. This construction is found in [**Bu**, Propositions 2.2 and 2.3] and, using different techniques, in [**Ma**, Theorem 1].

THEOREM 2.7. *Let R be a ring and Y a boolean space equipped with a continuous surjection $\tau\colon Y\to X=X(R)$. Let the boolean algebra corresponding to Y be B. Then, if R is almost clean so is the generic extension R_B.*

PROOF. The method of [**Bu**, Proposition 2.2] is most convenient here although [**Ma**, Theorem 1 (3)] can also be used. The ring R_B is the ring of sections of the inverse image sheaf $(\mathcal{T}, Y)$ of the Pierce sheaf $(\mathcal{R}, X)$ of R via τ (see, for example, [**G**, Chapter 2, §1.12]). This means that the stalks R_B are copies of those of R; more exactly, if $\tau(y)=x$ then $(R_B)_y=R_x$. Moreover, the topology of $\mathcal{T}$ has as basic open sets those of the following form. We fix a $r\in R$, U an open subset of X and V an open subset of Y; then, for $y\in V\cap\tau^{-1}(U)$, the element of $\mathcal{T}$ lying over y is r_x, where $\tau(y)=x$. Notice that R embeds in R_B by using the section with value r_x at y, when $\tau(y)=x$.

The new sheaf is the Pierce sheaf of its ring of sections (the remark preceding [**Bu**, Proposition 2.2]). The stalks are also stalks of R and, by Theorem 2.4, these are almost clean. However, the same theorem says that we must verify for $s\in R_B$ and $y\in Y$ that s is a non-zero divisor on a neighbourhood of y or that $1-s$ is a non-zero divisor on a neighbourhood of y. We know that if $\tau(y)=x$ then there is $r\in R$ so that $s_y=r_x$ and that s and (the image of) r coincide on a Y neighbourhood M of y. There are two cases which are treated in the same manner. Suppose there is an $r\in R$ which is a non-zero divisor on an neighbourhood N of x. Then, s is a non-zero divisor on $M\cap\tau^{-1}(N)$, as required. □

We have seen that "stalks almost clean" is not sufficient to have the ring almost clean. This does work in a special case.

Proposition 2.8. *Suppose R is such that for each $x \in X$, R_x is almost clean, $Q_{cl}(R)$ is clean and $\mathbf{B}(Q_{cl}(R)) = \mathbf{B}(R)$. Then, R is almost clean.*

Proof. We need to verify the NZDC. Given $r \in R$, $r_x \in \mathfrak{U}(Q_{\mathrm{cl}}(R)_x)$ or $r_x - 1_x \in \mathfrak{U}(Q_{\mathrm{cl}}(R)_x)$. It will suffice to show, for $r \in R$ with $r_x \in \mathfrak{U}(Q_{\mathrm{cl}}(R)_x)$, that r_x is the image of a non-zero divisor from R. We can write $r_x = a_x/b_x$, for some $a, b \in \mathfrak{R}(R)$. Hence, for any lifting of r_x to $r \in R$, there is $e^2 = e \notin x$ with $rebe = ae$. Since $a, b \in \mathfrak{R}(R)$, for $y \in X$, $e \notin y$, $r_y \in \mathfrak{R}(R_y)$. □

Recall that a ring R is called a *p.p. ring* if principal ideals are projective. This is equivalent to saying that the annihilator of each element is generated by an idempotent. (The terms "*weak Baer*" and "*Rickart*" are also used.) Any p.p. ring is almost clean. This is deduced in [**M1**, Proposition 16] from a result of Endo. In this context the conclusion also follows immediately from Theorem 2.4 and the Pierce sheaf characterization of p.p. rings in [**B**, Lemma 3.1 (ii)] which says that the Pierce stalks are domains and the support of an element is clopen.

The first of the next pair of examples illustrates that the condition on the supports in Theorem 2.4(2) is necessary. Recall from [**NR**, Theorem 2.2] that the Pierce stalks of a ring R are all domains if and only if for each $r \in R$, $\mathrm{ann}(r)$ is generated by its idempotents. These rings are also called *almost PP-rings*. As we will see, this condition does not suffice to imply R almost clean. (However, *cf.* Proposition 3.5.)

Examples 2.9. *(i) There is a ring R whose Pierce stalks are all domains but which is not almost clean. Moreover, all the stalks but one are fields and $Q_{cl}(R) = R$. (ii) There is an example of an almost clean ring R whose stalks are all domains such that $Q_{cl}(R)$ is clean but not a regular ring.*

Proof. (i) We write $\mathbb{N}$ as a disjoint union of infinitely many infinite subsets N_k, $k \in \mathbb{N}$, where each N_k is well-ordered $\{n_{k1}, n_{k2}, \ldots\}$. The ring S is defined as a product $\prod_{i\in\mathbb{N}} F_i$, where, for $i \in \mathbb{N}$, $F_i = \mathbb{Z}/(p_j)$, when $i = n_{kj}$, for some $k \in \mathbb{N}$ and $j \in \mathbb{N}$ and p_j is the j^{th} prime. We define a subring R of S as the set of those sequences $(\overline{z_1}, \overline{z_2}, \ldots)$ such that, for some $z \in \mathbb{Z}$, some $m \in \mathbb{N}$ and all $i \geq m$, $\overline{z_i} = \overline{z}$. Since $\{i \geq m\}$ meets infinitely many elements of each N_k, the integer z in an element of R is uniquely determined and we call it the *constant part* of r.

The space $X = \mathrm{Spec}\,\mathbf{B}(R)$ is the one-point compactification $\mathbb{N} \cup \{\infty\}$ and the Pierce stalks of R are the fields F_i, $i \in \mathbb{N}$ and $R_\infty = \mathbb{Z}$.

Notice that the only non-zero divisors are those elements non-zero in each component and, hence, in particular, with constant part 1 or -1. Moreover, an idempotent must have constant part 0 or 1. If r has constant part z such that $z \neq -1, 0, 1$ and $z - 1 \neq -1, 0, 1$ then neither r nor $r - e$, for $e \in \mathbf{B}(R)$, can be a non-zero divisor.

The claim about $Q_{\mathrm{cl}}(R)$ follows since non-zero divisors have constant part 1 and are already units in R.

(ii) Consider the product $\Pi = \prod_{n\in\mathbb{N}} R_n$ such that when n is odd, $R_n = \mathbb{Z}/(p_n)$ where p_n is the nth prime, and when n is even, $R_n = \mathbb{Z}/(3)$. Let R be the subring of Π of sequences of the form $(\overline{z_1}, \overline{z_2}, \ldots)$ where, for some $z \in \mathbb{Z}$ and $m \in \mathbb{N}$, for all $i \geq m$, $z_i = z$. As in previous examples, the Pierce stalks are the fields R_n and $R_\infty = \mathbb{Z}$. If the constant part of $r \in R$ is not divisible by 3 then r is a unit except in finitely many components and r can be written $u + e$, where $u \in R$ is a non-zero divisor and $e \in \mathbf{B}(R)$. When the constant part is divisible by 3, $r - 1$ is a unit except in finitely many components and hence, for some $e \in \mathbf{B}(R)$, $r - e$ is a non-zero divisor.

Non-zero divisors in R have constant part not divisible by 3. It follows that $S = Q_{\mathrm{cl}}(R)$ is the ring of sequences from Π which are eventually "constant" of the form a/b, where $3 \nmid b$ (meaning that the terms are eventually of the form $\bar{a}\bar{b}^{-1}$). The Pierce stalks of S are R_n, for $n \in \mathbb{N}$ and $S_\infty = \mathbb{Z}_{(3)}$. Since S_∞ is not a field, $Q_{\mathrm{cl}}(R)$ is not regular. □

The second of the above examples suggests that "gap" between having stalks domains and p.p. should be looked at. The referee has pointed out that the following is found in [**Al2**]. The short proof is included for the readers' convenience.

