#### **ORIGINAL ARTICLE**



# Dynamics of a non-smooth epidemic model with three thresholds

Aili Wang<sup>1</sup> · Yanni Xiao<sup>2</sup> · Robert Smith?<sup>3</sup>

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#### Abstract

A non-smooth epidemic model with piecewise incidence rate dependent on the derivative of the case number is proposed for the transmission dynamics of an infectious disease with media coverage, enhanced vaccination and treatment policy. This is an implicitly defined system, which is converted into an explicit system with three thresholds by employing the properties of the Lambert *W* function. We first analyze the dynamics of the proposed model for the limiting case, which induces two non-smooth but continuous models. The dynamic analysis of the model demonstrates that either one of the two generalized equilibria or the pseudo-equilibrium is globally asymptotically stable if the disease does not die out. This suggests that the case number can be contained either at an a priori level or at a high/low level, depending on the threshold, which governs whether the enhanced vaccination and treatment policies are implemented. Media coverage cannot help eradicate the disease, but it significantly delays the epidemic peak and lowers the peak case number. Hence, a good threshold policy and continuously updating the awareness of case numbers are required to combat the disease successfully.

Keywords Media coverage · Multiple thresholds · Non-smooth model · Sliding dynamics · Global dynamics

## Introduction

During the outbreak of an infectious disease, pharmaceutical interventions may not provide protection for the public, since an effective vaccine may not exist or antiviral drugs may not be available in sufficient quantities. Nonpharmaceutical control measures may thus play an important role in fighting against these diseases, especially for developing countries. Considering media coverage as a kind of non-pharmaceutical measure of heightened information awareness helps both the government and the public respond to and implement measures to curb the disease.

Robert Smith? rsmith43@uottawa.ca

- <sup>1</sup> School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji 721013, People's Republic of China
- <sup>2</sup> School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China
- <sup>3</sup> Department of Mathematics and Faculty of Medicine, The University of Ottawa, Ottawa, ON K1N 6N5, Canada

In 1973, cholera was curbed within a few days in southeast Italy, because media coverage led the local people to adopt strict measures to avoid being infected (Capasso and Serio 1978). In 2009, media coverage helped governments make interventions to contain the spread of swine flu (H1N1) (Funk et al. 2010; Xiao et al. 2015). Further, the degree of AIDS awareness in educated women and men is 4.69 and 77.73 times of those uneducated women and men in Bangladesh, and the probability of being aware of AIDS in individuals regularly watching TV is about 8.6 times of the probability in those who never watch TV (Rahman and Rahman 2007). It is also suggested that those placing risk perception foremost in their minds are more inclined to conduct precautionary behaviors, such as wearing masks, frequent hand washing, avoiding travel and social distancing (Tracy et al. 2009; Brewer et al. 2007; Kristiansen et al. 2007).

Media coverage triggers behavior changes in the population, which consequently leads to a reduction in the effective contact rates. Understanding how media coverage affects the transmission of an epidemic during an outbreak is vital (Al Basir et al. 2018; Misra et al. 2018; Liu et al. 2007; Cui et al. 2008a; Sahu and Dhar 2015; Wang and Xiao 2014; Xiao et al. 2013; Tchuenche and Bauch 2012; Li and Cui 2009; Sun et al. 2011; Tchuenche et al. 2011;

Cui et al. 2008b; Wang et al. 2016; Berrhazi et al. 2017; Chen et al. 2018; Khan et al. 2018; Song and Xiao 2018). Mathematically, there are two ways of modeling the impact of media awareness. One method is to consider the media campaign as a separate state variable to address the cumulative number of TV, media advertisements, Internet information and posters (Al Basir et al. 2018; Misra et al. 2018). The other method is to incorporate an explicit function in the transmission term of the model to describe the media effect (Liu et al. 2007; Cui et al. 2008a; Sahu and Dhar 2015; Wang and Xiao 2014; Xiao et al. 2013; Tchuenche and Bauch 2012; Li and Cui 2009; Sun et al. 2011; Tchuenche et al. 2011; Cui et al. 2008b; Wang et al. 2016; Berrhazi et al. 2017; Chen et al. 2018; Khan et al. 2018; Song and Xiao 2018). It follows that the precise functioning of media coverage becomes very important. Negative exponential forms, such as  $\beta e^{-\alpha I}$ and  $\beta e^{-\alpha_1 E - \alpha_2 I - \alpha_3 H}$ , have been adopted to describe the reduction factor (Liu et al. 2007; Cui et al. 2008a; Sahu and Dhar 2015). Multiple positive equilibria, multiple outbreaks and rich dynamics have been observed in these studies. Another functioning method is to use the nonlinear function  $\beta - \frac{\beta_1 I}{m+I}$  to reflect the intrinsic characters of media coverage (Li and Cui 2009; Tchuenche et al. 2011; Wang et al. 2016; Berrhazi et al. 2017; Chen et al. 2018; Khan et al. 2018), which can be expanded to a general contact rate  $\beta_1 - \beta_2 f(I)$  (Sun et al. 2011; Cui et al. 2008b). Alternatively, non-smooth models with piecewise smooth transmission functions have been used to evaluate the effect of media coverage (Xiao et al. 2013, 2015; Wang and Xiao 2014; Tchuenche and Bauch 2012; Chen et al. 2018). These results have shown that media coverage can have a significant impact on the epidemic, such as delaying the peak and reducing the severity of the outbreak.

However, almost all of these formulations are based on a large number of infectious cases. How human behavior and social responses vary depends on not only the reported case number but also the change in the case number, especially in the early stage of outbreak (Jones and Salathe 2009). Significantly increasing the case number can cause individuals to engage in protective behavior. In order to better understand the impact of the rate of the case number on disease transmission, Xiao et al. (2013) and Tchuenche and Bauch (2012) proposed epidemic models with piecewise smooth incidence rates dependent on both the case number and its rate of change. This implicitly defined model has been converted into a non-smooth and continuous model with one threshold (Xiao et al. 2013).

