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Chapter 2

GENDER DIFFERENCES IN HETEROSEXUAL TRANSMISSION OF HIV IN URBAN AND RURAL POPULATIONS

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Abstract

We explore effect of disease spread in both urban and rural populations for heterosexual transmission. We develop a two-sex model for the spread of HIV using ordinary differential equations. We then use two methods to calculate the basic reproductive ratio (R_0) and demonstrate that one is more biologically reasonable than the other. Furthermore, including gender differences can have a large quantitative effect on our choice of intervention strategies. We use numerical simulations to explore the impact of several possible intervention strategies against the disease. These results suggest that focusing on the "weaker" sex, i.e., the sex with the higher risk of being infected, will have a greater impact on slowing the spread of the disease than focusing on the more infectious sex. We also demonstrate that infection in the rural population can be sustained by sexual mixing in urban centers.

Keywords: HIV, mathematical model, differential equations, gender, urban, rural.

1 Introduction

Currently, HIV infects approximately 33 million individuals worldwide, 68% of whom are in sub-Saharan Africa [1]. The epidemic in southern Africa, which is spreading largely

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through heterosexual exposure, is driven by high rates of labour migration, concurrent sexual partnerships and gender inequalities [2]. Women are infected in greater numbers than men in sub-Saharan Africa [3]. Intravaginal practices, such as washing, douching, wiping and inserting substances into the vagina have also been associated with a higher prevalence of HIV [4, 5]. The highest risk group in rural Uganda are young married women [6]. Factors such as economic dependence, gender discrimination and neglect of women's sexuality increase a woman's risk for HIV [3]. Marriage has been implicated as a risk factor for HIV infection in young African women [7], while HIV acquisition is significantly higher for pregnant women [8].

In many developing countries, shifting population demographics have driven rural men to seek work in urban centres [9]. Prevalence and risk factors of HIV-1 and HIV-2 infections vary in urban and rural areas [10]. Migrant men in South Africa are 2.4 times more likely to be HIV infected than non-migrant men [11]. While HIV prevalence in rural areas is relatively low, rural men engaging in risky sex in urban areas subsequently infect women in rural areas, who can then infect further rural men [12].

A number of mathematical models have been developed to account for these effects. Blower *et al.* [13] demonstrated that the allocation of HIV medication between urban and rural settings has a significant effect on the outcome. Optimal public-health outcomes can be achieved by allocating all resources in the urban centres. However, this is not ethical and unlikely to be implemented. Consequently, the authors demonstrate that there is an optimal division of resources between urban and rural settings that facilitates equitable access to medication for all infected individuals. Renton *et al.* [14] use a two-sex model to demonstrate the importance of promoting sexually transmitted disease control as a major element of HIV prevention. Robinson *et al.* [15] used a simulation model to asses the impact of a variety of intervention strategies in rural Uganda. Gregson *et al.* [16] used mathematical models to analyse sexual mixing patterns in rural Zimbabwe. Mekonnen *et al.* [9] modelled the demographic impact of HIV in urban Ethiopia. Coffee *et al.* [17] modelled the impact of migration in South Africa and showed that frequent return of migrants is an important risk factor for HIV.

Here, we develop a mathematical model to examine the effects of gender differences in urban-rural populations. We model heterosexual sex, since that is the primary route of adult HIV transmission in Africa [12]. We pose the following research questions: 1. Can urban transmission sustain infection in rural areas? 2. Which intervention strategies will have the greatest effect on the outcome? 3. How do gender differences affect our choice of intervention strategy?

This chapter is organised as follows. In section 2, we introduce the mathematical model. In section 3, we analyse the model. In section 4, we illustrate the results with numerical simulations. In section 5, we explore the effect of possible intervention strategies. Finally, in section 6, we discuss the implications of the results. Matlab codes used to generate the figures are displayed in the Appendix.

2 The Model

The model consists of an urban community, in equations indicated by an index u, and a rural community, indicated by r. In each division, the population N is split into female (X) and

male (Y). Susceptible individuals and infected ones are denoted by S and I, respectively. The removal rate from the sexually active population is denoted by μ and the (constant) rate in which individuals join the sexually active population is η.

The parameter β is the product of the contact rate between the two genders and the transmission probability. Nicolosi *et al.* [18] point out that the probability of transmission depends on which partner is infected. For convenience, this will be reflected by the parameter α, the degree of differential infection, that occurs in all the female terms.

Critical to the model is the assumption that individuals from the rural community travel (for example to work) to the urban community and have sexual contact there. We denote the mixing probability by c, such that c_y, for example, is the probability that a male from the rural community has contact with an urban female within the urban region. The much more unlikely case of an urban individual having contacts in the rural community is not considered.

We will now introduce the model for the urban region. The change rate of the susceptibles consists of three parts. First, there is the constant inflow η^u. Second, we have the rate by which (healthy) individuals conclude their sexually active phase, μ_s. Third, there is the rate that describes the loss towards the infected population. This is a product of the contact rate β and the probability of actually having an infected partner. An urban woman may meet either an urban man, or a rural man, who has the probability c_y of having a contact in the city. Thus, the chance that an urban woman encounters an infected man in the city is

$$\frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r}\,.$$

Hence, the urban model is

$$\begin{aligned}
N^u &= X^u + Y^u && (2.1)\\
X^u &= I_x^u + S_x^u && (2.2)\\
Y^u &= I_y^u + S_y^u && (2.3)\\
\frac{dS_x^u}{dt} &= \eta^u - \mu_{sx} S_x^u - \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot \beta \cdot S_x^u && (2.4)\\
\frac{dI_x^u}{dt} &= \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot \beta \cdot S_x^u - \mu_{ix} I_x^u && (2.5)\\
\frac{dS_y^u}{dt} &= \eta^u - \mu_{sy} S_y^u - \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} \cdot \beta \cdot S_y^u && (2.6)\\
\frac{dI_y^u}{dt} &= \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} \cdot \beta \cdot S_y^u - \mu_{iy} I_y^u\,. && (2.7)
\end{aligned}$$

The equations are similar in the rural community. The difference is that rural individuals may have both rural contacts as well as urban contacts. Additionally, they could also meet rural individuals in the urban region. Thus, the chance that a rural woman encounters an infected urban man is

$$\frac{I_y^r}{Y^r} + c_x \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r}\,.$$

Hence, the rural model is

$$
\begin{aligned}
N^r &= X^r + Y^r && (2.8)\\
X^r &= I_x^r + S_x^r && (2.9)\\
Y^r &= I_y^r + S_y^r && (2.10)\\
\frac{dS_x^r}{dt} &= \eta^r - \mu_{sx} S_x^r - \left[\frac{I_y^r}{Y^r} + c_x \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r}\right] \cdot \alpha \cdot \beta \cdot S_x^r && (2.11)\\
\frac{dI_x^r}{dt} &= \left[\frac{I_y^r}{Y^r} + c_x \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r}\right] \cdot \alpha \cdot \beta \cdot S_x^r - \mu_{ix} I_x^r && (2.12)\\
\frac{dS_y^r}{dt} &= \eta^r - \mu_{sy} S_y^r - \left[\frac{I_x^r}{X^r} + c_y \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r}\right] \cdot \beta \cdot S_y^r && (2.13)\\
\frac{dI_y^k}{dt} &= \left[\frac{I_x^r}{X^r} + c_y \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r}\right] \cdot \beta \cdot S_y^r - \mu_{iy} I_y^r \,. && (2.14)
\end{aligned}
$$

The model is illustrated in Figure 1.

3 Analysis

We will now derive the basic reproductive ratio for our model. Suppose the system is in a state where almost no individuals are infected. Let the proportion of females in the infected population be denoted by $p \in [0,1]$, i.e. $I_x^u = p \cdot I^u$. Then, for the urban population, we find

$$
\begin{aligned}
\frac{dI^u}{dt} &= \frac{dI_x^u}{dt} + \frac{dI_y^u}{dt}\\
&= \left(\frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot S_x^u + \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} S_y^u\right)\beta - (\mu_{ix} I_x^u + \mu_{iy} I_y^u)\\
&= \left(\frac{I_y^u}{Y^u + c_y Y^r} + c_y \frac{I_y^r}{Y^u + c_y Y^r}\right) \cdot \alpha \cdot \beta \cdot S_x^u\\
&\quad + \left(\frac{I_x^u}{X^u + c_x X^r} + c_x \frac{I_x^r}{X^u + c_x X^r}\right) \cdot \beta \cdot S_y^u\\
&\quad - (\mu_{ix} \cdot p + \mu_{iy} \cdot (1-p)) I^u\\
&= \left(\frac{S_x^u}{Y^u + c_y Y^r} \cdot I_y^u + c_y \frac{S_x^u}{Y^u + c_y Y^r} \cdot I_y^r\right) \cdot \alpha \cdot \beta\\
&\quad + \left(\frac{S_y^u}{X^u + c_x X^r} \cdot I_x^u + c_x \frac{S_y^u}{X^u + c_x X^r} \cdot I_x^r\right) \cdot \beta\\
&\quad - (\mu_{ix} \cdot p + \mu_{iy} \cdot (1-p)) I^u
\end{aligned}
$$

Approximation 3.1. *Since the number of rural individuals is much smaller than the number of urban individuals, we set*

$$
\frac{S_x^u}{Y^u + c_y Y^r} \approx \frac{S_x^u}{Y^u} \qquad \textit{and} \qquad \frac{I_y^r}{Y^u} = \frac{I_x^r}{X^u} \approx 0\,.
$$