PROPOSITION 2.10. *Let R be a ring. (i) If the Pierce stalks of R are domains then $\mathbf{B}(Q_{cl}(R)) = \mathbf{B}(R)$ and the Pierce stalks of $Q_{cl}(R)$ are domains. (ii) If the Pierce stalks of R are domains and $Q_{cl}(R)$ is regular then R is p.p.*

PROOF. (i) We first show that the stalks of $Q_{\mathrm{cl}}(R)$ are domains. Let $u/v \in Q_{\mathrm{cl}}(R)$, $u, v \in R$, $v \in \mathfrak{R}(R)$. It must be shown that $\mathrm{ann}_{Q_{\mathrm{cl}}(R)}(u/v)$ is generated by idempotents. However, $\mathrm{ann}_{Q_{\mathrm{cl}}(R)}(u/v) = (\mathrm{ann}_R(u))Q_{\mathrm{cl}}(R)$, which is generated by the idempotents of $\mathrm{ann}_R(u)$. Now let $e \in \mathbf{B}(Q_{\mathrm{cl}}(R))$ where $e = u/v$. As we have seen, $\mathrm{ann}_{Q_{\mathrm{cl}}(R)}(e) = (1-e)Q_{\mathrm{cl}}(R) = (\mathrm{ann}_R(u))Q_{\mathrm{cl}}(R)$. In particular, there exists $f \in \mathbf{B}(R)$ with $fe = 0$ and $1 - e = (1-e)f$. Hence, $1 - e = f \in \mathbf{B}(R)$. Finally, $e = 1 - f$.

(ii) If $Q_{\mathrm{cl}}(R)$ is regular, then the annihilator of an element of $Q_{\mathrm{cl}}(R)$ is generated by a single idempotent. Hence, by (i), for $r \in R$, $\mathrm{ann}_R(r)$ is also generated by a single idempotent and, thus, R is a p.p. ring. □

3. pm rings and pm$^+$ rings.

3.1. On pm rings. In this section it will be shown that for pm rings, the classes of clean rings, almost clean rings and weakly clean rings coincide.

REMARK 3.1. *A ring R is a pm ring if and only if for each $x \in X(R)$, R_x is a pm ring. Moreover, if S is an extension of a pm ring by idempotents then S is a pm ring.*

PROOF. This is clear from looking at the primes ideals in R and the factor rings R_x. The second statement follows from the first by Corollary 1.6 since a homomorphic image of a pm ring is a pm ring. □

PROPOSITION 3.2. *If R is a weakly clean pm ring then it is clean.*

PROOF. Note that a homomorphic image of a pm ring is a pm ring. A ring T is clean if and only if it is pm and $\mathrm{Max}\, T$ is zero-dimensional ([**J**, Theorem V 3.9]). Now consider an indecomposable weakly clean ring S which is not clean. Then, [**AA**, Theorem 1.3] says that S has exactly two maximal ideals. However, if S is the non-clean Pierce stalk (see Proposition 2.1) of a weakly clean pm ring then it is pm and $\mathrm{Max}\, S$ is discrete – hence, zero-dimensional – making S clean, a contradiction. □

COROLLARY 3.3. *For any space X, if $C(X)$ is weakly clean, it is clean.*

The pm property is a very powerful one in this context. We know that for $C(X)$ ([**M1**, Theorem 13]) almost clean implies clean. However, this holds in a more general setting. The following also generalizes [**V**, Theorem 5.6].

THEOREM 3.4. *Let R be a pm ring. If R is almost clean it is clean.*

PROOF. Assume R is almost clean. We know from Theorem 2.4 that the stalks of R are almost clean and they are also pm. It suffices to show that an indecomposable almost clean pm ring is local. Now assume that R is indecomposable. By [**AA**, Theorem 2.3] we know that the sum of two ideals consisting of zero divisors is not all of R.

Suppose $\mathfrak{p}_1 \neq \mathfrak{p}_2$ are minimal primes of R. For $i = 1, 2$ let $\mathfrak{m}_i$ be the unique maximal ideal containing $\mathfrak{p}_i$. Since $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathfrak{Z}(R)$, $\mathfrak{p}_1 + \mathfrak{p}_2$ is a proper ideal contained in some maximal ideal $\mathfrak{m}_3$. We get that $\mathfrak{m}_1 = \mathfrak{m}_2 = \mathfrak{m}_3$. Hence, all the minimal primes are contained in the same maximal ideal. Since any prime ideal contains a minimal prime ideal, we see that there is only one maximal ideal, as required. $\square$

The following is found in [**Al1**, Theorem 2.3]; however, the methods developed above allow for a quick proof.

PROPOSITION 3.5. *Let R be a pm ring whose stalks are domains. Then, R is clean.*

PROOF. Since each stalk is a domain and a pm ring, it must be local (the zero ideal is in a unique maximal ideal). Hence, R is clean. $\square$

3.2. On pm$^+$ rings. Rings of continuous functions have a stronger property than "pm"; in fact in a ring $C(X)$ where X is a topological space, the prime ideals containing a given prime ideal form a chain. By contrast, any local domain is a pm ring but would have this stronger property only if all its prime ideals formed a chain, as, for example, in a valuation domain. The rings satisfying the equivalent conditions of [**C**, Theorem III.4] are pm$^+$ -rings; one of these conditions is that every indecomposable module is cyclic. There does not seem to be a standard name for this stronger property.

DEFINITION 3.6. *A ring R such that for each $\mathfrak{p} \in \operatorname{Spec} R$, the prime ideals containing $\mathfrak{p}$ form a chain is said to be a* pm$^+$ ring.

We will see that a ring R is a pm$^+$ ring if and only if its Pierce stalks are pm$^+$ rings; hence, Corollary 1.6 says that an extension of a pm$^+$ ring by idempotents is a pm$^+$ ring. Various characterizations, including an element-wise one, are presented. The criterion for a pm ring [**Ca1**, Theorem 4.1] is used in localizations. We first note a simple lemma.

LEMMA 3.7. *Suppose R is a pm$^+$ ring, I an ideal of R and $S \subseteq R$ a multiplicatively closed set with $0 \notin S$. Then, R/I and RS^{-1} are pm$^+$ rings.*

PROOF. We see that $\operatorname{Spec} R/I$ may be identified with a subspace of $\operatorname{Spec} R$ which preserves inclusions. Hence, R/I is a pm$^+$ ring. Using this fact, we may assume that $R \to RS^{-1}$ is one-to-one. Then, $\operatorname{Spec} RS^{-1} \to \operatorname{Spec} R$ is again injective and preserves inclusions. Hence, also, RS^{-1} is a pm$^+$ ring. $\square$

PROPOSITION 3.8. *The following are equivalent for a ring R.*
(1) R is a pm$^+$ ring.
(2) For each multiplicatively closed set $S \subseteq R$, $0 \notin S$, RS^{-1} is a pm ring.
(3) For each multiplicatively closed set $S \subseteq R$, $0 \notin S$, if $a, b \in R$ and $s = a + b \in S$ there there are $u \in S$ and $c, d \in R$ with $(u - ac)(u - bd) = 0$.

(4) If $a, b \in R$ and $s = a + b$ is not nilpotent there are $k \in \mathbb{N}$, $c, d \in R$ with $(s^k - ac)(s^k - bd) = 0$.

(5) For each $x \in X(R)$, R_x is a pm^+ ring.

Proof. (1) $\Rightarrow$ (2). This is from Lemma 3.7.

(2) $\Rightarrow$ (3). We use the criterion [**Ca1**, Theorem 4.1]. If $s = a + b \in S$ then, in the pm ring RS^{-1}, $as^{-1} + bs^{-1} = 1$; hence, there are $ct^{-1}, dt^{-1} \in RS^{-1}$ (using a common denominator) with $(1 - as^{-1}ct^{-1})(1 - bs^{-1}dt^{-1}) = 0$. Then, since $st \in S$, there is $v \in S$ with $(st - ac)(st - bd)v = 0$. We get $(stv - acv)(stv - bdv) = 0$.