Furthermore, vaccination and antiviral drugs serve as the two most effective ways to prevent and control disease transmission. However, if the strain is either unknown or novel at the initial stage, these pharmaceutical resources may not be available or may be limited (Kabineh et al. 2018; Wandeler et al. 2018; Jiang and Zhou 2018). Wang proposed a piecewise function to model limited resource capacity, which resulted in a non-smooth continuous model (Wang 2006). Saturated incidence functions have been incorporated into classical smooth epidemic models to explore how limited medical resources affect the disease transmission (Zhou and Fan 2012; Abdelrazec et al. 2016). In Wang et al. (2019), we embedded a piecewise-defined function into a continuous epidemic model, giving a nonsmooth model with one threshold, to quantify the effect of threshold level and the limiting capacity on disease spread. To examine the effect of hospital resources on disease control, we formulated a Filippov model with one threshold to examine the impact of medical resource constraints on disease control (Wang et al. 2018). In recent years, Filippov systems have gained a substantial amount of attention and have been extensively applied to examine the effect of threshold policies on disease control (Xiao et al. 2012; Chong et al. 2016; Tang et al. 2016; Zhou et al. 2016; Qin et al. 2016; Chen et al. 2018) and pest management (Tang et al. 2012).

The main purpose of this paper is to construct a negative exponential function to describe the reduction factor induced by changes in the case number; i.e., dI/dt. Additionally, we will consider the vaccination and treatment measures driven by the case number, which is defined as follows: when the case number exceeds a threshold level, the enhanced vaccination and treatment policies are conducted; otherwise, the general vaccination policy is adopted without any treatment measure. We will convert the implicitly defined system to an explicitly defined system by using the properties of the Lambert W function in the next section. The resulting system is a non-smooth system with three thresholds. In the third section, we will address the dynamics of the subsystem with only the media coverage considered. We will examine the sliding dynamics and global dynamics of our targeted model for the whole parameter space in the fourth and fifth sections, respectively. Concluding remarks are presented in the last section.

#### Model formulation

We consider the impact of media coverage, vaccination and treatment on the disease spread. Media coverage affects individuals' awareness by reporting the case number and its changing rate. When the case number increases, the public will take measures to avoid being infected. This results in a reduction in contact rates and ultimately a decrease in transmission. Thus, we adopt a function dependent on the changing rate of case number (i.e.,  $\exp[-\alpha \psi_1(dI/dt)])$  to describe the impact of media coverage on the disease. A general vaccination proportion (p) is enhanced by a proportion f when the case number increases and exceeds a threshold level  $I_c$ . A proportion of infected individuals  $(\gamma)$  are treated and return to the susceptible class in this case. Then, the model takes the form

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= \mu - \beta \exp\left(-\alpha\psi_1\left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)\right) SI - \mu S - [p + f\psi_2(I)]S + \gamma\psi_2(I)I,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta \exp\left(-\alpha\psi_1\left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)\right) SI - \mu I - \delta I - \gamma\psi_2(I)I,\\ \frac{\mathrm{d}V}{\mathrm{d}t} &= [p + f\psi_2(I)]S - \mu V, \end{aligned}$$
(1)

with

$$\psi_1\left(\frac{\mathrm{d}I}{\mathrm{d}t}\right) = \max\left\{0, \quad \frac{\mathrm{d}I}{\mathrm{d}t}\right\} \tag{2}$$

and

$$\psi_2(I) = \begin{cases} 0, & I < I_c, \\ 1, & I > I_c, \end{cases}$$
(3)

where S(t), I(t) and V(t) are the proportions of susceptible, infected and immune individuals at time *t*, respectively. All other parameters are positive constants, where  $\mu$  represents natural birth (death) rate,  $\beta$  denotes the basic transmission rate, *p* (or *f*) stands for the basic (or enhanced) vaccination rate,  $\gamma$  is the treatment rate and  $\delta$  is the disease-induced death rate.

Note that the vaccinated class in (1) does not affect the susceptible and infected class, so we only need to consider the following system

$$\frac{dS}{dt} = \mu - \beta \exp\left(-\alpha\psi_1\left(\frac{dI}{dt}\right)\right)SI - \mu S - [p + f\psi_2(I)]S + \gamma\psi_2(I)I,$$
$$\frac{dI}{dt} = \beta \exp\left(-\alpha\psi_1\left(\frac{dI}{dt}\right)\right)SI - \mu I - \delta I - \gamma\psi_2(I)I.$$
(4)

It is easy to show  $\Omega \equiv \{(S, I) \in R^2_+ : S + I \le 1\}$  is an attraction region of system (4). Next, we introduce the properties of the Lambert *W* function, which will be used in the rest of this paper (Corless et al. 1996).

**Definition 1** The Lambert W function is a multi-valued inverse of the function  $x \mapsto xe^x$  with

Lambert  $W(x) \exp($  Lambert W(x)) = x.

We further have

Lambert  $W'(x) = \frac{\text{Lambert } W(x)}{x(1 + \text{ Lambert } W(x))}.$ 

In fact, since  $(x \exp(x))' = (x + 1) \exp(x)$  is positive for x > -1, the inverse function of  $x \exp(x)$  restricted on  $[-1, +\infty)$  is defined as Lambert W(0, x) = Lambert W(x). The inverse function of  $x \exp(x)$  on  $(-\infty, -1)$  is similarly defined as Lambert W(-1, x).