We thus have

$$\frac{dI^u}{dt} \approx \left(\alpha\cdot(1-p)\cdot\frac{S_x^u}{Y^u}+p\cdot\frac{S_y^u}{X^u}\right)\cdot\beta\cdot I^u-(\mu_{ix}\cdot p+\mu_{iy}\cdot(1-p))I^u\,.$$

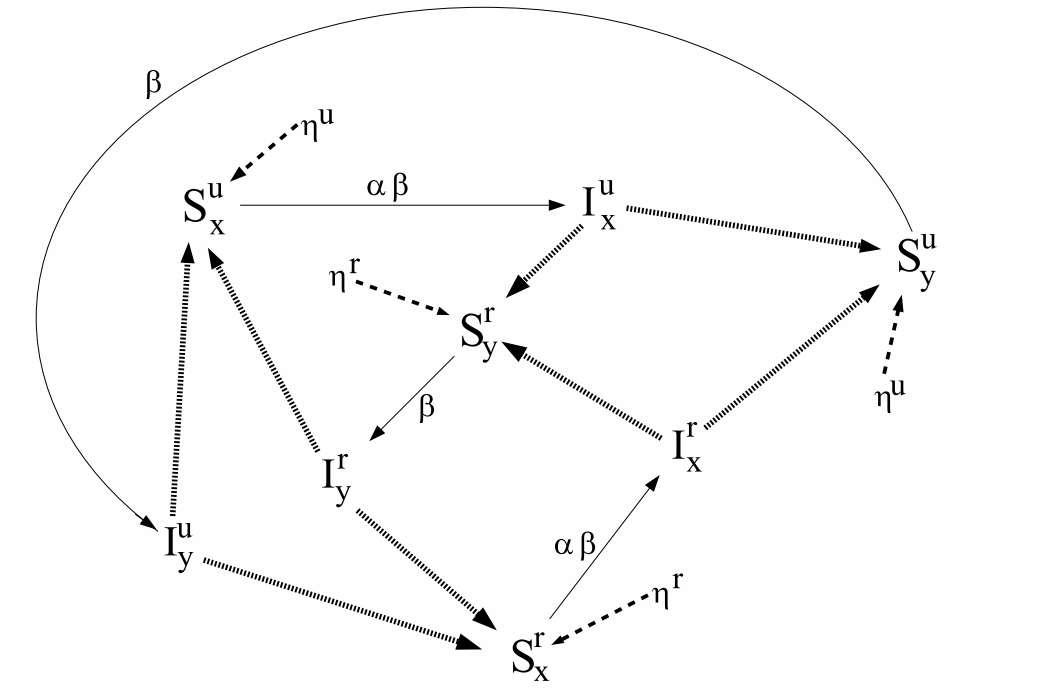

Figure 1: The diagram visualises the relationships between the eight compartments. A dashed arrow indicates the inflow rate which affects only the susceptible population. A solid thin arrow shows where the susceptible people may convert to; the probability is proportional to β and $\alpha\beta$, respectively, depending on the sex. The thick arrows point out which infected population influences which susceptible population. In order to not overload the diagram, we left out the removal rates, which would affect every compartment. The removal rates differ between the sexes, and between the infected population and the susceptible population.

Approximation 3.2. *We make the further approximation that the male and female population are roughly equal in size. Thus*

$$\frac{S_x^u}{Y^u}\approx 1 \qquad \textit{and} \qquad p=\frac{1}{2}\,.$$

The same approximations are also valid if x and y, or X and Y, are interchanged.

Hence,

$$\frac{dI^u}{dt}\approx\frac{1}{2}\left((\alpha+1)\beta-\mu_{ix}-\mu_{iy}\right)I^u\equiv r_{0,1}\cdot I^u\,.$$

The value of the intrinsic growth rate $r_{0,1}$ is critical for the behaviour of the disease. If $r_{0,1}$ is positive, then, the number of infected people increases. Rephrasing, this means that the disease will persist if

$$R_{0,1} = \frac{(\alpha+1)\beta}{\mu_{ix}+\mu_{iy}}$$

is greater than 1, whereas the disease will be eradicated if $R_{0,1} < 1$. The same calculation can be done for the rural population and, using Approximations 3.1 and 3.2 accordingly, leads to the same result. In general, the important parameter R_0 is defined as the average number of susceptible people that an infected person infects during their sexually active phase. Then, it is clear that the total number of infected people can only decrease if $R_0 < 1$. However, as Heffernan *et al.* [19] make clear, one should be very careful when it comes to such a concrete meaning for R_0; it is often not clear what the appropriate choice would be. Surrogate thresholds like $R_{0,1}$, although not necessarily the average number of secondary infections, retain the same threshold property: persistence results if the value is greater than 1 and eradication results if the value is less than 1.

However, as we shall see, there are potential issues with this value. Thus, we shall try another approach to find a better R_0. A widely used method is invasion analysis with Jacobian matrices. The interested reader can find a useful introduction in [20]. To do invasion analysis, let J be the 8×8 Jacobian matrix for our system. Then, $J = (J_1|J_2|J_3|J_4)$ with

$$J_1 = \begin{pmatrix}
-\mu_{sx} - \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot \beta & 0 \\
\frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot \beta & -\mu_{ix} \\
\frac{I_x^u + c_x I_x^r}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^u & -\frac{(X^u + c_x X^r) - (I_x^u + c_x I_x^r)}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^u \\
\frac{I_x^u + c_x I_x^r}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^u & \frac{(X^u + c_x X^r) - (I_x^u + c_x I_x^r)}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^u \\
0 & 0 \\
0 & 0 \\
c_y \frac{I_x^u + c_x I_x^r}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^r & -c_y \frac{(X^u + c_x X^r) - (I_x^u + c_x I_x^r)}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^r \\
-c_y \frac{I_x^u + c_x I_x^r}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^r & c_y \frac{(X^u + c_x X^r) - (I_x^u + c_x I_x^r)}{(X^u + c_x X^r)^2} \cdot \beta \cdot S_y^r
\end{pmatrix}$$

$$J_2 = \begin{pmatrix}
\frac{I_y^u + c_y I_y^r}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^u & -\frac{(Y^u + c_y Y^r) - (I_y^u + c_y I_y^r)}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^u \\
-\frac{I_y^u + c_y I_y^r}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^u & \frac{(Y^u + c_y Y^r) - (I_y^u + c_y I_y^r)}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^u \\
-\mu_{sy} - \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} \cdot \beta & 0 \\
\frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} \cdot \beta & -\mu_{iy} \\
c_x \frac{I_y^u + c_y I_y^r}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^r & -c_x \frac{Y^u + c_y Y^r - (I_y^u + c_y I_y^r)}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^r \\
-c_x \frac{I_y^u + c_y I_y^r}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^r & c_x \frac{Y^u + c_y Y^r - (I_y^u + c_y I_y^r)}{(Y^u + c_y Y^r)^2} \cdot \alpha \cdot \beta \cdot S_x^r \\
0 & 0 \\
0 & 0
\end{pmatrix}$$

$$J_3 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ \frac{c_x(I_x^u+c_xI_x^r)}{(X^u+c_xX^r)^2}\cdot\beta\cdot S_y^u & -\frac{c_x(X^u+c_xX^r)-(I_x^u+c_xI_x^r)c_x}{(X^u+c_xX^r)^2}\cdot\beta\cdot S_y^u \\ -\frac{c_x(I_x^u+c_xI_x^r)}{(X^u+c_xX^r)^2}\cdot\beta\cdot S_y^u & \frac{c_x(X^u+c_xX^r)-(I_x^u+c_xI_x^r)c_x}{(X^u+c_xX^r)^2}\cdot\beta\cdot S_y^u \\ -\mu_{sx}-\left[\frac{I_y^r}{Y^r}+c_x\frac{I_y^u+c_yI_y^r}{Y^u+c_yY^r}\right]\cdot\alpha\cdot\beta & 0 \\ \left[\frac{I_y^r}{Y^r}+c_x\frac{I_y^u+c_yI_y^r}{Y^u+c_yY^r}\right]\cdot\alpha\cdot\beta & -\mu_{ix} \\ \left[\frac{I_x^r}{(X^r)^2}+c_y\frac{I_x^u+c_xI_x^r}{(X^u+c_xX^r)^2}\right]\cdot\beta\cdot S_y^r & -\left[\frac{1}{X^r}+c_y\frac{(X^u+c_xX^r)c_x-c_x(I_x^u+c_xI_x^r)}{(X^u+c_xX^r)^2}\right]\cdot\beta\cdot S_y^r \\ -\left[\frac{I_x^r}{(X^r)^2}+c_y\frac{I_x^u+c_xI_x^r}{(X^u+c_xX^r)^2}\right]\cdot\beta\cdot S_y^r & \left[\frac{1}{X^r}+c_y\frac{(X^u+c_xX^r)c_x-c_x(I_x^u+c_xI_x^r)}{(X^u+c_xX^r)^2}\right]\cdot\beta\cdot S_y^r \end{vmatrix}$$