(3) $\Rightarrow$ (4). The statement (4) is a special case of (3) using $S = \{s^m \mid m \in \mathbb{N}\}$.

(4) $\Rightarrow$ (1). We assume (4) and suppose $\mathfrak{q}, \mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec} R$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \subseteq \mathfrak{p}'$ such that $\mathfrak{p}$ and $\mathfrak{p}'$ are not comparable. We pick $a \in \mathfrak{p} \setminus \mathfrak{p}'$ and $b \in \mathfrak{p}' \setminus \mathfrak{p}$ and set $s = a + b$. If s were nilpotent then $s \in \mathfrak{q}$, showing that $b \in \mathfrak{p}$, a contradiction. Then, (4) is applied to $S = \{s^m \mid m \in \mathbb{N}\}$ to get $k \in \mathbb{N}$ and $c, d \in R$ with $(s^k - ac)(s^k - bd) = 0$. One of the factors is in $\mathfrak{q}$; suppose $s^k - ac \in \mathfrak{q}$. Now $a \in \mathfrak{p}$ implies $s^k \in \mathfrak{p}$ and, thus, $s \in \mathfrak{p}$. This is impossible because it would imply $b \in \mathfrak{p}$. The contradiction shows that two unrelated prime ideals cannot contain a common prime ideal. It follows that R is a pm^+ ring.

(5) $\Leftrightarrow$ (1). If R is a pm^+ ring so is R_x for each $x \in X(R)$ since R_x is a homomorphic image of R. In the other direction will be shown that (5) implies (4). We assume $a, b \in R$ and that $s = a + b$ is not nilpotent. This means that for some $x \in X(R)$, s_x is not nilpotent. In any case, if, for $x \in X(R)$, s_x is nilpotent, say with $s_x^{k(x)} = 0_x$ then $(s^{k(x)} - as^{k(x)})_x(s^{k(x)} - bs^{k(x)})_x = 0_x$. Otherwise, there are $k(x) \in \mathbb{N}$, $c(x), d(x) \in R$ with $(s^{k(x)} - ac(x))_x(s^{k(x)} - bd(x))_x = 0_x$. Hence, for each $x \in X(R)$ there is a clopen neighbourhood N_x of x, $k(x) \in \mathbb{N}$, $c(x), d(x) \in R$ such that for all $y \in N_x$, $(s^{k(x)} - ac(x))_y(s^{k(x)} - bd(x))_y = 0_y$. The usual Pierce method for building elements of R can now be applied to find $k \in \mathbb{N}$, $c, d \in R$ such that $(s^k - ac)(s^k - bd) = 0$, proving (4). $\square$

Statements (1) and (4) of Proposition 3.8 and Corollary 1.6 combine to show that if S is an extension of a pm^+ ring R by idempotents, then S is a pm^+ ring.

Corollary 3.9. *Let X be a strongly 0-dimensional space. For every $f \in C(X)$, $C^*(X)[f]$ is clean. Hence, every ring between $C^*(X)$ and $C(X)$ is clean.*

Proof. The referee pointed out that $C^*(X)[f] = C^*(X)S^{-1}$ for a suitable multiplicatively closed set S; indeed, $S = \{1/(1 + f^2)^n \mid n \in \mathbb{N}\}$ works. Since $C^*(X) \cong C(\beta X)$, $C^*(X)$ is a pm^+ ring and $C^*(X)[f]$ is a pm ring by Proposition 3.8(2). Since $C(X)$ is clean, each $g \in C^*(X)[f]$ can be written $g = v + e$, $e \in \mathbf{B}(C(X))$, $v \in \mathfrak{U}(C(X))$. However, $e \in C^*(X)$ and, hence, $v \in C^*(X)[f]$. Because $v \in \mathfrak{R}(C^*(X)[f])$ it follows that $C^*(X)[f]$ is almost clean. Then, $C^*(X)[f]$ is clean by Theorem 3.4. $\square$

As an application of Lemma 3.7 we have a strengthening of the result ([**BR**, Proposition 2.1] and [**KLM**, Proposition 5.19]) which says that if $C(X)$ is clean so is $Q_{\mathrm{cl}}(X)$. We need that a cozero-set in a strongly 0-dimensional space is strongly 0-dimensional. This is shown in the proof of [**KLM**, Proposition 5.19].

Theorem 3.10. *Let X be a strongly 0-dimensional space. Then, for any multiplicatively closed $S \subseteq C(X)$, $0 \notin S$, $C(X)S^{-1}$ is a clean ring.*

PROOF. It will suffice to show that for any $0 \neq f \in C(X)$ the ring $A = C(X)[f^{-1}]$ is clean. We may assume that f is not a unit in $C(X)$; in that case $V = \operatorname{coz} f$ is a proper cozero-set. As quoted above, V is strongly 0-dimensional and, hence, $C(V)$ is also a clean ring. Moreover, A embeds naturally in $C(V)$.

We first assume that f is bounded and show that the idempotents of $C(V)$ are in A. Let $e^2 = e \in C(V)$. Since f is bounded, $ef|_V$ extends to some $g \in C(X)$. Hence, $e = g/f \in A$. For an arbitrary $h/f^n \in A$, $h/f^n = u + e$ for some $u \in \mathfrak{U}(C(V))$ and $e \in \mathbf{B}(C(V))$. Again, since $e \in A$, $u \in A$, as well. Then, $u \in \mathfrak{R}(A)$. This shows that A is an almost clean ring. It is also a pm ring (Lemma 3.7) and then, by Theorem 3.4, A is clean.

We now drop the assumption that f is bounded and set $g = f/(1 + f^2)$. Because $1/(1 + f^2) \in C(X)$, it follows that $C(X)[f^{-1}] = C(X)[g^{-1}]$. This shows that $C(X)[f^{-1}]$ is clean, since g is bounded. $\square$

It is not, however, the case that, when X is strongly 0-dimensional, every ring between $C(X)$ and $Q_{\mathrm{cl}}(X)$ is clean. Recall that a space is *perfectly normal* if every open set in it is a cozero-set. The class of perfectly normal spaces properly contains that of metric spaces.

PROPOSITION 3.11. *Let X be a perfectly normal which has an infinite convergent sequence. Then, there is $f \in Q_{cl}(X)$ such that $C(X)[f]$ is not clean. This applies, in particular, to the strongly 0-dimensional space $X = \mathbb{Q}$.*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of distinct points in X converging to $a \in X$ and let $Y = X \setminus \{a\}$. Since X is perfectly normal, the dense open subset Y is a cozero-set. Since $S \cup \{a\}$ is compact, S is closed in Y. It follows that S is C*-embedded in the perfectly normal space Y ([**GJ**, 3D]). We assign values to the elements of S in such a way that each of 0 and $1/m$, $m \in \mathbb{N}$, occurs infinitely many times, and let f be a continuous bounded extension to Y. Then, f is not a unit since it sometimes has value 0. Note that for each $m \in \mathbb{N}$, there is a subsequence S_m of S, converging to $a \in X$, on which f is constantly $1/m$.