It is worth emphasizing that the equations in model (4) are dependent on the sign of dI/dt, which makes them implicit equations. To simplify them to explicit equations, it is crucial to determine the conditions under which the sign of dI/dt is positive. If  $\frac{dt}{dt} > 0$ , we have  $\psi_1(\frac{dt}{dt}) = \frac{dt}{dt}$ , and so the second equation of model (1) takes the form

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta \exp\left(-\alpha \frac{\mathrm{d}I}{\mathrm{d}t}\right) SI - (\mu + \delta + \gamma \psi_2(I))I.$$

Multiplying both sides of the above equation by  $\alpha$  and moving the second term on the right-hand side to the left-hand side, we get

$$\alpha(\mu + \delta + \gamma \psi_2(I))I + \alpha \frac{\mathrm{d}I}{\mathrm{d}t} = \alpha\beta SI \exp\left(-\alpha \frac{\mathrm{d}I}{\mathrm{d}t}\right)$$

It follows that

$$\alpha \left[ (\mu + \delta + \gamma \psi_2(I))I + \frac{dI}{dt} \right]$$
  
 
$$\times \exp \left( \alpha \frac{dI}{dt} + \alpha (\mu + \delta + \gamma \psi_2(I))I \right)$$
  
 
$$= \alpha \beta SI \exp(\alpha (\mu + \delta + \gamma \psi_2(I))I).$$

Employing the properties of Lambert W function yields

$$\frac{dI}{dt} = \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta + \gamma\psi_2(I))I))$$
  
-  $(\mu + \delta + \gamma\psi_2(I))I.$  (5)

Setting dI/dt = 0, we get

$$S = \frac{\mu + \delta + \gamma \psi_2(I)}{\beta} = \begin{cases} \frac{\mu + \delta}{\beta} \equiv S_{c1}, \\ \frac{\mu + \delta + \gamma}{\beta} \equiv S_{c2}, \end{cases}$$
(6)

according to the definition of  $\psi_2(I)$ . Note that  $\frac{dI}{dt}$  defined by (5) is strictly monotonic with respect to *S*, so  $\frac{dI}{dt} > 0$  is mathematically equivalent to  $S > S_{c1}$  (or  $S > S_{c2}$ ). It follows that

$$\frac{\mathrm{d}I}{\mathrm{d}t} > 0 \Longleftrightarrow S > S_c$$

where  $S_c$  denotes  $S_{c1}$  or  $S_{c2}$ . Then model (4) becomes as follows, which we shall refer to as the targeted model:

Before further examining the dynamics of model (7), we introduce some technical definitions (Utkin 1992; Filippov 1988).

Denote  $R_+^2 = \{Z = (x, y) : x \ge 0, y \ge 0\}$ . A general planar non-smooth continuous system takes the form

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \begin{cases} F_{G_1}(Z), & \sigma(Z) \le 0, \\ F_{G_2}(Z), & \sigma(Z) > 0, \end{cases}$$

$$\tag{8}$$

with  $F_{G_1}(Z) \equiv F_{G_2}(Z)$  for  $Z \in \{Z : \sigma(Z) = 0\}$ , where  $\sigma(Z)$  is a smooth scalar function.

**Definition 2** Let  $Z^*$  be such that  $F_{G_i}(Z^*) = 0$  (i = 1, 2). Then,  $Z^*$  is called a *real equilibrium* of system (8) if it belongs to  $G_i$  and a *virtual equilibrium* if it belongs to  $G_j, j \neq i$ . Both the real equilibrium and virtual equilibrium are called *regular equilibria*.

**Definition 3** A point  $Z^*$  is called a *generalized singular* point of model (8) if  $F_{G_1}\sigma(Z^*)F_{G_2}\sigma(Z^*) \leq 0$ , where  $F_{G_i}\sigma(Z^*) = F_{G_i}(Z) \cdot \text{grad } \sigma(Z^*)$  (i = 1, 2) represents the Lie derivative of  $\sigma$  with respect to the vector field  $F_{G_i}$  at the point  $Z^*$ . The generalized singular point is also called an *irregular singular point*.

A general Filippov system reads

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \begin{cases} F_{G_1}(Z), & \sigma(Z) < 0, \\ F_{G_2}(Z), & \sigma(Z) > 0, \end{cases}$$

$$\tag{9}$$

where  $\sigma(Z)$  is a smooth scalar function. Besides the regular equilibria, another type of equilibrium may exist for Filippov system (9), which is defined as follows.

**Definition 4** A point  $Z^*$  is called a *pseudo-equilibrium* of Filippov system (9) if it satisfies  $\lambda F_{G_1}(Z^*) + (1 - \lambda)F_{G_2}(Z^*) = 0, \sigma(Z^*) = 0$  with  $0 < \lambda < 1$  and

$$\lambda = \frac{F_{G_2}\sigma(Z^*)}{(F_{G_2} - F_{G_1})\sigma(Z^*)}$$

This suggests a pseudo-equilibrium is indeed an equilibrium of the sliding-mode dynamics.

#### Media impact switching policy

Two threshold strategies occur in model (1), which are guided by the case number or the derivative of the case number. In particular, the derivative of the case number guides whether the media impact occurs in the disease control, while the case number itself guides whether the enhanced vaccination and treatment are carried out. In this section, we first examine the dynamics of system (7) with only the media impact switching policy. This type of switching policy is guided by the sign of the derivative of case number, which is mathematically converted to the switching policy guided by the susceptible numbers according to the "Model formulation" section. In this case, there are two possibilities to consider:  $I_c = +\infty$  and  $I_c = 0$ . For  $I_c = +\infty$ , no enhanced vaccination or treatment strategy is carried out, so the system takes the form

$$\frac{dS}{dt} = \mu - \beta SI - \mu S - pS, \\
\frac{dI}{dt} = \beta SI - \mu I - \delta I,
\end{cases} S \le S_{c1},$$

$$\frac{dS}{dt} = \mu - \beta \exp\left(-\alpha \frac{dI}{dt}\right) SI - \mu S - pS, \\
\frac{dI}{dt} = \beta \exp\left(-\alpha \frac{dI}{dt}\right) SI - \mu I - \delta I,
\end{cases} S > S_{c1},$$
(10)

which is a non-smooth but continuous model. In this case, the media coverage takes effect only when the proportion of susceptible individuals is above the threshold value  $S_{c1}$ ; otherwise, the classic model is present. For  $I_c = 0$ , enhanced vaccination and treatment are always adopted, so the system takes the form

$$\frac{dS}{dt} = \mu - \beta SI - \mu S - (p+f)S + \gamma I, \\
\frac{dI}{dt} = \beta SI - \mu I - \delta I - \gamma I, \\
\frac{dS}{dt} = \mu - \beta \exp\left(-\alpha \frac{dI}{dt}\right)SI - \mu S - (p+f)S + \gamma I, \\
\frac{dI}{dt} = \beta \exp\left(-\alpha \frac{dI}{dt}\right)SI - \mu I - \delta I - \gamma I, \\
\end{bmatrix} S > S_{c2}.$$
(11)