and $J_4 =$

$$\begin{vmatrix} \frac{c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\cdot\alpha\cdot\beta\cdot S_x^u & -\frac{(Y^u+c_yY^r)c_y-c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\cdot\alpha\cdot\beta\cdot S_x^u \\ -\frac{c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\cdot\alpha\cdot\beta\cdot S_x^u & \frac{(Y^u+c_yY^r)c_y-c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\cdot\alpha\cdot\beta\cdot S_x^u \\ 0 & 0 \\ 0 & 0 \\ \left[\frac{I_y^r}{(Y^r)^2}+c_x\frac{c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\right]\cdot\alpha\cdot\beta\cdot S_x^r & -\left[\frac{Y^r-I_y^r}{(Y^r)^2}+c_x\frac{c_y(Y^u+c_yY^r)-c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\right]\cdot\alpha\cdot\beta\cdot S_x^r \\ -\left[\frac{I_y^r}{(Y^r)^2}+c_x\frac{c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\right]\cdot\alpha\cdot\beta\cdot S_x^r & \left[\frac{Y^r-I_y^r}{(Y^r)^2}+c_x\frac{c_y(Y^u+c_yY^r)-c_y(I_y^u+c_yI_y^r)}{(Y^u+c_yY^r)^2}\right]\cdot\alpha\cdot\beta\cdot S_x^r \\ -\mu_{sy}-\left[\frac{I_x^r}{X^r}+c_y\frac{I_x^u+c_xI_x^r}{X^u+c_xX^r}\right]\cdot\beta\cdot S_y^r & 0 \\ \left[\frac{I_x^r}{X^r}+c_y\frac{I_x^u+c_xI_x^r}{X^u+c_xX^r}\right]\cdot\beta\cdot S_y^r & -\mu_{iy} \end{vmatrix}.$$

The disease-free state satisfies

$$(S_x^u, I_x^u, S_y^u, I_y^u, S_x^r, I_x^r, S_y^r, I_y^r) = \left(\frac{\eta^u}{\mu_{sx}}, 0, \frac{\eta^u}{\mu_{sy}}, 0, \frac{\eta^r}{\mu_{sx}}, 0, \frac{\eta^r}{\mu_{sy}}, 0\right).$$

Then, the Jacobian evaluated at this point becomes $J(S_x^u, 0, S_y^u, 0, S_x^r, 0, S_y^r, 0) = (J_5|J_6)$, where

$$J_5 = \begin{pmatrix} -\mu_{sx} & 0 & 0 & -\frac{1}{Y^u+c_yY^r}\cdot\beta\cdot S_x^u \\ 0 & -\mu_{ix} & 0 & \frac{1}{Y^u+c_yY^r}\cdot\beta\cdot S_x^u \\ 0 & -\frac{1}{X^u+c_xX^r}\cdot\beta\cdot S_y^u & -\mu_{sy} & 0 \\ 0 & \frac{1}{X^u+c_xX^r}\cdot\beta\cdot S_y^u & 0 & -\mu_{iy} \\ 0 & 0 & 0 & -c_x\frac{1}{Y^u+c_yY^r}\cdot\beta\cdot S_x^r \\ 0 & 0 & 0 & c_x\frac{1}{Y^u+c_yY^r}\cdot\beta\cdot S_x^r \\ 0 & -c_y\frac{1}{X^u+c_xX^r}\cdot\beta\cdot S_y^r & 0 & 0 \\ 0 & c_y\frac{1}{X^u+c_xX^r}\cdot\beta\cdot S_y^r & 0 & 0 \end{pmatrix}$$

and

$$J_6 = \begin{pmatrix} 0 & 0 & 0 & -\frac{c_y}{Y^u+c_yY^r}\cdot\beta\cdot S_x^u \\ 0 & 0 & 0 & \frac{c_y}{Y^u+c_yY^r}\cdot\beta\cdot S_x^u \\ 0 & -\frac{c_x}{X^u+c_xX^r}\cdot\beta\cdot S_y^u & 0 & 0 \\ 0 & \frac{c_x}{X^u+c_xX^r}\cdot\beta\cdot S_y^u & 0 & 0 \\ -\mu_{sx} & 0 & 0 & -\left[\frac{1}{Y^r}+c_x\frac{c_y}{Y^u+c_yY^r}\right]\cdot\beta\cdot S_x^r \\ 0 & -\mu_{ix} & 0 & \left[\frac{1}{Y^r}+c_x\frac{c_y}{Y^u+c_yY^r}\right]\cdot\beta\cdot S_x^r \\ 0 & -\left[\frac{1}{X^r}+c_y\frac{c_x}{X^u+c_xX^r}\right]\cdot\beta\cdot S_y^r & -\mu_{sy} & 0 \\ 0 & \left[\frac{1}{X^r}+c_y\frac{c_x}{X^u+c_xX^r}\right]\cdot\beta\cdot S_y^r & 0 & -\mu_{iy} \end{pmatrix}$$

Four of the eigenvalues are $-\mu_{sx}$, $-\mu_{sy}$, $-\mu_{sx}$ and $-\mu_{sy}$, and are thus negative. For the behaviour of the steady state $(\frac{\eta^u}{\mu_{sx}},0,\frac{\eta^u}{\mu_{sy}},0,\frac{\eta^r}{\mu_{sx}},0,\frac{\eta^r}{\mu_{sy}},0)$, we need to find the eigenvalues of $H=(H_1|H_2)$, where

$$H_1 = \begin{pmatrix} -\mu_{ix} & \frac{\mu_{sy}}{\mu_{sx}}\cdot\frac{\eta^u}{\eta^u+c_y\eta^r}\cdot\alpha\cdot\beta \\ \frac{\mu_{sx}}{\mu_{sy}}\cdot\frac{\eta^u}{\eta^u+c_x\eta^r}\cdot\beta & -\mu_{iy} \\ 0 & c_x\frac{\mu_{sy}}{\mu_{sx}}\cdot\frac{\eta^r}{\eta^u+c_y\eta^r}\cdot\alpha\cdot\beta \\ c_y\frac{\mu_{sx}}{\mu_{sy}}\cdot\frac{\eta^r}{\eta^u+c_y\eta^r}\cdot\beta & 0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 0 & c_y\frac{\mu_{sy}}{\mu_{sx}}\cdot\frac{\eta^u}{\eta^u+c_y\eta^r}\cdot\alpha\cdot\beta \\ c_x\frac{\mu_{sx}}{\mu_{sy}}\cdot\frac{\eta^u}{\eta^u+c_x\eta^r}\cdot\beta & 0 \\ -\mu_{ix} & \frac{\mu_{sy}}{\mu_{sx}}\left[1+c_xc_y\frac{\eta^r}{\eta^u+c_y\eta^r}\right]\cdot\alpha\cdot\beta \\ \frac{\mu_{sx}}{\mu_{sy}}\left[1+c_yc_x\frac{\eta^r}{\eta^u+c_x\eta^r}\right]\cdot\beta & -\mu_{iy} \end{pmatrix}$$

In order to simplify matters, we will use the following approximation:

Approximation 3.3. *As there are many more urban individuals than rural, the constant urban inflow rate η^u is much bigger than the rural inflow rate, η^r. Furthermore, c_x and c_y are both much smaller than 1, so we shall use the approximations*

$$\frac{\eta^u}{\eta^u+c_b\eta^r}\approx 1 \qquad \textit{and} \qquad \frac{\eta^r}{\eta^u+c_b\eta^r}\approx 0$$

for $b\in\{x,y\}$.

For convenience, abbreviate $\gamma=\frac{\mu_{sx}}{\mu_{sy}}$. Then, H simplifies to

$$\tilde{H} = \begin{pmatrix} -\mu_{ix} & \alpha\beta\gamma^{-1} & 0 & c_y\alpha\beta\gamma^{-1} \\ \beta\gamma & -\mu_{iy} & c_x\beta\gamma & 0 \\ 0 & 0 & -\mu_{ix} & \alpha\beta\gamma^{-1} \\ 0 & 0 & \beta\gamma & -\mu_{iy} \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2}=\frac{1}{2}\left(-\mu_{ix}-\mu_{iy}\pm\sqrt{(\mu_{ix}-\mu_{iy})^2+4\alpha\beta^2}\right)$$

(each one with algebraic multiplicity 2). As the expression under the square root is non-negative, all eigenvalues will be real. Thus, we have six negative eigenvalues. In order for the seventh and eighth (they are the same) to be negative, we would need

$$\begin{aligned} \sqrt{(\mu_{ix}-\mu_{iy})^2+4\alpha\beta^2} &> \mu_{ix}+\mu_{iy} \\ \iff \qquad \mu_{ix}\mu_{iy} &> \alpha\beta^2 . \end{aligned}$$

Thus, we define

$$R_{0,2} \equiv \frac{\alpha\beta^2}{\mu_{ix}\mu_{iy}} = \frac{\alpha\beta}{\mu_{iy}} \cdot \frac{\beta}{\mu_{ix}}$$

Again, the value $R_{0,2} = 1$ is the critical point. If $R_{0,2} > 1$, then we expect the disease to spread as we have an unstable disease-free steady state.

The value of $R_{0,1}$ has a threshold when

$$\begin{aligned} \beta-\mu_{ix}+\alpha\beta-\mu_{iy} &= 0 \\ \frac{1}{\mu_{iy}}\left(\frac{\beta}{\mu_{ix}}-1\right)+\frac{1}{\mu_{ix}}\left(\frac{\beta}{\mu_{iy}}-1\right) &= 0 . \end{aligned}$$

However, when $R_{0,1} = 1$, it is not true in general that $R_{0,2} = 1$. For example, if $\beta = 9$, $\mu_{ix} = 10$, $\alpha = 111/9$ and $\mu_{iy} = 100$, then,

$$\begin{aligned} R_{0,1} &= \frac{12}{11} > 1 \\ R_{0,2} &= 0.999 < 1 . \end{aligned}$$

We shall argue that $R_{0,2}$ is more biologically meaningful.