Recall that $C(Y)$ may be viewed as a subring of $Q_{\mathrm{cl}}(X)$ ([**FGL**, 2.6 Theorem (2)]). The ring we propose is $A = C(X)[f] \subseteq C(Y)$. Suppose that $f = u + e$ where $u \in \mathfrak{U}(A)$ and $e \in \mathbf{B}(A)$. The function e can only have values 0 and 1. Moreover, $e(x_n) = 0$ when $f(x_n) = 1$ and $e(x_n) = 1$ when $f(x_n) = 0$. Hence, e does not extend to an element of $C(X)$. Because u is invertible in A there is an equation

$$(*) \quad (f - e)(g_0 + g_1 f + \cdots + g_n f^n) = 1,$$

with $g_0, \ldots, g_n \in C(X)$. Let the second factor in $(*)$ be denoted w. For fixed $m \in \mathbb{N}$, w extends continuously to $S_m \cup \{a\}$ since the $g_i \in C(X)$ and f is constant on S_m. However, for $x_k \in S_m$, $w(x_k) = m$ if $e(x_k) = 0$ and $w(x_k) = m/(1 - m)$ if $e(x_k) = 1$. Hence, e is eventually 0 on S_m or is eventually 1. Suppose, for convenience that e is eventually 0 on S_m for infinitely many $m \in \mathbb{N}$ and consider the real polynomial $F(z) = z(g_0(a) + g_1(a)z + \cdots + g_n(a)z^n) - 1$. For infinitely many $m \in \mathbb{N}$, $F(1/m) = 0$, which is absurd. Hence, there can be no expression $f = u + e$. When e is eventually 1, similar reasoning applies but with a different polynomial. $\square$

A key argument used in Corollary 3.9 and in Theorem 3.10 has other applications. There are many ways of embedding a ring into a clean ring. See Construction 4.6 and the remarks preceding it.

THEOREM 3.12. *Every ring R has an extension by idempotents which is almost clean. Moreover, every pm ring R has an extension by idempotents which is clean. However, if R is not a pm ring, no integral extension (in particular, no extension by idempotents) of R can be a pm ring.*

PROOF. Let T be an extension of R which is a clean ring and let S be the subring of T generated by R and $\mathbf{B}(T)$. For any $s \in S$, there are $u \in \mathfrak{U}(T)$ and $e \in \mathbf{B}(T)$ such that $s = u + e$. Since $e \in S$, $u \in S$ as well, and $u \in \mathfrak{R}(S)$, proving the first statement. If, in addition, R is a pm ring, then so is S by Remark 3.1. Hence, Theorem 3.4 shows that S is clean.

For the last part, suppose R is not a pm ring and that some $\mathfrak{p} \in \operatorname{Spec} R$ is in two distinct maximal ideals, $\mathfrak{m}$ and $\mathfrak{n}$. If S is an integral extension of R then there are $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{m}', \mathfrak{n}' \in \operatorname{Max} S$ containing $\mathfrak{q}$ with $\mathfrak{m}' \cap R = \mathfrak{m}$, $\mathfrak{n}' \cap R = \mathfrak{n}$ and $\mathfrak{q} \cap R = \mathfrak{p}$ ([**Mat**, Theorem 9.4]). Hence, S is not a pm ring. $\square$

4. Uniquely clean rings and J-clean rings.

Categorical language in this section will follow that of Mac Lane's book [**ML**].

4.1. F-semiperfect rings whose idempotents are central. At the start of this section we temporarily drop the assumption that our rings are commutative; in this setting $\mathbf{B}(R)$ stands for the boolean algebra of *central* idempotents. In [**NZ**], the authors give a thorough description of uniquely clean rings, including the noncommutative case. The methods suggest a generalization which will turn out to have categorical properties in the commutative case. Before going on to our generalization it is useful to put the rings to be discussed into context.

The same class of (noncommutative) rings has appeared under two different names. The first is that of the *F-semiperfect rings* of Oberst and Schneider, i.e., rings such that every finitely presented module (left or right) has a projective cover (see [**OS**]). The second is the class of *D-rings* introduced by Contessa ([**Ca2**]) and further studied by Contessa and Lesieur ([**CL**]): A ring R is a *left D-ring* is for all $r \in R$ there are $e^2 = e \in R$ and an $a \in R$ such that $e = ar$ and $r - re \in \mathbf{J}(R)$. A *right D-ring* is defined similarly ([**CL**, Definition 11.1]). Then [**CL**, Théorème II.1.3] shows that left D-rings, right D-rings and F-semiperfect rings coincide. Other equivalent conditions and properties of these rings can be found in [**CL**]. Not all regular rings are clean and, hence, not all F-semiperfect rings are clean.

We shall study F-semiperfect rings whose idempotents are central. Recall that a ring R is *abelian regular* (or *strongly regular*) if for each $r \in R$ there is $s \in R$ with $r^2 s = r$. In an abelian regular ring all idempotents are central. Some parts of the proof of the next result are adapted from that of [**NZ**, Theorem 20].

THEOREM 4.1. *Let R be any ring. The following statements are equivalent.*

(1) $R/\mathbf{J}(R)$ is abelian regular and idempotents lift uniquely modulo $\mathbf{J}(R)$.

(2) $R/\mathbf{J}(R)$ is regular, idempotents lift modulo $\mathbf{J}(R)$, and all idempotents of R are central.

(2′) R is F-semiperfect and all idempotents of R are central.

(3) R is a clean ring with idempotents central and for all $x \in X(R)$, $\mathbf{J}(R_x) = \mathbf{J}(R)_x$.

(4) For each $r \in R$, there is $e \in \mathbf{B}(R)$ such that $re + (1-e) \in \mathfrak{U}(R)$ and $r(1-e) \in \mathbf{J}(R)$.

Proof. Throughout, "modulo $\mathbf{J}(R)$" will be indicated by a bar and $X = X(R)$.

(2) $\Leftrightarrow$ (2′). One of the characterizations of an F-semiperfect ring (sometimes its definition) is that $R/\mathbf{J}(R)$ is regular and idempotents lift modulo the radical.

(1) $\Rightarrow$ (2). Once we have shown that idempotents of R are central, the fact that idempotents lift modulo $\mathbf{J}(R)$ will yield that $\overline{R}$ is abelian regular. Following [**NZ**], if $e^2 = e \in R$ then for $r \in R$, $e + (re - ere)$ is an idempotent with the same image as e. Hence, by the uniqueness, $re = ere$ and, similarly, $er = ere$.

(2) $\Rightarrow$ (1). We only need show the uniqueness. Suppose $e, f \in \mathbf{B}(R)$ with $\bar{e} = \bar{f}$. Thus, $e - f \in \mathbf{J}(R)$ and then $1 - (e - ef) \in \mathfrak{U}(R)$. It follows that $e = ef$ and, similarly, $f = ef$.

(2) $\Rightarrow$ (3). We have that $\overline{R}$ is abelian regular and idempotents lift. Thus, R is a clean ring ([**HN**, Proposition 6]). By [**BS**, Proposition 1.2], since idempotents are central, each R_x has a unique maximal left ideal. Given $r \in R$ there is $s \in R$ such that $\bar{r}^2\bar{s} = \bar{r}$. Then, the idempotent $\bar{r}\bar{s}$ lifts to some $e \in \mathbf{B}(R)$. When $e_x = 1_x$ then $r_xs_x - 1_x \in \mathbf{J}(R_x)$ which implies that $r_xs_x \in \mathfrak{U}(R_x)$ and, hence, that $r_x \in \mathfrak{U}(R_x)$. When $e_x = 0_x$, $r_xs_x \in \mathbf{J}(R_x)$ and, hence, $r_x^2s_x \in \mathbf{J}(R_x)$; moreover, $r_x^2s_x - r_x \in \mathbf{J}(R_x)$ as well. It follows that $r_x \in \mathbf{J}(R_x)$. Then, $r_x \in \mathfrak{U}(R_x)$ for $x \in \mathrm{Supp}(e)$ and $r_x \in \mathbf{J}(R_x)$ for $x \in \mathrm{Supp}(1-e)$. Thus, $r(1-e) \in \mathbf{J}(R)$ and if, for some $y \in X$, $r_y \in \mathbf{J}(R_y)$, we have that $r_y = (r(1-e))_y \in \mathbf{J}(R)_y$.

(3) $\Rightarrow$ (4). We have that each stalk R_x has a unique maximal left ideal $M^{(x)}$ ([**BS**, Proposition 1.2]). For $r \in R$, $\{x \in X \mid r_x \in \mathfrak{U}(R_x)\}$ is open in X. The condition on the radicals implies that $\{x \in X \mid r_x \in \mathbf{J}(R_x)\}$ is open as well. Since either $r_x \in \mathfrak{U}(R_x)$ or $r_x \in \mathbf{J}(R_x)$, the two open sets are complements of each other. Hence there is $e \in \mathbf{B}(R)$ such that for all $x \in \mathrm{Supp}(e)$, $r_x \in \mathfrak{U}(R)$ and for all $x \in \mathrm{Supp}(1-e)$, $r_x \in \mathbf{J}(R_x)$. This is the desired idempotent.