For convenience, we denote  $Z = (S, I)^{T}$  in the rest of this work. System (10) can be rewritten as

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \begin{cases} F_{M_1}, & S \le S_{c1}, \\ F_{M_2}, & S > S_{c1}, \end{cases}$$
  
with  $F_{M_1} = (F_{11}, F_{12})^{\mathrm{T}}, F_{M_2} = (F_{21}, F_{22})^{\mathrm{T}}.$  Denote  
 $G_{M_1} = \{(S, I) \in R_+^2 : S \le S_{c1}\}, \quad G_{M_2} = \{(S, I) \in R_+^2 : S > S_{c1}\}$ 

and  $\Sigma_{M_1} = \{(S_{c1}, I) : I \ge 0\}$ . We denote the subsystem of (10) determined by  $F_{M_i}(i = 1, 2)$  as subsystem  $S_{M_i}$ .

For subsystem  $S_{M_1}$ , there is one disease-free equilibrium  $E_{01} = (\mu/(\mu + p), 0)$ , which is locally asymptotically stable if  $R_{01} < 1$ , where  $R_{01} = \beta \mu/(\mu + p)(\mu + \delta)$ . There exists one endemic equilibrium  $E_1 = (S_1, I_1)$  for  $R_{01} > 1$  with

$$S_1 = \frac{\mu + \delta}{\beta}, \quad I_1 = \frac{\beta \mu - (\mu + p)(\mu + \delta)}{\beta(\mu + \delta)}.$$

The Jacobian of subsystem  $S_{M_1}$  is

$$J_{M_1}(S,I) = \begin{pmatrix} -(\mu+p) - \beta I & -\beta S \\ \beta I & -(\mu+\delta) + \beta S \end{pmatrix}.$$

Then,  $E_1$  is locally asymptotically stable provided it is feasible. We can further preclude the existence of limit cycles for  $S_{M_1}$  by defining the classic Dulac function  $B_1 = 1/SI$ . The endemic equilibrium  $E_1$  (or disease-free equilibrium  $E_{01}$ ) is thus globally asymptotically stable for  $R_{01} > 1$  (or  $R_{01} < 1$ ).

For subsystem  $S_{M_2}$ , it follows from the properties of the Lambert *W* function that

$$\exp\left(-\alpha \frac{dI}{dt}\right) = \frac{\text{Lambert } W(\alpha\beta SI \exp(\alpha(\mu+\delta)))}{\alpha\beta SI},$$

so subsystem  $S_{M_2}$  can be rewritten as

$$\frac{dS}{dt} = \mu - \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))SI - (\mu + p)S,$$

$$\frac{dI}{dt} = \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))SI - (\mu + \delta)I.$$
(12)

There is one disease-free equilibrium  $E_{01}$  for system (12). The endemic equilibrium of (12) satisfies

$$\begin{cases} \mu - \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))SI - (\mu + p)S = 0\\ \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))SI - (\mu + \delta)I = 0. \end{cases}$$

We derive

Lambert 
$$W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))SI = \alpha(\mu + \delta)I \Longrightarrow S = \frac{\mu + \delta}{\beta}$$

from the last equation, so it is also  $E_1$  that acts as the endemic equilibrium of subsystem  $S_{M_2}$ . Note that both the disease-free equilibrium and the endemic equilibrium coincide with their counterpart of subsystem  $S_{M_1}$ . Denote

$$\mathcal{A}_{1} = \frac{\text{Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))}{\alpha S[1 + \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))]},$$
  
$$\mathcal{A}_{2} = \frac{[1 + \alpha(\mu + \delta)I] \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))}{\alpha I[1 + \text{ Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))]}.$$

The Jacobian of subsystem  $S_{M_2}$  takes the form

$$J_{M_2}(S,I) = egin{pmatrix} -\mathcal{A}_1 - (\mu + p) & -\mathcal{A}_2 \ \mathcal{A}_1 & \mathcal{A}_2 - (\mu + \delta) \end{pmatrix}.$$

The disease-free equilibrium  $E_{01}$  is locally asymptotically stable for subsystem  $S_{M_2}$  when  $R_{01} < 1$ , while the endemic equilibrium  $E_1$  is locally asymptotically stable for the subsystem  $S_{M_2}$  when  $R_{01} > 1$ .

We now rule out the existence of limit cycles for subsystem  $S_{M_2}$ . Define the Dulac function

$$B_2 = \frac{\alpha\beta}{\text{Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))},$$

and it follows that

$$\begin{aligned} \frac{\partial (B_2 F_{21})}{\partial S} &= -\frac{\alpha \beta (\mu + p)}{\mathcal{W}_1} - \frac{\alpha \beta (\mu - \alpha \beta (\mu + p)S}{\mathcal{W}_1 (1 + \mathcal{W}_1)S}, \\ \frac{\partial (B_2 F_{22})}{\partial I} &= -\frac{\alpha \beta (\mu + \delta)(1 + \mathcal{W}_1) - (\mu + \delta)[\alpha \beta + \alpha^2 \beta (\mu + \delta)I]}{\mathcal{W}_1 (1 + \mathcal{W}_1)} \end{aligned}$$

with

$$W_1$$
 = Lambert  $W(\alpha\beta SI \exp(\alpha(\mu + \delta)I))$ 

Thus, we get

$$\frac{\partial(B_2F_{21})}{\partial S} + \frac{\partial(B_2F_{22})}{\partial I} = -\frac{\alpha\beta(2\mu+p+\delta)W_1S + \alpha\beta\mu - \alpha^2\beta(\mu+\delta)^2SI}{W_1(1+W_1)S}$$

Since

$$S > S_{c1} \Longrightarrow \mathcal{W}_1 > \alpha(\mu + \delta)I,$$

we have

$$\frac{\partial(B_2F_{21})}{\partial S} + \frac{\partial(B_2F_{22})}{\partial I} < -\frac{\alpha\beta(\mu+p)\mathcal{W}_1S + \alpha\beta\mu}{\mathcal{W}_1(1+\mathcal{W}_1)S} < 0$$

for  $S_c > S_{c1}$ . This suggests that no limit cycle exists for the subsystem  $S_{M_2}$  if  $S_c > S_{c1}$ .