The first term of $R_{0,2}$ is the number of females that an infected male infects in his active phase, the second term the number of males a female infects. Suppose, for example, a male infects four females and a female infects two males, on average. Then, the proper R_0 in this process will be 8, as shown in Figure 2.

Of course, the same result is gained if, in Figure 2, we had started from an infected female. Interestingly enough, this example shows that, for the reproductive ratio, it does not matter whether the factor α is associated with either males or females. Thus, henceforth, when we refer to R_0, we are referring to $R_{0,2}$.

4 Numerical Simulations

4.1 The choice of parameter values

A typical African city, like Durban in KwaZulu-Natal, has around 3.5 million inhabitants. Blower *et al.* [13] suggest 400-4,000 villages, with populations ranging from 1,300 to 13,000 individuals. For convenience, we will choose 3,500 inhabitants. The average number of new sex partners per person per year is given as 0.5-1.5 in [13], but ranges from 0-18.1 in [14]. The latter also suggests a per-partnership transmission probability ranging from 0.01 to 0.1, so we model the worst-case scenario by using the latter.

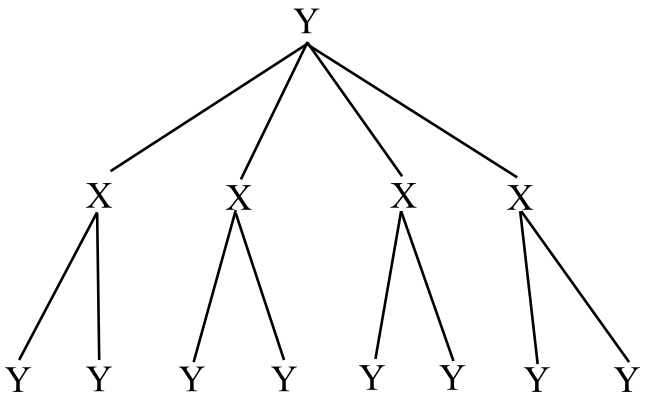

Figure 2: An example of $R_{0,2}$.

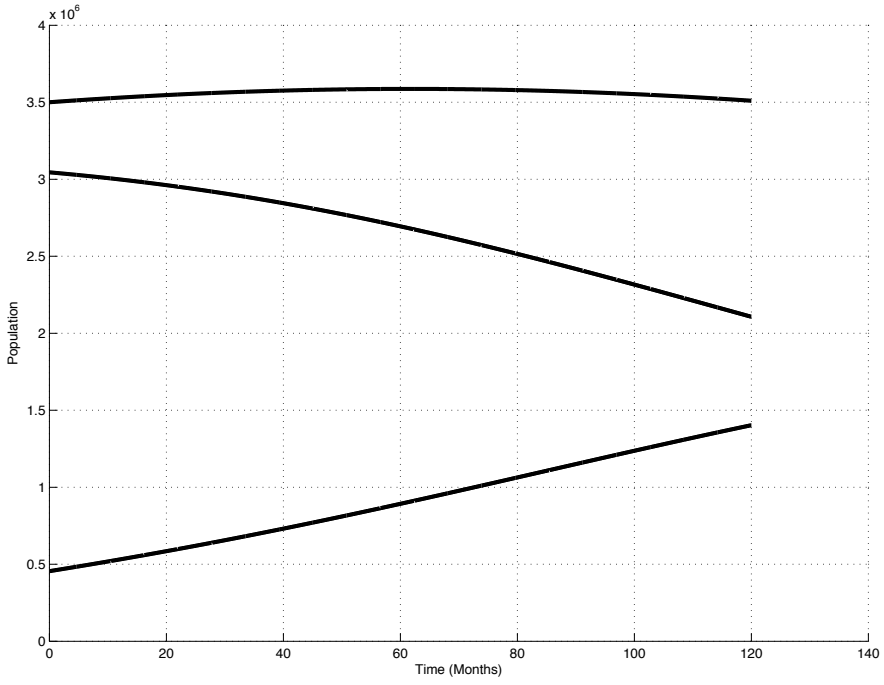

Figure 3: Ten year urban timecourse without intervention. The top curve represents the total urban population, the middle one shows the number of susceptible urban individuals and the bottom curve shows the number of infected urban individuals. The curves for the rural population are similar, but on a much smaller scale (not shown).

The time period that susceptible and infected individuals are sexually active is 30 and 10 years, respectively, as suggested in [14]. For the urban-rural contact rate, [13] uses several random variables, namely $c = \frac{fc}{1+d}$, where $f \in (0,1)$, $c \in (0.5, 1.5)$ and $d \in (10, 100)$. As a mean, we will use $\frac{0.5 \cdot 1}{50} = 0.01$ (per year), or 0.00085 per month. As it is rare that females travel to the city, we will set $c_x = 0$. See Table 1.

Table 1: Abbreviations and parameter values

Variable	Meaning	Value	Source
N^a	Total number of individuals in community $a \in \{u,r\}$	$N^u(0) = 3,500,000$ $N^r(0) = 3,500$	[13]
X^a	Total number of females in community $a \in \{u,r\}$	50%	[21]
Y^a	Total number of males in community $a \in \{u,r\}$	50%	[21]
S_a^b	Total number of susceptibles in community $a \in \{u,r\}$, gender $b \in \{x,y\}$	$S_b^u = 0.87 \times B^u$ $S_b^r = 0.91 \times B^r$	[21]
I_a^b	Total number of infected in community $a \in \{u,r\}$, gender $b \in \{x,y\}$	$I_b^u = 0.13 \times B^u$ $I_b^r = 0.09 \times B^r$	[21]
α	The degree of differential infection	2	[18]
β	(contact rate between the two genders) × (transmission probability)	$\frac{1}{60}$ = 1/6 partners per month × 0.1 transmission probability	[13]
η^a	Constant rate of individuals in community $a \in \{u,r\}$ joining the sexually active phase	$\eta_u = 7,500$; $\eta_r = 7.5$	[21]
μ_{sb}	Constant removal rate of susceptibles of gender $b \in \{x,y\}$	30 active years $\Rightarrow \mu_{sb} = \frac{1}{360}$	[14]
μ_{ib}	Constant removal rate of infected of gender $b \in \{x,y\}$	10 active years $\Rightarrow \mu_{ib} = \frac{1}{120}$	[14]
c_b	Degree of sexual cross-mixing of individuals of gender $b \in \{x,y\}$ between urban and rural areas	$c_y = 0.00085$; $c_x = 0$	[13]

Using the values in Table 1, we have

$$\begin{aligned} R_{0,1} &= \frac{(\alpha+1)\beta}{\mu_{ix}+\mu_{iy}} = \frac{(2+1)\frac{1}{60}}{\frac{1}{120}+\frac{1}{120}} = 3 \\ R_{0,2} &= 8\,. \end{aligned}$$

4.2 Numerical results

We use the parameters given in Table 1 and let the system run for ten years. The method used for all numerical analysis is Runge-Kutta (3,4) with adaptive stepsize. The results are

Table 2: Changes in infection after ten years

Parameter	Starting value	Ending value
S_x^u	$1.52 \cdot 10^6$	$9.34 \cdot 10^5$
I_x^u	$2.28 \cdot 10^5$	$7.88 \cdot 10^5$
S_y^u	$1.52 \cdot 10^6$	$1.17 \cdot 10^6$
I_y^u	$2.28 \cdot 10^5$	$6.15 \cdot 10^5$
S_x^r	$1.59 \cdot 10^3$	$1.13 \cdot 10^3$
I_x^r	$1.58 \cdot 10^2$	$6.62 \cdot 10^2$
S_y^r	$1.59 \cdot 10^3$	$1.34 \cdot 10^3$
I_y^r	$1.58 \cdot 10^2$	$5.05 \cdot 10^2$
N^u	$3.50 \cdot 10^6$	$3.51 \cdot 10^6$
N^r	$3.50 \cdot 10^3$	$3.63 \cdot 10^3$
S	$3.05 \cdot 10^6$	$2.11 \cdot 10^6$
I	$4.55 \cdot 10^5$	$1.40 \cdot 10^6$
p_{ux}	50.00%	49.06%
p_{ui}	13.00%	39.97%
p_{rx}	50.00%	49.26%
p_{ri}	9.00%	32.15%

shown in Figure 3.

The numerical changes are given in Table 2, where p_{ab}, $a \in (u,r)$, $b \in (x,y)$ is the percentage of individuals of a given gender in a given region.

It is interesting to note that p_{ux} and p_{rx}, the percentage of the female population in the city and the village, respectively, stays close to 50%, although females are infected twice as easily as men.

4.3 Long-term behaviour and a second steady state

In order to understand the dynamics of the system better, we simulated fifty years of the epidemic. The results are shown in Figure 4. We see that the trend indicated in the first ten years, will be continued for quite a while and that infected individuals will eventually outnumber susceptible individuals. However, at some point the number of susceptible individuals stops decreasing so drastically and seems to asymptotically approach a steady state from above. Also, the infected population seems to approach a steady state from above. Note that the graph for the infected population overshoots and reaches its peak slightly before the decrease of the susceptible population slows down. This phenomenon is typical for SIR-models.