(4) $\Rightarrow$ (2). We first show that $\overline{R}$ is abelian regular. For $r \in R$ we find $e \in \mathbf{B}(R)$ as in the statement. Suppose $(re + (1-e))v = 1$; then, $rev = e$ and $r - re \in \mathbf{J}(R)$. Thus, $\bar{r} = \bar{r}\bar{e} = \bar{r}^2\bar{e}\bar{v}$, as required. Next, idempotents of R are central; suppose $g = g^2$. We find $e \in \mathbf{B}(R)$ where $ge + (1-e) \in \mathfrak{U}(R)$ and $g(1-e) \in \mathbf{J}(R)$. Since $g(1-e)$ is an idempotent, it is zero and $g = ge$. The unit $ge + (1-e) = g + (1-e)$ is also an idempotent, hence, it is 1. Thus, $g = e$.

Finally, we need to lift idempotents. Suppose $r^2 - r \in \mathbf{J}(R)$. We find $e \in \mathbf{B}(R)$ as in the statement. The claim is that $\bar{r} = \bar{e}$. We have $(e - er)^2 = (e - er) + e(r^2 - r)$. Hence, $\bar{e} - \bar{e}\bar{r}$ is an idempotent. We also have $\bar{r} = \bar{e}\bar{r}$ since $r(1-e) \in \mathbf{J}(R)$. From this, $\bar{e}\bar{r} + (\bar{1} - \bar{e}) = \bar{1} - (\bar{e} - \bar{e}\bar{r}) \in \mathfrak{U}(\overline{R})$. Hence, $\bar{e} - \bar{e}\bar{r} = \bar{0}$ and $\bar{e} = \bar{e}\bar{r} = \bar{r}$. $\square$

We note that the idempotent e in the expression (4) of the theorem is uniquely determined. Suppose $f \in \mathbf{B}(R)$, $rf + 1 - f \in \mathfrak{U}(R)$, $r(1-f) \in \mathbf{J}(R)$ and that, for some $x \in X(R)$, $e_x = 1_x$ and $f_x = 0_x$. In this case $r_x(1_x - f_x) = r_x \in \mathbf{J}(R_x)$ while $(re + 1 - e)_x = r_x \in \mathfrak{U}(R_x)$, which is impossible. Similarly $e_x = 0_x$ and $f_x = 1_x$ is impossible.

We will see shortly that the rings of Theorem 4.1 include the uniquely clean rings. The statement "R is F-semiperfect and all idempotents of R are central" is too long and so we give a shorter name.

DEFINITION 4.2. *A ring R satisfying the equivalent conditions of Theorem 4.1 will be call a* J-clean ring.

We recall the characterization of uniquely clean rings in [**NZ**, Theorem 20]: *The following are equivalent for a ring R: (1) R is uniquely clean; (2) $R/\mathbf{J}(R)$ is boolean and idempotents lift uniquely modulo $\mathbf{J}(R)$; (3) $R/\mathbf{J}(R)$ is boolean, idempotents lift modulo $\mathbf{J}(R)$, and idempotents of R are central; (4) for all $a \in R$ there exists a unique $e \in \mathbf{B}(R)$ such that $e - a \in \mathbf{J}(R)$.*

COROLLARY 4.3. *The following are equivalent for a ring R: (i) R is uniquely clean; (ii) R is J-clean and $R/\mathbf{J}(R)$ is boolean; (iii) for each $x \in X(R)$, R_x is local (has a unique maximal one-sided ideal) and $R_x/\mathbf{J}(R_x) \cong \mathbb{Z}/2\mathbb{Z}$.*

PROOF. The equivalence of (i) and (ii) is clear using [**NZ**, Theorem 20]. Moreover, (i) and (ii), using Theorem 4.1 (2), imply that if R is clean and idempotents are central, then the stalks of R are local in the strong sense of having a unique maximal one-sided ideal and Theorem 4.1 (3) yields that $R_x/\mathbf{J}(R_x) \cong \mathbb{Z}/2\mathbb{Z}$. On the other hand, (iii) implies that R is clean and each R_x is uniquely clean, as can be checked either directly or by using [**AC**, Corollary 22], which works equally well in the non-commutative case. It is also clear that if all the stalks are uniquely clean then, so is R because two different expressions of $r \in R$ as a clean element would have to differ in some stalk. $\square$

4.2. Commutative J-clean rings. At this point we revert to the convention that *all rings are assumed to be commutative.*

A useful property of a J-clean ring R, in the commutative case, is that $X(R)$ and $\operatorname{Max} R$ are homeomorphic boolean spaces. Indeed, for the regular ring $R/\mathbf{J}(R)$, $\operatorname{Spec} R/\mathbf{J}(R)$ coincides with $\operatorname{Spec}\mathbf{B}(R/\mathbf{J}(R))$ and, since idempotents lift uniquely modulo $\mathbf{J}(R)$, these spaces coincide with $\operatorname{Spec}\mathbf{B}(R)$; moreover, $\operatorname{Max} R$ coincides with $\operatorname{Max}(R/\mathbf{J}(R))$.

The following lists some examples of J-clean rings.

- All local rings. More generally, for a local ring A and X a boolean space, the ring of sections of the simple sheaf (X, A).
- All 0-dimensional rings.
- A direct product of J-clean rings.
- An extension of a J-clean ring by idempotents (using Theorem 4.1 (4)).

On the other hand, a semiprimitive ring R (i.e., $\mathbf{J}(R) = \mathbf{0}$) is J-clean if and only if R is regular. In particular, $C(X)$ is J-clean if and only if X is a P-space.

Recall that a reflective subcategory is closed under limits and, hence, under equalizers. The subcategory of J-clean rings, which we call $\mathcal{JC}$, is not a reflective subcategory of the category of (commutative) rings, as we will see. It does, however, have some closure properties.

PROPOSITION 4.4. *(i) The categories of J-clean and of clean rings are closed under products and homomorphic images. (ii) Let R and S be local rings and $\alpha, \beta\colon R \to S$ be homomorphisms. Let E be the equalizer of α and β, then E is local. However, equalizers in $\mathcal{JC}$ are not J-clean or even clean.*

Proof. (i) Both parts for $\mathcal{JC}$ follow readily from Theorem 4.1(4) (see also [**Ca2**, Proposition 5.3(3)]). (ii) Suppose $r \in E$. If $r \in \mathfrak{U}(R)$ then its inverse is also in E. Now suppose r is not invertible in R. Then, for all $t \in E$, $tr - 1 \in \mathfrak{U}(R)$ and its inverse is also is in E. It follows that $r \in \mathbf{J}(E)$ and E is local. For the second part of (ii) consider $R = \mathbb{Z}_{(3)} \times \mathbb{Z}_{(5)}$ and $S = \mathbb{Q}$ with α given by the projection onto $\mathbb{Z}_{(3)}$ followed by inclusion, while β is the other projection followed by inclusion. The equalizer is isomorphic to the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$, which is not clean. □

In the last paragraph of [**AC**], the authors ask if the subcategory of uniquely clean rings is closed under homomorphic images. This was answered in the positive in [**NZ**, Theorem 22]. The proof of Proposition 4.4 (iii) applies equally to noncommutative J-clean rings and, when specialized to uniquely clean rings, again yields the same conclusion.