We now examine the dynamics of the non-smooth continuous system (10). If  $R_{01} < 1$ , the disease-free equilibrium  $E_{01} \in G_{M_1}$ , which is the disease-free equilibrium of both  $S_{M_1}$  and  $S_{M_2}$ , is locally asymptotically stable. If  $R_{01} > 1$ , the endemic equilibrium  $E_1 \in \Sigma_{M_1}$ , which is an endemic equilibrium of both  $S_{M_1}$  and  $S_{M_2}$ , is a generalized equilibrium of the non-smooth continuous system (10). By adopting the generalized Jacobian proposed by Leine (2006) and Clarke et al. (1998), we get the following Jacobian at the generalized equilibrium  $E_1$  for system (10):

$$J_M(S_{11}, I_{11}) = \begin{pmatrix} -q(\mu + p + \beta I_{11}) - (1 - q) \left[ \frac{(\mu + \delta)I_{11}}{[1 + \alpha(\mu + \delta)I_{11}]S_{11}} + (\mu + p) \right] & -(\mu + \delta) \\ q\beta I_{11} + (1 - q) \frac{(\mu + \delta)I_{11}}{[1 + \alpha(\mu + \delta)I_{11}]S_{11}} & 0 \end{pmatrix}$$

where  $q \in [0, 1]$ . It follows that the generalized equilibrium  $E_1$  is locally asymptotically stable for the non-smooth system (10). To further obtain the global stability of both the equilibria  $E_{01}$  and  $E_1$ , we examine the existence of limit cycles for system (10). In fact, no limit cycles totally in the subregion  $G_{M_i}$  (i = 1, 2) exist for system (10) by the above discussion, so we only need to exclude the existence of crossing cycles.

**Lemma 1** There is no crossing cycle containing pieces of the trajectories of subsystems  $S_{M_1}$  and  $S_{M_2}$  for the non-smooth continuous system (10).

#### Proof Let

$$B_M = \frac{\exp\left(\alpha \max\left\{0, \frac{dI}{dt}\right\}\right)}{SI},$$

which is continuous and equivalent to

$$B_M = \begin{cases} \frac{1}{SI} \equiv B_1, & S \leq S_{c1} \\ \frac{\alpha \beta}{\text{Lambert } W(\alpha \beta SI \exp(\alpha(\mu + \delta)I))} \equiv B_2, & S > S_{c1}. \end{cases}$$

Denote  $Q = \{(S, I) \in \mathbb{R}^2 : S > 0, I > 0\}$  and

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = F_M = \begin{cases} F_{M_1}, & S \le S_{c1}, \\ F_{M_2}, & S > S_{c1}, \end{cases}$$

and we derive

$$B_M F_M = \begin{cases} B_1 F_{M_1}, & S \le S_{c1}, \\ B_2 F_{M_2}, & S > S_{c1}. \end{cases}$$

It is easy to verify that  $F_M$  has the following properties:

- Q is open in  $\mathbb{R}^2$ , which is divided into two subregions  $Q_1 = \{(S, I) \in Q : S \leq S_{c1}\}$  and  $Q_2 = \{(S, I) \in Q : S > S_{c1}\}$  with  $\overline{Q}_1 \cup \overline{Q}_2 = \overline{Q}$ .
- $\overline{Q}_1 \cap \overline{Q}_2 = \Sigma_{M_1}$ , where  $F_M$  is piecewise smooth.
- $F_M$  is Lipschitz in both  $Q_1$  and  $Q_2$  and continuous along  $\Sigma_{M_1}$ .
- For every point  $Z_c \in \Sigma_{M_1}$ ,  $F_M$  specifies into which  $Q_i$  (i = 1, 2) the flow is directed.

Thus, we can prove this lemma based on the idea proposed by Melin (2004). Let  $\chi_i$  (i = 1, 2) be the characteristic function of  $Q_i$ . We get

$$B_M F_M = \chi_1 B_1 F_{M_1} + \chi_2 B_2 F_{M_2}$$
  
=  $(\chi_1 B_1 F_{11} + \chi_2 B_2 F_{21}, \chi_1 B_1 F_{12} + \chi_2 B_2 F_{22})^{\mathrm{T}}.$ 

Suppose that there is a non-smooth continuous limit cycle  $\Gamma_M$  in Q with tangent vector  $(\dot{S}, \dot{I})$ . Denote the interior of  $\Gamma_M$  as  $D_M$  and the normal vector of  $\Gamma_M$  as  $N = (-\dot{I}, \dot{S})$ . Let t be in some  $\Delta_i$  (i = 1, 2) and consider the part of the line integral  $\int_{\Gamma_M} \langle B_M F_M, N \rangle ds$  in  $Q_i$  (i = 1, 2). We get

$$\int_{\Gamma_{M}} \langle \chi_{1}B_{1}F_{M_{1}}, N \rangle ds = \int_{\Delta_{1}} \left( -B_{1}F_{11}\dot{I} + B_{1}F_{12}\dot{S} \right) dt$$
$$= \int_{\Delta_{1}} \left( -B_{1}F_{11}F_{12} + B_{1}F_{12}F_{11} \right) dt$$
$$= 0,$$
$$\int_{\Gamma_{M}} \langle \chi_{2}B_{2}F_{M_{2}}, N \rangle ds = \int_{\Delta_{2}} \left( -B_{2}F_{21}\dot{I} + B_{2}F_{22}\dot{S} \right) dt$$
$$= \int_{\Delta_{2}} \left( -B_{2}F_{21}F_{22} + B_{2}F_{22}F_{21} \right) dt$$