Any long timescale simulation we performed showed the effect of all three functions (the total, the susceptibles and the infected population) seemingly approaching a steady state. However, we haven't shown that this state is actually a stable steady state, so we will refer to it as a "quasi steady state".

From a biological point of view, however, such a long timescale is not realistic. The parameters may very well change over time and some new dynamics may enter the system

to change it completely. That is the reason why the focus in this chapter is on the shorter timescale of ten years.

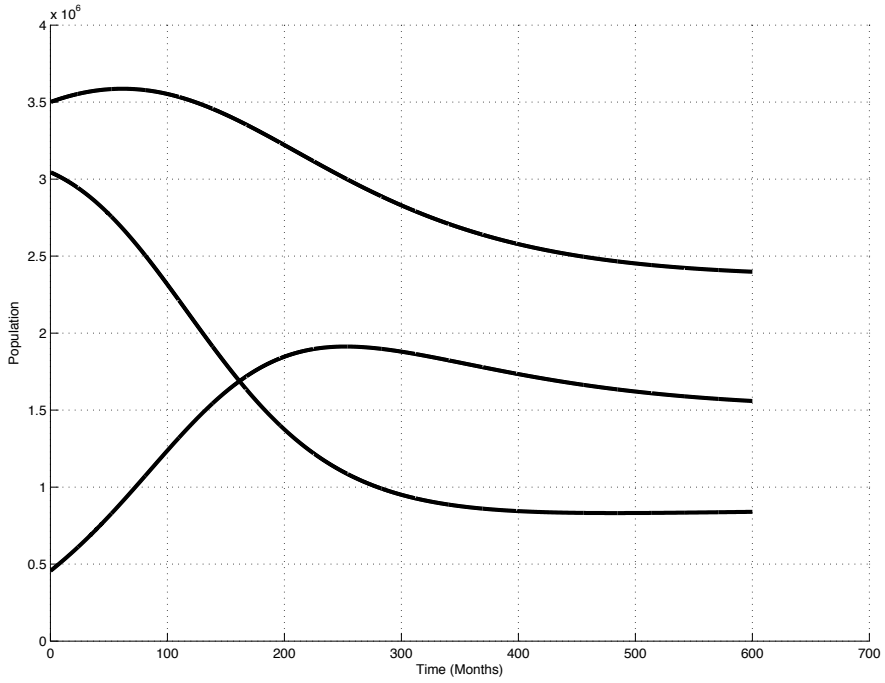

Figure 4: Fifty year urban timecourse without intervention.

5 Intervention Strategies

5.1 Modelling intervention strategies

We wish to include a variety of intervention strategies, in order to compare their effectiveness. When modelling intervention strategies, there are two main approaches. The first is to alter our differential equations, such as including a recovery term or subtracting a certain term from the transmission probability. The advantage of this method is that it provides a general approach. The disadvantage is that it complicates the model. The second method is to change some parameter values. While this approach cannot give us a general picture, it is straightforward and allows for a variety of possible intervention strategies to be explored numerically.

From our analysis of R_0, the parameters β, μ_{ix} and μ_{iy} are clearly the most important. For comparison purposes, we will always make changes in these parameters in such a way that $R_{0,2}$ reduces to 2. Although this value is not below one (and will thus not lead to eradication), it is nevertheless a significant reduction from 8. Consequently, we are comparing intervention strategies that reduce the prevalence of the disease, but which do not lead to eradication.

AIDS-awareness education and/or an increased condom use will decrease the transmission probabilty and/or the contact rate, resulting in a decreased β. If AIDS tests were available that could inform infected people about their status, we could hope that infected people stop having unsafe contact with others and thus drop out of our model. The result would be an increase of μ_{ix} and/or μ_{iy}. Antiretroviral drugs would decrease β, while simultaneously increasing μ_{ix} and μ_{iy}.

A critical question is whether or not any of these changes would influence the system in the long run. We can obviously hope that the disease will spread slower with a smaller R_0, but will it in the long run still reach this second steady state we saw in the previous section? In order to address this question, let us look at Figure 5.

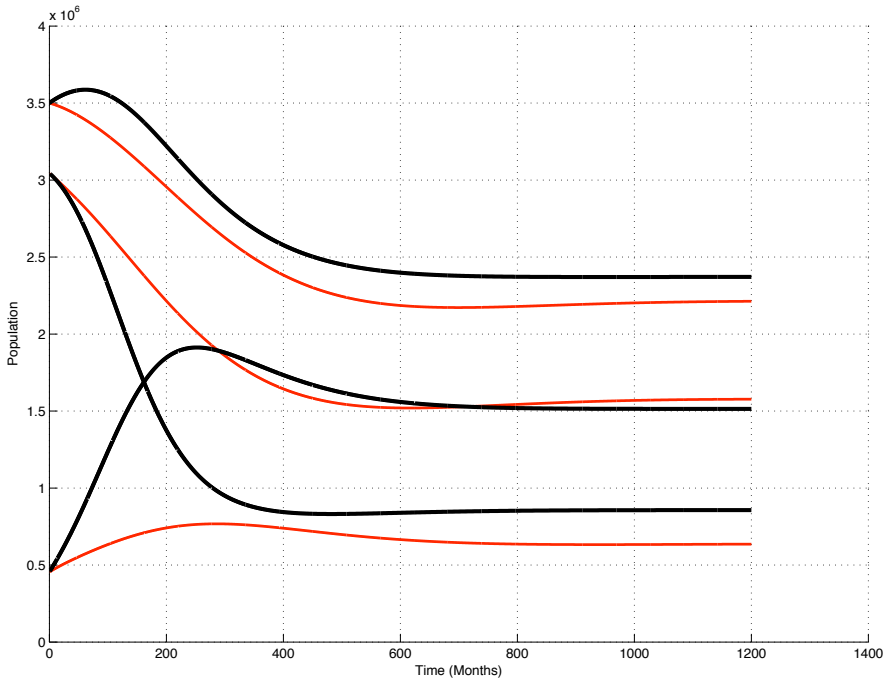

Figure 5: The system with doubled removal rates. The solid curves show the behaviour of the system without intervention; the thinner red lines give the same functions for a doubled μ_{ix} and μ_{iy}.

Thus, if we double both μ_{ix} and μ_{iy} and compare it to the behaviour of the system without intervention, we see a significant difference. While we still have the overshooting of the infected population, the actual numbers are drastically different. The infected population never outnumbers the susceptible population. Note that the overall population is lower, since we are assuming many more people are removed from the system.

Changes resulting from an increased β are shown in Figure 6. As expected, the disease spreads much quicker with the higher β. Besides that, the functions behave similarly. Again, we see an overshoot of the infected population and both systems approaching a steady state

in the long run. However, we see that these two steady states differ and that with the smaller β the approached steady state is much better, due to a smaller amount of infected people and a larger amount of susceptibles.

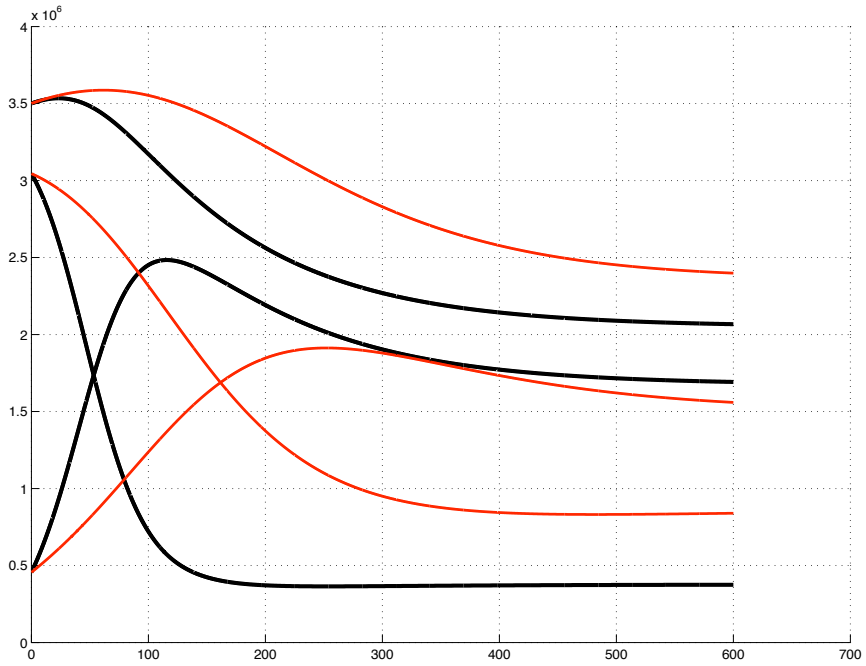

Figure 6: The system with β halved. The thin red curves show the behaviour of the system without intervention, the solid black curves give the same functions for β halved.

5.2 Strategy comparison

We want to compare the following intervention strategies:

- Strategy I: Halving β through increasing education.
- Strategy II: Doubling μ_{ix} and μ_{iy}, due to more widespread testing.

The result are shown in Figure 7. Clearly, strategy I is superior to strategy II. The total number of susceptible individuals increases in strategy I, while it decreases in strategy II. Even in the total number of infected individuals, strategy II shows a worse performance although infected individuals leave the system twice as often! These two effects, taken together, result in a huge difference in the AIDS prevalence after ten years. The actual numbers are found in the Table 3.