4.3. The functor $\mathfrak{jc}$. One of the reasons for giving a special name to the rings of Theorem 4.1 is that there is a functorial way of associating a J-clean ring extension to each ring; the functor is not a reflector in the category of rings (it is not the identity on $\mathcal{JC}$) but has some of the properties of a reflector, including a universal property. Construction 4.6, below, is reminiscent of the construction of the universal regular ring (see [**H**] and [**W**]) and, indeed, the two are closely related as will be shown. The key point is that if R is a J-clean ring then, for $r \in R$, $\{x \in X \mid r_x \in \mathfrak{U}(R_x)\}$ is clopen in X.

Other methods of embedding R into a J-clean ring are studied in [**Ca2**] and these will be revisited later. A construction and a result from [**Ca2**] are useful at this point ([**Ca2**, Theorems 5.11 and 6.3]).

Let R be a subring of a direct product $P = \prod_{\alpha \in A} L_\alpha$ of local rings. Let the maximal ideal of L_α be $\mathfrak{m}_\alpha$. For $r = (r_\alpha) \in R$, define $r^* \in P$ by $(r^*)_\alpha = r_\alpha^{-1}$ if $r_\alpha \notin \mathfrak{m}_\alpha$ and $r^*_\alpha = 0$ if $r_\alpha \in \mathfrak{m}_\alpha$. Then $D_P(R)$ is defined to be the subring of P generated by R and $\{r^* \mid r \in R\}$.

Lemma 4.5. [**Ca2**, Theorem 6.3]. *Let R be a subring of a product P of local rings. Then, the subring $D_P(R)$ of P is a J-clean ring extending R.*

Construction 4.6. *There is a functor $\mathfrak{jc}\colon \mathcal{CR} \to \mathcal{JC}$ such that (1) for each $R \in \mathcal{CR}$, $\iota_R\colon R \to \mathfrak{jc}(R)$ is a monomorphism, (2) For a homomorphism $\phi\colon R \to S$ in $\mathcal{CR}$, $\mathfrak{jc}(\phi)(\mathbf{J}(\mathfrak{jc}(R))) \subseteq \mathbf{J}(\mathfrak{jc}(S))$, and (3) the functor $\mathfrak{jc}$ followed by reduction modulo the Jacobson radical is equivalent to the universal regular ring functor T.*

Proof. (1) We begin by describing the ring $\mathfrak{jc}(R)$ for $R \in \mathcal{CR}$. The *constructible* (or *patch*) *topology* on the set $\operatorname{Spec} R$ (see [**H**, Section 2] and also [**W**] or [**J**, Proposition 4.5]) has as sub-basic open sets those of the form $D(a) = \{\mathfrak{p} \in \operatorname{Spec} R \mid a \notin \mathfrak{p}\}$, $a \in R$ and $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}$, I a finitely generated ideal. The set with the new topology is denoted $\operatorname{Spec}_c R$ and is always a boolean space ([**H**, Theorem 1]). This space will serve as the base space of the Pierce sheaf of $\mathfrak{jc}(R)$; to distinguish between $\mathfrak{p} \in \operatorname{Spec} R$ and the corresponding point in $X(\mathfrak{jc}(R))$, we write $x(\mathfrak{p})$ for the latter.

For $x(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec} R$, the corresponding stalk $\mathfrak{jc}(R)_{x(\mathfrak{p})}$ will be $R_\mathfrak{p}$. The topology on the *espace étalé* needs to be specified. A sub-basic open set of $\operatorname{Spec}_c R$ has the form $N(a, I) = D(a) \cap V(I)$, $a \in R$ and I a finitely generated ideal; such a set is also closed. Given $N(a, I)$ and $c, d \in R$, $d \notin \mathfrak{p}$ for all $\mathfrak{p} \in N(a, I)$, the set $U(a, I, c, d) = \{cd^{-1} \in R_\mathfrak{p} \mid \mathfrak{p} \in N(a, I)\}$ is decreed to be open. These sets may be viewed as partial sections. A key feature of a Pierce sheaf is that if two (partial) sections have a non-empty intersection then they coincide over a neighbourhood of the base space. We verify that property here: suppose $U(a_1, I_1, c_1, d_1)$ and $U(a_2, I_2, c_2, d_2)$ are open sets and for some $\mathfrak{p} \in N(a_1, I_1) \cap N(a_2, I_2)$ we have $c_1 d_1^{-1} = c_2 d_2^{-1}$ in $R_\mathfrak{p}$. Then, for some $s \notin \mathfrak{p}$, $(c_1 d_2 - c_2 d_1)s = 0 \in R$ and $U(a_1 a_2 s, I_1 + I_2, c_1, d_1)$ is an appropriate neighbourhood in the intersection. The ring of sections of this new sheaf is denoted $\mathfrak{jc}(R)$. Given $cd^{-1} \in R_\mathfrak{p}$, for some $\mathfrak{p} \in \operatorname{Spec} R$, there is a partial section using $U(d, \mathbf{0}, c, d)$ which, because $D(d)$ is clopen, extends to a global section using, for example, $U(1, (d), 0, 1)$. Hence, the stalk of $\mathfrak{jc}(R)$ at $x(\mathfrak{p})$ is $R_\mathfrak{p}$.

We next note that R has a natural embedding into $\mathfrak{jc}(R)$; for $r \in R$ let $\hat{r}$ be defined by $\hat{r}_{x(\mathfrak{p})} = r1^{-1}$. This is clearly a section and $\hat{r} = 0$ only if $r = 0$. The resulting monomorphism is denoted $\iota_R \colon R \to \mathfrak{jc}(R)$.

The ring $\mathfrak{jc}(R)$ is a J-clean ring as may be verified directly. However, it also follows from Lemma 4.5 by viewing $\mathfrak{jc}(R)$ as a subring of the product $P = \prod_{\mathfrak{p} \in \operatorname{Spec} R} R_\mathfrak{p}$. Then, $\mathfrak{jc}(R)$ is readily seen to be the J-clean ring $D_P(R)$.

Given a homomorphism $\phi \colon R \to S$ in $\mathcal{CR}$ we need to define $\mathfrak{jc}(\phi) \colon \mathfrak{jc}(R) \to \mathfrak{jc}(S)$. This is done by specifying the value of $\mathfrak{jc}(\phi)(\sigma)$ at some $x(\mathfrak{q})$, $\mathfrak{q} \in \operatorname{Spec} S$. Suppose that $\sigma_{x(\phi^{-1}(\mathfrak{q}))} = ab^{-1}$, then $\mathfrak{jc}(\phi)(\sigma)_\mathfrak{q} = \phi(a)\phi(b)^{-1}$. Since σ has value ab^{-1} on a neigbourhood, say N of $x(\phi^{-1}(\mathfrak{q}))$, $\mathfrak{jc}(\phi)(\sigma)$ will have value $\phi(a)\phi(b)^{-1}$ on the preimage of N in $\operatorname{Spec_c} S$. (Recall that the function $\operatorname{Spec} S \to \operatorname{Spec} R$ induced by ϕ is also continuous in the constructible topology.) This shows that $\mathfrak{jc}(\phi)$ is a well-defined ring homomorphism.

(2) As in the construction of $\mathfrak{jc}(\phi)$, for $\mathfrak{q} \in \operatorname{Spec} S$, $\phi(a)\phi(b)^{-1} \in \mathbf{J}(S_\mathfrak{q})$ exactly when $a \in \phi^{-1}(\mathfrak{q})$. Since we are dealing with J-clean rings, this is enough to show that $\phi(\mathbf{J}(\mathfrak{jc}(R))) \subseteq \mathbf{J}(\mathfrak{jc}(S))$.

(3) The construction of the functor T is very much like that of $\mathfrak{jc}$ but for $\mathfrak{p} \in \operatorname{Spec} R$, the corresponding stalk of $T(R)$ is $Q_{\mathrm{cl}}(R/\mathfrak{p})$. However, $Q_{\mathrm{cl}}(R/p) \cong R_\mathfrak{p}/(\mathfrak{p})$ in a natural way. This, combined with (2), gives all that is required. □

We have already seen in Proposition 4.4 that $\mathcal{JC}$ is not a reflective subcategory of $\mathcal{CR}$ and, hence, $\mathfrak{jc}$ cannot be a reflector. As further evidence, note that $\mathfrak{jc}$ is not the identity on $\mathcal{JC}$. Indeed $\mathfrak{jc}(R) \cong R$ only when R is 0-dimensional. (When R is not 0-dimensional $\mathfrak{jc}(R)$ acquires new maximal ideals; when R is 0-dimensional, $\operatorname{Spec} R$ and $\operatorname{Spec_c} R$ coincide.)