= 0,

so

$$\begin{split} \int_{\Gamma_M} \langle B_M F_M, N \rangle \mathrm{d}s &= \int_{\Gamma_M} \langle (\chi_1 B_1 F_{M_1} + \chi_2 B_2 F_{M_2}), N \rangle \mathrm{d}s \\ &= \int_{\Gamma_M} \langle \chi_1 B_1 F_{M_1}, N \rangle \mathrm{d}s + \int_{\Gamma_M} \langle \chi_2 B_2 F_{M_2}, N \rangle \mathrm{d}s \\ &= 0. \end{split}$$

Since both  $F_M(S, I)$  and  $B_M(S, I)$  are continuous in Q, the function  $B_M F_M(S, I) \in L^1(Q)$  in our case. According to Hörmander (1990), the Gauss–Green formula holds for  $B_M F_M(S, I)$ , which implies

$$\int \int_{D_M} \operatorname{div} B_M F_M \mathrm{d}S \mathrm{d}I = - \int_{\Gamma_M} \langle B_M F_M, N \rangle = 0 \qquad (13)$$

on the one hand. On the other hand, denote the Heaviside function

$$U_M(S) = \begin{cases} 0, & S \leq S_{c1}, \\ 1, & S > S_{c1}, \end{cases}$$

and so its general derivative is the following Dirac function:

$$\delta_M(S) = \begin{cases} 0, & S \neq S_{c1}, \\ +\infty, & S = S_{c1}. \end{cases}$$

Thus, the function  $B_M$  and the non-smooth system (10) can be written as

$$B_M = \frac{\exp(\alpha U_M(S)dI/dt)}{SI} = \frac{\exp(U_M(S)\mathcal{W}_1 - \alpha(\mu + \delta)U_M(S)I)}{SI},$$

with

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu - \beta \exp\left(-\alpha U_M(S)\frac{\mathrm{d}I}{\mathrm{d}t}\right)SI - \mu S - pS \equiv F_M^1$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta \exp\left(-\alpha U_M(S)\frac{\mathrm{d}I}{\mathrm{d}t}\right)SI - \mu I - \delta I \equiv F_M^2.$$

It follows that

$$\begin{aligned} \frac{\partial (B_M F_M^1)}{\partial S} &+ \frac{\partial (B_M F_M^2)}{\partial I} \\ &= -\frac{\mu}{S^2 I} \mathcal{A}_3 - \frac{\mu + \delta}{S} U_M(S) \mathcal{A}_3 \left[ \frac{1 + \alpha(\mu + \delta)I}{I} \frac{\mathcal{W}_1}{1 + \mathcal{W}_1} - \alpha(\mu + \delta) \right] \\ &+ \left( \frac{\mu}{SI} - \frac{\mu + p}{I} \right) \mathcal{A}_3 \left[ \delta_M(S) \mathcal{W}_1 + \frac{U(S) \mathcal{W}_1}{S(1 + \mathcal{W}_1)} - \alpha(\mu + \delta) \delta_M(S) I \right], \end{aligned}$$

with

$$\mathcal{A}_3 = \exp(U_M(S)\mathcal{W}_1 - \alpha(\mu + \delta)U_M(S)I).$$

Set  $A_{31} = \exp(W_1 - \alpha(\mu + \delta)I)$ . According to the properties of  $U_M(S)$  and  $\delta_M(S)$ , we get

$$\begin{split} & \frac{\partial (B_M F_M^1)}{\partial S} + \frac{\partial (B_M F_M^2)}{\partial I} \\ & = \begin{cases} & -\frac{\mu}{S^2 I}, & S < S_{c1}, \\ & -\mathcal{W}_{31} \left\{ \frac{\mu}{S^2 I 1 + \mathcal{W}_1} + \frac{\mu + p}{SI} \frac{\mathcal{W}_1}{1 + \mathcal{W}_1} + \frac{\mu + \delta}{S} [\mathcal{W}_1 - \alpha(\mu + \delta)I] \right\}, & S > S_{c1}, \\ & & -\frac{\mu}{S_{c1}^2 I}, & S = S_{c1}, \end{cases} \end{split}$$

so  $\frac{\partial(B_M F_M^1)}{\partial S} + \frac{\partial(B_M F_M^2)}{\partial I} < 0$  for  $(S, I) \in Q$ , which contradicts equality (13). Hence, no crossing cycle exists for the non-smooth continuous system (10).

We thus have the following theorem.

**Theorem 2** For system (10), the disease dies out for  $R_{01} < 1$ , while it becomes endemic for  $R_{01} > 1$ . In particular, the disease-free equilibrium  $E_{01}$  or the generalized endemic equilibrium  $E_1$  is globally asymptotically stable for  $R_{01} < 1$  or  $R_{01} > 1$ , as shown in Fig. 1.

For the non-smooth continuous system (11), implementing a similar procedure yields one disease-free equilibrium  $E_{02} = \left(\frac{\mu}{\mu+p+f}, 0\right)$ , which is globally asymptotically stable for  $R_{02} < 1$  with

$$R_{02} = \frac{\beta\mu}{(\mu + p + f)(\mu + \delta + \gamma)}$$

If  $R_{02} > 1$ , there exists one generalized endemic equilibrium  $E_2 = (S_2, I_2)$ , which is globally asymptotically stable, where

$$S_2 = \frac{\mu + \delta + \gamma}{\beta}, \quad I_2 = \frac{\beta \mu - (\mu + p + f)(\mu + \delta + \gamma)}{\beta(\mu + \delta)}.$$

### Sliding dynamics of the targeted model

To explore the global behavior of the targeted model (7), we examine the trajectory through a point  $Z \in \Sigma_j$  (j = 1, 2, 3). There are three types of regions crossing region, escaping region and sliding region on a discontinuous boundary—according to whether or not the vector field points toward it. See Wang and Xiao (2013) for the detailed definition of the three discontinuous boundaries.