While the number of infected individuals increases in both strategies, it should nevertheless be noted that R_0 is still greater than 1. If strategy I is implemented instead of

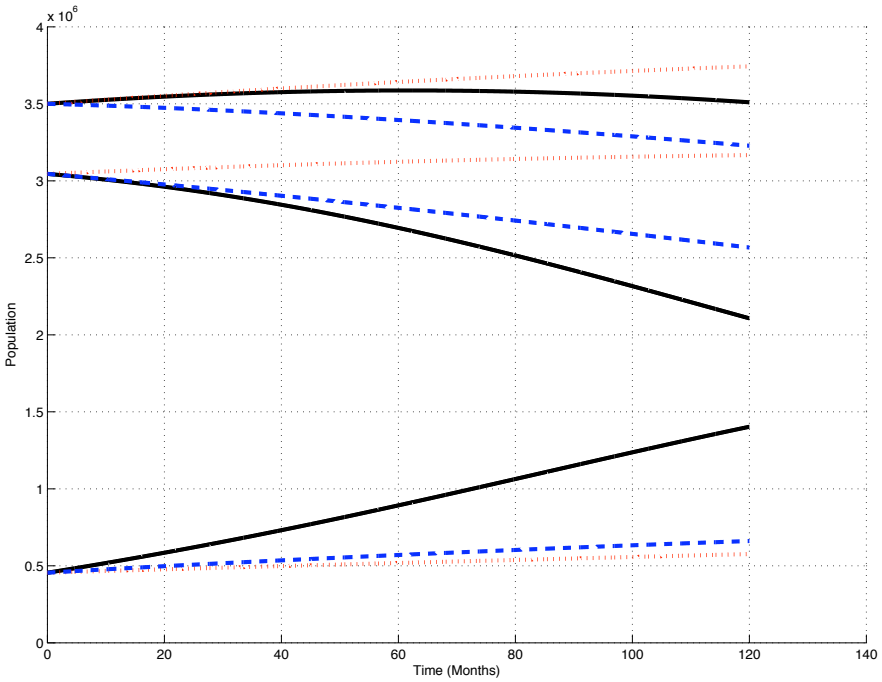

Figure 7: Comparison of the two intervention strategies. The solid black curves show the system without intervention. The dotted red lines show the effects of halving β (strategy I). The dashed blue curves show the effects of doubling μ_{ix} and μ_{iy} (strategy II).

Table 3: Changes in infection after ten years, using strategies I and II.

	Starting value	No intervention	Strategy I	Strategy II
S_x^u	$1.52 \cdot 10^6$	$9.34 \cdot 10^5$	$1.53 \cdot 10^6$	$1.19 \cdot 10^6$
I_x^u	$2.28 \cdot 10^5$	$7.88 \cdot 10^5$	$3.29 \cdot 10^5$	$3.73 \cdot 10^5$
S_y^u	$1.52 \cdot 10^6$	$1.17 \cdot 10^6$	$1.64 \cdot 10^6$	$1.38 \cdot 10^6$
I_y^u	$2.28 \cdot 10^5$	$6.15 \cdot 10^5$	$2.47 \cdot 10^5$	$2.88 \cdot 10^5$
S_x^r	$1.59 \cdot 10^3$	$1.13 \cdot 10^3$	$1.66 \cdot 10^3$	$1.38 \cdot 10^3$
I_x^r	$1.58 \cdot 10^2$	$6.62 \cdot 10^2$	$2.43 \cdot 10^2$	$3.00 \cdot 10^2$
S_y^r	$1.59 \cdot 10^3$	$1.34 \cdot 10^3$	$1.74 \cdot 10^3$	$1.53 \cdot 10^3$
I_y^r	$1.58 \cdot 10^2$	$5.05 \cdot 10^2$	$1.81 \cdot 10^2$	$2.26 \cdot 10^2$
N^u	$3.50 \cdot 10^6$	$3.51 \cdot 10^6$	$3.74 \cdot 10^6$	$3.23 \cdot 10^6$
N^r	$3.50 \cdot 10^3$	$3.63 \cdot 10^3$	$3.83 \cdot 10^3$	$3.43 \cdot 10^3$
S	$3.05 \cdot 10^6$	$2.11 \cdot 10^6$	$3.17 \cdot 10^6$	$2.57 \cdot 10^6$
I	$4.55 \cdot 10^5$	$1.40 \cdot 10^6$	$5.77 \cdot 10^5$	$6.61 \cdot 10^5$
p_{ux}	50.00%	49.06%	49.57%	48.45%
p_{ui}	13.00%	39.97%	15.40%	20.48%
p_{rx}	50.00%	49.26%	49.69%	48.84%
p_{ri}	9.00%	32.15%	11.08%	15.30%

Table 4: Changes in infection after ten years, using all four strategies

	Starting value	No Intervention	Strategy I	Strategy II	Strategy III	Strategy IV
S_x^u	$1.52 \cdot 10^6$	$9.34 \cdot 10^5$	$1.53 \cdot 10^6$	$1.19 \cdot 10^6$	$1.37 \cdot 10^6$	$1.13 \cdot 10^6$
I_x^u	$2.28 \cdot 10^5$	$7.88 \cdot 10^5$	$3.29 \cdot 10^5$	$3.73 \cdot 10^5$	$4.42 \cdot 10^5$	$2.18 \cdot 10^5$
S_y^u	$1.52 \cdot 10^6$	$1.17 \cdot 10^6$	$1.64 \cdot 10^6$	$1.38 \cdot 10^6$	$1.36 \cdot 10^6$	$1.49 \cdot 10^6$
I_y^u	$2.28 \cdot 10^5$	$6.15 \cdot 10^5$	$2.47 \cdot 10^5$	$2.88 \cdot 10^5$	$1.60 \cdot 10^5$	$3.59 \cdot 10^5$
S_x^r	$1.59 \cdot 10^3$	$1.13 \cdot 10^3$	$1.66 \cdot 10^3$	$1.38 \cdot 10^3$	$1.53 \cdot 10^3$	$1.33 \cdot 10^3$
I_x^r	$1.58 \cdot 10^2$	$6.62 \cdot 10^2$	$2.43 \cdot 10^2$	$3.00 \cdot 10^2$	$3.38 \cdot 10^2$	$1.80 \cdot 10^2$
S_y^r	$1.59 \cdot 10^3$	$1.34 \cdot 10^3$	$1.74 \cdot 10^3$	$1.53 \cdot 10^3$	$1.52 \cdot 10^3$	$1.63 \cdot 10^3$
I_y^r	$1.58 \cdot 10^2$	$5.05 \cdot 10^2$	$1.81 \cdot 10^2$	$2.26 \cdot 10^2$	$1.28 \cdot 10^2$	$2.71 \cdot 10^2$
N^u	$3.50 \cdot 10^6$	$3.51 \cdot 10^6$	$3.74 \cdot 10^6$	$3.23 \cdot 10^6$	$3.33 \cdot 10^6$	$3.20 \cdot 10^6$
N^r	$3.50 \cdot 10^3$	$3.63 \cdot 10^3$	$3.83 \cdot 10^3$	$3.43 \cdot 10^3$	$3.51 \cdot 10^3$	$3.40 \cdot 10^3$
S	$3.05 \cdot 10^6$	$2.11 \cdot 10^6$	$3.17 \cdot 10^6$	$2.57 \cdot 10^6$	$2.73 \cdot 10^6$	$2.62 \cdot 10^6$
I	$4.55 \cdot 10^5$	$1.40 \cdot 10^6$	$5.77 \cdot 10^5$	$6.61 \cdot 10^5$	$6.02 \cdot 10^5$	$5.77 \cdot 10^5$
p_{ux}	50.00%	49.06%	49.57%	48.45%	54.43%	42.13%
p_{ui}	13.00%	39.97%	15.40%	20.48%	18.07%	18.05 %
p_{rx}	50.00%	49.26%	49.69%	48.84%	53.18%	44.27%
p_{ri}	9.00%	32.15%	11.08%	15.30%	13.27%	13.26%

strategy II, there is a difference of approximately 80,000 fewer infected after 10 years; the percentage reduction in urban infectionis 20.5% under strategy II, but only 15.4% under strategy I.

5.3 The effect of gender differences

We now propose intervention strategies that affect only one gender, in order to determine some of the effects of gender on the outcome. Instead of doubling the removal rates uniformly, as in strategy II, we instead propose quadrupling only one gender-specific removal rate. This reflects the situation where testing of one gender results in them ceasing to find new sexual partners and thus leaving the sexually active pool.

- Strategy III: Available AIDS tests for males only to quadruple μ_{iy}.

- Strategy IV: Available AIDS tests for females only to quadruple μ_{ix}.

We compare all four intervention strategies in Figure 8. In order to get a better view of the development of the number of infected individuals, we zoom in. See Figure 9. The values are given in Table 4, where p_{ui} and p_{ri} are the percentages of infected individuals in the urban and rural areas, respectively.

When comparing the percentage of infected people in the population, we see that strategy III and strategy IV perform about equally (approximately 18%), which is between strategy I (15.4%) and strategy II (20.5%). However, if we concentrate on the total amount of infected people, we see that strategy II (6.61×10^5) performs worse than strategy IV (5.77×10^5), which actually gives us the same result as strategy I. So the difference between strategy I and strategy IV really is only in the number of susceptible individuals. The number of infected people in strategy IV is always less than the number of infected people

of strategy I for the first ten years. The number of infected people in strategy III is less than that in strategy I for five years and thereafter is greater. Strategy II always produces more infected individuals than any other strategy.