The next simple example illustrates how $\mathfrak{jc}(R)$ and $T(R)$ are related.

Example 4.7. *Let R be the ring of sequences from $\mathbb{Q}$ which are eventually constant in $\mathbb{Z}_{(p)}$, for a prime integer p. Then, R is clean but not J-clean, $\mathfrak{jc}(R) = S \times \mathbb{Z}_{(p)}$, where S is the ring of sequences from $\mathbb{Q}$ which are eventually constant, and $\mathfrak{jc}(R)/\mathbf{J}(\mathfrak{jc}(R)) = S \times \mathbb{Z}/(p) = T(R)$. Moreover, $R \to \mathfrak{jc}(R)$ is not an epimorphism in $\mathcal{CR}$.*

Proof. Let $\mathfrak{m} = \{(q_1, q_2, \ldots) \mid \text{eventually constant in } (p)\}$, a maximal ideal. Since $\{\mathfrak{m}\} = V((p, p, \ldots))$, $\{\mathfrak{m}\}$ is an isolated point in $\operatorname{Spec_c} R$ and $R_\mathfrak{m} \cong \mathbb{Z}_{(p)}$.

To prove the last remark, let $\alpha, \beta \colon \mathfrak{jc}(R) \to \mathbb{Q}$ be given by $\alpha((s,t)) = q$, where $s \in S$ is eventually $q \in \mathbb{Q}$ and $\beta((s,t)) = t \in \mathbb{Z}_{(p)} \subseteq \mathbb{Q}$. Clearly $\alpha \neq \beta$ but $\alpha \cdot \iota = \beta \cdot \iota$. □

In this context we note that if R is local with finitely generated maximal ideal $\mathfrak{m}$, a copy of R will split off from $\mathfrak{jc}(R)$ because $\mathfrak{m}$ is an isolated point in $\operatorname{Spec_c} R$.

4.4. The universal property of the functor $\mathfrak{jc}$. We next show that, for a ring R, $\mathfrak{jc}(R)$ satisfies a universal property. The universal property of $R^{\mathcal{D}}$ ([**Ca2**, Theorem 6.10]) will be subsumed in that for $\mathfrak{jc}(R)$. We need a preliminary lemma.

LEMMA 4.8. *Let R be a J-clean ring. For any $r \in R$, $\{x \in X(R) \mid r_x \in \mathbf{J}(R_x)\}$ is a clopen set in $X = X(R)$. Moreover, $\operatorname{Spec} R$ and $\operatorname{Spec_c} R$ induce the same topology on the subset $\operatorname{Max} R$.*

PROOF. Given $r \in R$ there is $e \in \mathbf{B}(R)$ with $u = re + 1 - e \in \mathfrak{U}(R)$ and $r(1-e) \in \mathbf{J}(R)$. For $x \in X$, $u_x = r_x \in \mathfrak{U}(R_x)$ if $x \in \operatorname{Supp}(e)$ while $r_x \in \mathbf{J}(R_x)$ if $x \in \operatorname{Supp}(1-e)$. Hence, $\{x \in X \mid r_x \in \mathbf{J}(R_x)\} = \operatorname{Supp}(1-e)$, which is clopen in X. However, X and $\operatorname{Max} R$ are homeomorphic where $\operatorname{Max} R$ has the Zariski topology. To finish the proof it needs to be shown that the basic open sets $D(r) \cap \operatorname{Max} R$ are also closed. Indeed, $\{\mathfrak{m} \in \operatorname{Max} R \mid r \notin \mathfrak{m}\} = \{m \in \operatorname{Max} R \mid r_x \notin \mathbf{J}(R_x)$, where $x = \mathfrak{m} \cap X\}$, which has just been seen to be clopen. □

THEOREM 4.9. *Let R be a ring and $\kappa\colon R \to S$ a homomorphism where S is a J-clean ring. Then, there is a unique homomorphism $\theta\colon \mathfrak{jc}(R) \to S$ whose restriction to R is κ.*

PROOF. We define $\theta(\delta)$, for $\delta \in \mathfrak{jc}(R)$, locally and show that it is a well-defined homomorphism. Let $\phi\colon \max S \to \operatorname{Spec} R$ be induced by κ: The two topologies coincide on $\operatorname{Max} S$ (Lemma 4.8) and ϕ is continuous if its codomain is $\operatorname{Spec} R$ and if it is $\operatorname{Spec_c} R$. For $\mathfrak{m} \in \operatorname{Max} S$, we define $\theta(\delta)_{\mathfrak{m}} \in S_{\mathfrak{m}}$ as follows: Let $\delta_{\phi(\mathfrak{m})} = cd^{-1} \in R_{\phi(\mathfrak{m})}$, $c, d \in R$, $d \notin \phi(\mathfrak{m})$, and put $\theta(\delta)_{\mathfrak{m}} = \kappa(c)\kappa(d)^{-1}$.

In order to show that $\theta(\delta)$ is an element of S it suffices to show that this is true locally. Now δ coincides with cd^{-1} on a $\operatorname{Spec_c} R$-neighbourhood N of $\phi(\mathfrak{m})$. There is an $s \in S$ whose image in $S_{\mathfrak{m}}$ is $\kappa(c)\kappa(d)^{-1}$ and this occurs on a neighbourhood M of $\mathfrak{m}$. It follows that $\theta(\delta)$ and s coincide on $M \cap \phi^{-1}(N)$.

The proofs that θ is a well-defined homomorphism and the uniqueness are now straightforward. □

COROLLARY 4.10. *Let R be embedded in a product $P = \prod_{\alpha \in A} L_\alpha$ of local rings. Then, the homomorphism $\theta\colon \mathfrak{jc}(R) \to D_P(R)$ of Theorem 4.9 is surjective. In fact, the universal property applied to the J-clean ring P yields $\theta'\colon \mathfrak{jc}(R) \to P$ with image $D_P(R)$.*

PROOF. We only need to observe that both $\mathfrak{jc}(R)$ and $D_P(R)$ are generated by R and the elements s^*, $s \in R$ and that $\theta(s^*) = s^* \in D_P(R)$. (See the paragraph before Lemma 4.5 for the notation.) The second part follows for the same reason. □

The statement of [**Ca2**, Corollary 6.8] that if R is a J-clean ring and P is a direct product of local rings then $D_P(R) = R$ is false without more conditions. (Consider $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_{(p)} \times \mathbb{Q}$ via the diagonal map.) However, the claim is true for the *canonical* construction $R^{\mathcal{D}}$, which is done as follows. For any ring R, R embeds naturally in $M = \prod_{\mathfrak{m} \in \operatorname{Max} R} R_{\mathfrak{m}}$; the resulting J-clean ring $D_M(R)$ is denoted $R^{\mathcal{D}}$ in [**Ca2**] and called the *canonical D-ring* for R (and the *$\mathcal{D}$-enveloppe* in [**CL**]).

PROPOSITION 4.11. *Let R be a J-clean ring which is embedded via $\iota\colon R \to P$ in a direct product $P = \prod_{\alpha \in A} L_\alpha$ of local rings with the maximal ideal of L_α denoted by $\mathfrak{m}_\alpha$. For each $\beta \in A$, let $\mathfrak{n}_\beta = \{(l_\alpha) \in P \mid l_\beta \in \mathfrak{m}_\beta\}$. Suppose that for each*

$\alpha \in A$, $\iota^{-1}(\mathfrak{n}_\alpha) \in \operatorname{Max}(R)$. Then, $R = D_P(R)$. In particular, this occurs when $M = \prod_{\mathfrak{m}\in\operatorname{Max}(R)} R_\mathfrak{m}$, i.e., $R = R^\mathcal{D}$.