For system (7), there are three non-smooth switching boundaries:

$$\begin{split} & \Sigma_1 = \{ (S_{c1}, I) : 0 \le I < I_c \}, \quad & \Sigma_2 = \{ (S_{c2}, I) : I_c < I \le 1 \}, \\ & \Sigma_3 = \{ (S, I_c) : S \ge 0 \}. \end{split}$$

They divide the (S, I) space  $R_+^2$  into four subregions as follows:

$$G_{1} = \{(S, I) \in R_{+}^{2} : S \leq S_{c1} \text{ and } I < I_{c}\},\$$

$$G_{2} = \{(S, I) \in R_{+}^{2} : S > S_{c1} \text{ and } I < I_{c}\},\$$

$$G_{3} = \{(S, I) \in R_{+}^{2} : S \leq S_{c2} \text{ and } I > I_{c}\},\$$

$$G_{4} = \{(S, I) \in R_{+}^{2} : S > S_{c2} \text{ and } I > I_{c}\}.$$

The dynamics in subregion  $G_i$  (i = 1, 2, 3, 4) are governed by  $F_i \equiv (F_{i1}, F_{i2})^{\mathrm{T}}$ , which we call subsystem  $S_i$  for convenience.

For the discontinuous boundary  $\Sigma_1$  and  $\Sigma_2$ , only the crossing region is available. For the discontinuous boundary  $\Sigma_3$ , define  $H(S,I) = I - I_c$ ; direct calculation yields no escaping region. We now examine the existence of a sliding-mode region for  $\Sigma_3$ , which is defined as

$$\Sigma_S = \{ Z \in \Sigma_V : F_l H(Z) \ge 0, F_m H(Z) \le 0 \},\$$

where l = 1 or 2, m = 3 or 4 and  $F_jH(Z) = F_j \cdot$ grad H(Z) (j = 1, 2, 3, 4) is the Lie derivative of H with

(a) S=S<sub>c1</sub>  $g_1^{\circ}$ 0.9 0.8 0.7 I 0.6 0.5 0.4 0.3 0.2 E<sub>01</sub> 0.1 0 0.6 0.7 0.2 0.4 0.5 0.8 0.9 0 0.1 0.3 S



**Fig. 1** Phase plane S-I for the non-smooth continuous model (10), showing the asymptotic equilibrium (small square  $E_{01}$  and small circle  $E_1$ ) for different parameter sets. The isoclinic lines  $g_S^1$  (resp.  $g_S^2$ ) and  $S = S_{c1}$  are plotted for the subsystem  $S_{M_1}$  ( $S_{M_2}$ ). The curves

respect to  $F_j$ . We have the following three possibilities to consider:

- (a)  $\Sigma_S = \{(S, I_c) : S_{c_{a1}} \le S \le S_{c_{b1}} \text{ with } S_{c_{a1}}, S_{c_{b1}} \in [0, S_{c_1}]\},$
- (b)  $\Sigma_{S} = \{(S, I_{c}) : S_{c_{a2}} \le S \le S_{c_{b2}} \text{ with } S_{c_{a2}}, S_{c_{b2}} \in [S_{c1}, S_{c2}]\},\$ (c)  $\Sigma_{S} = \{(S, I_{c}) : S_{c_{a3}} \le S \le S_{c_{b3}} \text{ with } S_{c_{a3}} \le S \le S_{c_{b3}} \text{ with } S_{c_{b3}} + S_{c_{b3}}$

(c) 
$$Z_{S} = \{(S, I_{c}) : S_{c_{a3}} \leq S \leq S \\ S_{c_{a3}}, S_{c_{b3}} \in [S_{c2}, 1]\}.$$

For (a), it is sufficient to solve

$$\begin{cases} F_1 H(Z) \ge 0, \\ F_3 H(Z) \le 0, \\ S \le S_{c1}. \end{cases}$$

Direct calculation yields no solution, which leads to a null set  $\Sigma_S$ . No sliding-mode region exists for system (7) in this scenario.

For (b), it is sufficient to solve the inequalities

$$\begin{cases} F_2 H(Z) \ge 0, \\ F_3 H(Z) \le 0, \\ S_{c1} \le S \le S_{c2}. \end{cases}$$

Solving  $F_2H(Z) \ge 0$  yields  $S \ge \frac{\mu+\delta}{\beta}$ . Similarly, solving  $F_3H(Z) \le 0$  yields  $S \le \frac{\mu+\delta+\gamma}{\beta}$ . We thus derive the sliding-mode region as the following form:

$$\Sigma_{S} = \left\{ (S, I_{c}) : \frac{\mu + \delta}{\beta} \le S \le \min\left\{ \frac{\mu + \delta + \gamma}{\beta}, 1 \right\} \right\},$$
(14)

represent the orbits in the phase plane indicating the asymptotic equilibrium. Parameters values are  $\mu = 0.2, p = 0.1, \delta = 0.2, \alpha = 0.8$  with (a)  $\beta = 0.55$  and (b)  $\beta = 0.8$ 

which is also called a sliding segment.

For (c),  $\Sigma_S$  is empty, and so no sliding-mode region exists. As a result, (14) is the unique sliding-mode region. In particular, (14) takes the form

$$\Sigma_{S} = \left\{ (S, I_{c}) : \frac{\mu + \delta}{\beta} \le S \le \frac{\mu + \delta + \gamma}{\beta} \right\} \equiv \Sigma_{S_{1}}$$

for  $\frac{\mu+\delta+\gamma}{\beta} \leq 1$ , while it takes the form

$$\Sigma_{S} = \left\{ (S, I_{c}) : \frac{\mu + \delta}{\beta} \le S \le 1 \right\} \equiv \Sigma_{S_{2}}$$

for  $\frac{\mu+\delta+\gamma}{\beta} > 1 \ge \frac{\mu+\delta}{\beta}$ .