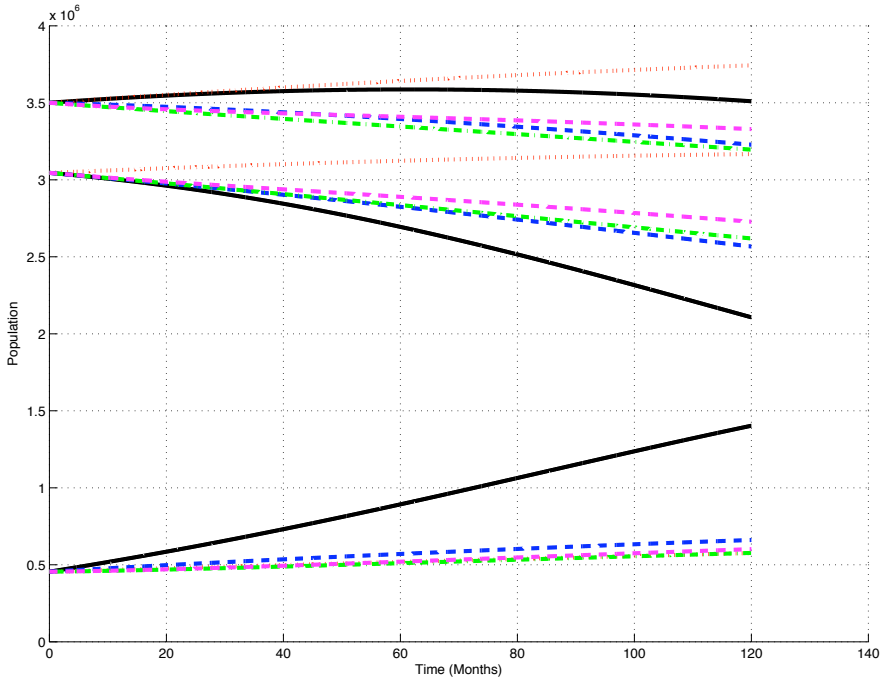

Figure 8: Comparison of all intervention strategies. The solid black curves show how the system behaves without intervention. The dotted red curves represent the system under the influence of strategy I. The dashed blue curves represent strategy II. The dashed pink curves represent strategy III and the dot-dashed green curves represent strategy IV.

5.4 Redefining "success"

The previous discussion raises the question of which measurement of "success" is appropriate. On the one hand, we want as few infected people to be sexually active as possible. On the other hand, the lower the percentage of infected individuals in the sexually active phase, the lower the chance for an individual to actually have sexual contacts with an infected person.

Besides the percentage or overall number of infected people in the population, another way to measure success is to look at the total number of newly infected people in a given time period. Rephrasing this, we want to count the cumulative number of susceptible people who become infected.

Consequently, we introduce a measure function M, that satisfies

$$\frac{dM}{dt} = \frac{I_y^u + c_y I_y^r}{Y^u + c_y Y^r} \cdot \alpha \cdot \beta \cdot S_x^u + \frac{I_x^u + c_x I_x^r}{X^u + c_x X^r} \cdot \beta \cdot S_y^u$$

$$+\left[\frac{I_y^r}{Y^r}+c_x\frac{I_y^u+c_yI_y^r}{Y^u+c_yY^r}\right]\cdot\alpha\cdot\beta\cdot S_x^r+\left[\frac{I_x^r}{X^r}+c_y\frac{I_x^u+c_xI_x^r}{X^u+c_xX^r}\right]\cdot\beta\cdot S_y^r$$

with $M(0)=0$. The four summation terms in the derivative of M are the number of new infections in, respectively, urban men, urban women, rural men and rural women.

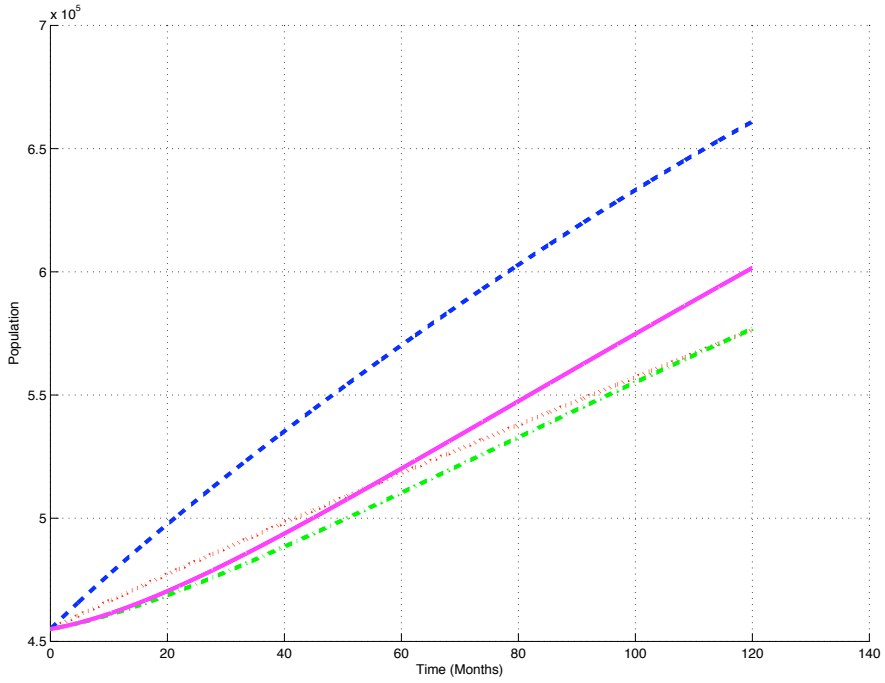

Figure 9: Comparison of intervention strategies for infected individuals. The dotted red curve shows the number of infected people under strategy I, the dashed blue curve shows the number of infected people under strategy II. The solid pink curve represents the number of infected people under strategy III and the dot-dashed green curve represents the number of infected people under strategy IV.

In Figure 10, we see the change in M when there are no intervention strategies, while Figure 11 compares the different measure functions for the four different strategies. While strategies II, III and IV are more or less equally "successful", strategy I is significantly better. Here, the number of new infections is less than half as big as the amount in any other strategy and is one third of the amount in the system without intervention.

It follows that, in terms of preventing new infections in all groups, strategy I (halving β) is clearly superior. However, the next best intervention is strategy IV (quadrupling the female removal rate), which is significantly better than quadrupling the male removal rate. This shows that gender differences have an important impact on the outcome.

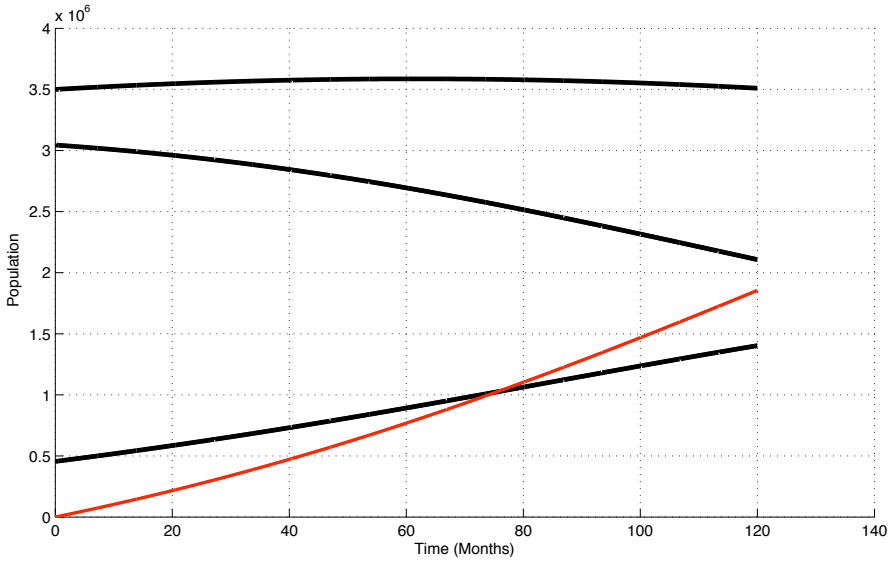

Figure 10: The model without intervention (black curves) and the measure function M (red curve).

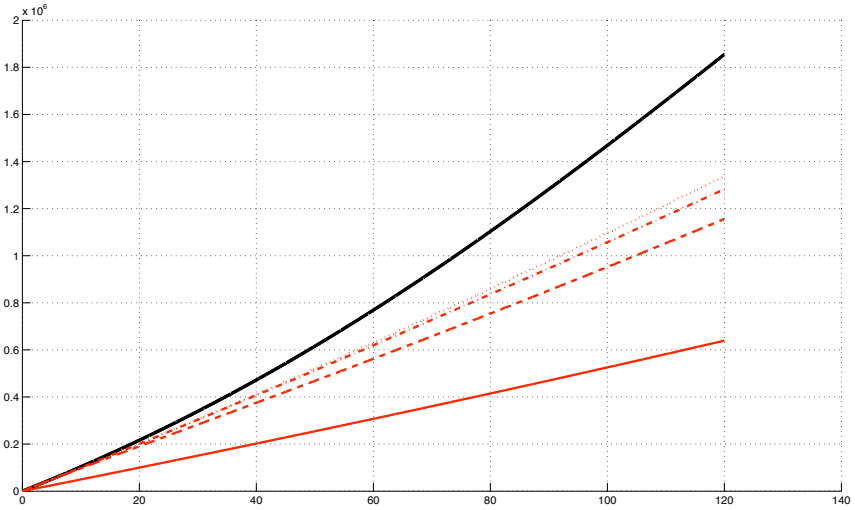

Figure 11: The measure function for all intervention strategies. The solid black and red curves show the system without intervention and with strategy I, respectively. The dotted curve demonstrates strategy II, the dotted-dashed curve strategy III and the dashed curve strategy IV.