Proof. The notation preceding Lemma 4.5 is again used. It needs to be shown that for each $r \in R$, $r^* \in \iota(R)$. There is $e \in \mathbf{B}(R)$ with $u = re + (1-e) \in \mathfrak{U}(R)$. Then, $e \in \mathfrak{m} \in \operatorname{Max}(R)$ if and only if $\mathfrak{m} \in D(r) \subseteq \operatorname{Spec} R$. It follows that $\iota(u^{-1}e)_\alpha$ is $\iota(r)_\alpha^{-1}$ exactly when $\iota^{-1}(\mathfrak{n}_\alpha) \in D(r)$ and is 0 otherwise. In other words, $\iota(u^{-1}e) = r^*$. □

The construction of $R^\mathcal{D}$, however, is not functorial since inverse images of maximal ideals are not necessarily maximal. We will now compare $\mathfrak{jc}(R)$ and $R^\mathcal{D}$. In [**CL**, Théorème I.2.3], it is shown that $\operatorname{Max} R$ embeds naturally in $\operatorname{Max} R^\mathcal{D}$ as a dense subset. The space $\operatorname{Max} R^\mathcal{D}$ is described in some specific cases ([**CL**, 3. Exemple]). We will supply a description of $\operatorname{Max} R^\mathcal{D}$ for any ring R.

Let us first recall the identification of $\operatorname{Max} R$ as a subspace of $\operatorname{Max} R^\mathcal{D}$ ([**CL**]). For $\mathfrak{m} \in \operatorname{Max} R$, $\mathfrak{m}' = \{\delta \in R^\mathcal{D} \mid \delta_\mathfrak{m} \in \mathfrak{m}R_\mathfrak{m}\}$. Let $\mathfrak{M}' = \{\mathfrak{m}' \mid \mathfrak{m} \in \operatorname{Max} R\}$. Then, $\mathfrak{M}'$ is Zariski dense in $\operatorname{Max} R^\mathcal{D}$ ([**CL**, Théorème I.2.3]). Moreover, the Zariski topology on $\mathfrak{M}'$, as a subspace of $\operatorname{Spec} R^\mathcal{D}$, is finer than that on $\operatorname{Max} R$ ([**CL**, Théorème I.2.3 (i)]). The homomorphism given by Theorem 4.9 applied to $R^\mathcal{D}$ will be denoted $\theta_\mathcal{D} \colon \mathfrak{jc}(R) \to R^\mathcal{D}$.

Theorem 4.12. *For a ring R, put $K = \operatorname{cl}_{\operatorname{Spec_c} R}(\operatorname{Max} R)$. The homomorphism $\theta_\mathcal{D} \colon \mathfrak{jc}(R) \to R^\mathcal{D}$ is surjective and $\operatorname{Max} R^\mathcal{D}$ is homeomorphic with K.*

Proof. Recall that $R^\mathcal{D}$ was defined as a subring of $M = \prod_{\mathfrak{m}\in\operatorname{Max} R} R_\mathfrak{m}$ and we define $\Theta \colon \mathfrak{jc}(R) \to R^\mathcal{D}$ by $\Theta(\delta)_\mathfrak{m} = cd^{-1} \in R_\mathfrak{m}$, where $\delta_\mathfrak{m} = cd^{-1} \in R_\mathfrak{m}$. When $\mathfrak{jc}(R)$ is viewed as the subring $D_P(R)$ of $P = \prod_{\mathfrak{p}\in\operatorname{Spec_c} R} R_\mathfrak{p}$, we see that $\Theta \colon D_P(R) \to D_M(R) = R^\mathcal{D}$ is a well-defined surjection.

By the uniqueness of $\theta_\mathcal{D}$ (Theorem 4.9) it follows that $\Theta = \theta_\mathcal{D}$. Now $\ker \theta_\mathcal{D} = \{\delta \in \mathfrak{jc}(R) \mid \delta_\mathfrak{m} = 0 \in R_\mathfrak{m} \text{ for all } \mathfrak{m} \in \operatorname{Max} R\}$. We can thus identify $\operatorname{Max} R^\mathcal{D}$ with $V(\ker\theta_\mathcal{D}) \cap \operatorname{Max}\mathfrak{jc}(R)$, i.e, $V(\ker\theta_\mathcal{D}) \cap \operatorname{Max}\mathfrak{jc}(R)$ as a subspace of $\operatorname{Spec_c} R$. We will show that this is K.

Clearly, if $\delta \in \mathfrak{jc}(R)$ is zero on K, it is in $\ker\theta_\mathcal{D}$. On the other hand, if $\mathfrak{p} \notin K$ then there is a $\operatorname{Spec_c} R$-neighbourhood N of $\mathfrak{p}$ not meeting $\operatorname{Max} R$. Since $\operatorname{Spec_c} R$ is a boolean space, we may assume that N is a clopen set. However, a clopen set of $\operatorname{Max}\mathfrak{jc}(R)$ corresponds to an idempotent $e \in \mathbf{B}(\mathfrak{jc}(R))$; i.e, $e \in \mathfrak{q}$ if and only if $\mathfrak{q} \in N$. Then, $1 - e \in \ker\theta_\mathcal{D}$ while $(1-e)_\mathfrak{p} \neq 0$. □

While Theorem 4.12 describes $\operatorname{Max} R^\mathcal{D}$ in general, it is instructive to look at an important special case, that of $C(X)$, X a topological space. Recall that an ideal of $C(X)$ determined by the zero-sets of its elements is called a *z-ideal* ([**GJ**, 2.7]). The set of prime z-ideals in $C(X)$ is denoted $\operatorname{Spec_Z} C(X)$ and, for $x \in X$, $M_x = \{f \in C(X) \mid f(x) = 0\}$ is a maximal ideal. We need the following information about $\operatorname{Spec_Z} C(X)$ ([**S**, Theorem 3.2]): $\operatorname{Spec_Z} C(X)$ is closed in $\operatorname{Spec_c} C(X)$ and $\{M_x \mid x \in X\}$ is dense in $\operatorname{Spec_Z} C(X)$ as a subset of $\operatorname{Spec_c} C(X)$. We have the following.

Example 4.13. *For a topological space X, $\operatorname{Max} C(X)^\mathcal{D}$ is $\operatorname{Spec_Z} C(X)$ as a subspace of $\operatorname{Spec_c} C(X)$.*

Each ring $C(X)$ can be embedded in a regular ring called $G(X)$, the smallest regular ring containing $C(X)$ lying in $F(X) = \prod_{x\in X}\mathbb{R}$. Its description in terms of elements (see [**HRW**, Theorem 1.1] and the references found there) shows that $G(X) = D_{F(X)}C(X)$. It is not hard to see, although the topic will not be pursued here, that $G(X) \cong C(X)^{\mathcal{D}}/\,\mathbf{J}(C(X)^{\mathcal{D}})$. It was already known that $\operatorname{Spec} G(X) = \operatorname{Max} G(X) = \operatorname{Spec}_{\mathrm{Z}} C(X)$, with the constructible topology ([**BBR**, Proposition 3.3]).

As a final observation we recall ([**Ca2**, Theorem 6.10]) that $R^{\mathcal{D}}$ also satisfies a universal property with respect to those $D_P(R)$ where R is embedded as a *subdirect product* in a product $P = \prod_{\alpha\in A} L_\alpha$ of local rings. Let $f\colon R^{\mathcal{D}} \to D_P(R)$ be as given in this universal property. Then, $f \circ \theta_{\mathcal{D}}\colon \mathfrak{jc}(R) \to D_P(R)$ coincides with $\theta\colon \mathfrak{jc}(R) \to D_P(R)$ given by Theorem 4.9 since θ is unique and $\theta_{\mathcal{D}}$ is surjective.

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