6 Conclusion

The best strategy for reducing the impact of the epidemic is reducing the infection probability. Failing that, the next best strategy is to increase the removal rate of females from the sexually active pool. This may be achieved through education, increased testing, improved condom use or prevention awareness. Interestingly, increasing the removal rate of females is significantly more likely to improve the outcome than increasing the removal rate of males (Figure 11).

Our model thus demonstrates that gender differences can have a significant effect on the outcome. We also demonstrated that transmission occurring predominantly in urban areas results in an increase from 13% to 40% in urban areas and an increase from 9% to 32% in rural areas after ten years (Table 2). It follows that urban transmission can sustain infection in rural areas.

We also analysed the effects of calculating the basic reproductive ratio under two scenarios. In the first, we assumed that the number of rural individuals is significantly smaller than the number of urban individuals, and also that the male and female populations are similar in size. In the second, we assumed that the inflow of urban individuals is large, while the inflow of rural individuals is small. In each case, the value depends only on the transmission probability, the removal rates of each gender and the degree of differential infection. While each value satisfies the threshold condition that the disease persists if $R_0 > 1$ and is eradicated if $R_0 < 1$, we showed that the second approximation led to a path-independent product of the individual reproductive numbers for each gender. This demonstrates the care that needs to be taken when calculating surrogate R_0-like thresholds from mathematical models (see [19] for more discussion).

Furthermore, the second calculation of R_0 demonstrated that the choice of whether the degree of differential infection is applied to males or females is arbitrary. Thus, the results can be generalised to note that removing the "weaker" sex – i.e., the one with the highest risk of being infected – has a greater impact on the outcome. This "removal" could be achieved via targeted education strategies, increased testing for one gender or through community organisations. Of course, education campaigns or AIDS testing should attempt to encompass both genders if possible, but the realities of existing cultural structures may make one gender more receptive to some stategies than others.

While the first two approximations lead to a "first guess" for R_0, this value is not biologically meaningful. By refining our approximations, we derived a biologically useful threshold condition. Approximation 3.3 implies that HIV can be sustained in urban areas alone. By using this key approximation, we were able to simplify an 8×8 system to two 2×2 systems. This makes the system mathematically tractable and allows the derivation of the second, more useful, R_0.

We are primarily interested in transient behaviour of the system. While analytical methods may determine long-term phenomena such as equilibria, the timescale of such behaviour may be much longer than the lifespan of an infected individual. Consequently, we use numerical simulations to examine the short-term dynamics of the system. In Tables 2 and 4, our starting values are a long way from the disease-free equilibrium and do not evolve to stable behaviour in a ten-year period. While stable behaviour is eventually seen, it takes approximately 50 years to reach (Figure 4), much longer than the timecourse of the

disease in individuals.

It should be noted that this model is only a partial snapshot of the epidemic, as it ignores many other important routes of transmission. Specifically, homosexual transmission, needle sharing and vertical transmission are not modelled.

In summary, the effects of gender differences can have a significant impact on which intervention strategy should be applied. Not all intervention strategies will have the same effect, even if they remove the same number of total individuals. Furthermore, the effects of urban and rural mixing have a significant effect on the outcome for rural individuals, who suffer a proportionally greater increase in prevalence than do urban areas. It follows that, before intervention strategies are implemented, the population-level impact should be carefully assessed.

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Appendix

Below are the codes used in generating the figures.

Associated function file (aidsf.m)

```
function z = aidsf(t,x0); global beta alpha etau usx usy cx cy uix uiy etar
z(1,:) = etau - usx*x0(1) - alpha*beta*((x0(4) + cy*x0(8))/(x0(4) + x0(3) + cy*(x0(8) + x0(7))))*x0(1);
z(2,:) = alpha*beta*((x0(4) + cy*x0(8))/(x0(4) + x0(3) + cy*(x0(8) + x0(7))))*x0(1) - uix*x0(2);
z(3,:) = etau - usy*x0(3) - beta*((x0(2) + cx*x0(6))/(x0(2) + x0(1) + cx*(x0(6) + x0(5))))*x0(3);
z(4,:) = beta*((x0(2) + cx*x0(6))/(x0(2) + x0(1) + cx*(x0(6) + x0(5))))*x0(3) - uiy*x0(4);
z(5,:) = etar - usx*x0(5) - alpha*beta*(x0(8)/(x0(8) + x0(7)) + cx*(x0(4) + cy*x0(8))/(x0(4) + x0(3)
+ cy*(x0(8) + x0(7))))*x0(5);
z(6,:) = alpha*beta*(x0(8)/(x0(8) + x0(7)) + cx*(x0(4) + cy*x0(8))/(x0(4) + x0(3) + cy*(x0(8) +
x0(7))))*x0(5) - uix*x0(6);
z(7,:) = etar - usy*x0(7) - beta*(x0(6)/(x0(6) + x0(5)) + cy*(x0(2) + cx*x0(6))/(x0(2) + x0(1) +
cx*(x0(6) + x0(5))))*x0(7);
z(8,:)= beta*(x0(6)/(x0(6) + x0(5)) + cy*(x0(2) + cx*x0(6))/(x0(2) + x0(1) + cx*(x0(6) + x0(5))))*x0(7)
- uiy*x0(8);
end
```

The main file (aids.m)

```
function aids(varargin);
% There are two special features included: The function 'aids' takes one or two optional arguments.
% They allow you to implement the counter strategies I,II,III or IV and allow you to choose the
% running time. If no argument is give the system runs on strategy 0 (= no counter measures)
% for Tmax = 120 months. If only one optional argument is given, that stands for the counter
% strategy:
%
% 0 = no counter measures
% 1 = Strategy I (half beta)
% 2 = Strategy II (double μix, μiy)
% 3 = Strategy III (quadruple μix)
% 4 = Strategy IV (quadruple μiy)
%
% If two optional arguments are given, the first one stands for the counter strategy and the second
% one for the time (in months) the simulation shall run. Examples:
%
% >> aids; % no counter strategy, ten years
```

```
% >> aids(2); % counter strategy II, ten years
% >> aids(3,240); % counter strategy III, twenty years
% >> aids(0,1200); % no counter strategy, one hundred years
%
% The second special feature is that you can plot several runs into one diagram. MatLab
% automatically plots the old graphs red and the new one blue, so you always know where you're
% at.

% global parameter values
global beta alpha cx cy usx usy uix uiy etau etar;
beta = 1/60;
alpha = 2;
cx = 0;
cy = 0.00085;
usx = 1/360;
usy = 1/360;
uix = 1/120;
uiy = 1/120;
etau = 7500;
etar = 7.5;

% further parameter values
strategy = 0;
Tmax = 120;
pui = 0.13;
pri = 0.09;
Nu = 3500000;
pux = 0.5;
Nr = 3500;
prx = 0.5;

% starting values
Sux = (1-pui)*pux*Nu;
Iux = pui*pux*Nu;
Suy = (1-pui)*(1-pux)*Nu;
Iuy = pui*(1-pux)*Nu;
Srx = (1-pri)*prx*Nr;
Irx = pri*prx*Nr;
Sry = (1-pri)*(1-prx)*Nr;
Iry = pri*(1-prx)*Nr;
M = 0;

% handling the optional argument
if nargin == 1
strategy = varargin1;
```

```
end
if nargin == 2
strategy = varargin1;
Tmax = varargin2;
end
if nargin > 2
disp('Too many arguments');
end

% resetting the parameter values according to the chosen strategy
if strategy == 1
beta = .5*beta;
end
if strategy == 2
uix = 2*uix;
uiy = 2*uiy;
end
if strategy == 3
uiy = 4*uiy;
end
if strategy == 4
uix = 4*uix;
end

% solving the ode
j=1;
tau=0.1;
t0=0;
x0=[Sux,Iux,Suy,Iuy,Srx,Irx,Sry,Iry];

for i=1:Tmax/tau
tspan=[t0 t0+tau];
[t,x] = ode45(@aidsf,tspan,x0);
n=length(t);
xnew=x(n,:);
for k=1:8
u(k,j)=xnew(k);
end
time(j)=t(n);
t0=t0+tau;
x0=x(n,:);
j=j+1;
end

% plotting the results
```

```
hold on
grid on
set(findobj('Type','line'),'color','r');
plot(time,u(1,:) + u(2,:) + u(3,:) + u(4,:));   % urban population
plot(time,u(2,:) + u(4,:));    % infected urban population
plot(time,u(1,:) + u(3,:));    % suspective urban population

plot(time,u(5,:) + u(6,:) + u(7,:) + u(8,:));    % rural population
plot(time,u(6,:) + u(8,:));    % infected rural population
plot(time,u(5,:) + u(7,:));    % suspective rural population
end
```