Denote the two endpoints  $((\mu + \delta)/\beta, I_c)$  and  $((\mu + \delta + \gamma)/\beta, I_c)$  of the sliding segment  $\Sigma_s$  as  $T_1$  and  $T_2$ , and we have  $T_1 = (S_{c1}, I_c)$  and  $T_2 = (S_{c2}, I_c)$ .

Now we determine the dynamics of system (7) on the sliding-mode region  $\Sigma_s$ . There are three mathematical methods—Utkin's equivalent control method Utkin (1992), Filippov convex method Filippov (1988) and a singular approach Claudio et al. (2006)—to explore the sliding dynamics. We shall adopt the Filippov convex method to compute the sliding dynamics of system (7) on  $\Sigma_s$ . Let

$$\lambda = \frac{\langle H_Z(Z), F_3(Z) \rangle}{\langle H_Z(Z), F_3(Z) - F_2(Z) \rangle}.$$

Then, the sliding dynamics are determined by

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \lambda F_2 + (1-\lambda)F_3 = \begin{pmatrix} \lambda F_{21} + (1-\lambda)F_{31} \\ \lambda F_{22} + (1-\lambda)F_{32} \end{pmatrix}.$$
 (15)

After some algebra, we get

$$\lambda = \frac{\beta SI_c - (\mu + \delta + \gamma)I_c}{\beta SI_c - \frac{1}{\alpha} \text{ Lambert } W(\alpha\beta SI_c \exp(\alpha(\mu + \delta)I_c)) - \gamma I_c}.$$
(16)

Substituting (16) into (15) and simplifying yield dI/dt = 0 and

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu - (\mu + p + f)S - (\mu + \delta)I_c + \frac{(\beta S - \mu - \delta - \gamma)fI_cS}{(\beta S - \gamma)I_c - \frac{1}{\alpha}\mathcal{W}_1}$$
(17)

where  $S \in \Sigma_S$  and  $W_1$  are as defined in the "Media impact switching policy" section. Equation (17) is a scalar equation defined on the sliding-mode region  $\Sigma_S$  and describes the sliding dynamics of system (7). Let

$$\begin{split} G(S) &\equiv [\mu - (\mu + p + f)S - (\mu + \delta)I_c] \\ &\times \left[ (\beta S - \gamma)I_c - \frac{1}{\alpha}\mathcal{W}_1 \right] + (\beta S - \mu - \delta - \gamma)fI_cS, \\ \tilde{G}(S) &\equiv \frac{G(S)}{(\beta S - \gamma)I_c - \frac{1}{\alpha}\mathcal{W}_1}, \end{split}$$

and so equation (17) can be rewritten as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \tilde{G}(S).$$

We first examine the monotonicity of the function  $\tilde{G}(S)$ . According to the properties of the Lambert *W* function, we have  $W_1 \ge \alpha(\mu + \delta)I_c$  for  $S \in \Sigma_S$ . It follows that

$$(\beta S - \gamma)I_c - \frac{1}{\alpha}\mathcal{W}_1 \le (\beta S - \mu - \delta - \gamma)I_c \le 0$$
(18)

for  $S \in \Sigma_S$ , which suggests  $(\beta S - \gamma)I_c - \frac{1}{\alpha}W_1 < 0$  for  $S \in \Sigma_S$ . Since

$$\frac{\partial \tilde{G}}{\partial S} = \frac{G_1 + G_2 + G_3}{G^2(S)},$$

where

$$G_{1} = -(\mu + p + f) \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha}W_{1} \right]^{2},$$
  

$$G_{2} = \left[ 2fI_{c}\beta S - fI_{c}(\mu + \delta + \gamma) \right] \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha}W_{1} \right],$$
  

$$G_{3} = -fI_{c}S(\beta S - \mu - \delta - \gamma) \left[ \beta I_{c} - \frac{W_{1}}{\alpha S(1 + W_{1})} \right],$$

the monotonicity of the function  $\tilde{G}(S)$  is determined by the relationship of  $G_1 + G_2 + G_3$  and 0. To this end, we consider the following two cases in terms of the relationship of  $\mu + \delta$  and  $\gamma$ .

Case (A)  $\mu + \delta \ge \gamma$ . In this case, we have

$$S \in (S_{c1}, S_{c2}) \Longrightarrow S > \frac{\mu + \delta + \gamma}{2\beta},$$
  
so  $[2fI_c\beta S - fI_c(\mu + \delta + \gamma)] > 0$  and  
 $G_2 < [2fI_c\beta S - fI_c(\mu + \delta + \gamma)](\beta S - \mu - \delta - \gamma)I_c.$ 

It follows that

$$G_{2} + G_{3} < I_{c}^{2} f \left(\beta S - \mu - \delta - \gamma\right)^{2} + I_{c} f \left(\beta S - \mu - \delta - \gamma\right) \frac{W_{1}}{\alpha S (1 + W_{1})}.$$

By (18), we have

$$G_1 < -(\mu + p + f)[(\beta S - \mu - \delta - \gamma)I_c]^2,$$

so

$$G_1 + G_2 + G_3 < -(\mu + p)[(\beta S - \mu - \delta - \gamma)I_c]^2$$
  
+  $I_c f(\beta S - \mu - \delta - \gamma) \frac{W_1}{\alpha S(1 + W_1)}$   
< 0.

For  $S = S_{cj}$  (j = 1, 2), we can easily get  $G_1 + G_2 + G_3 < 0$ . Hence,  $\frac{\partial \tilde{G}}{\partial S} < 0$  and  $\tilde{G}$  is monotonically decreasing in this case.

Case (B)  $\mu + \delta < \gamma$ .

In this case, we have  $S_{c1} < \frac{\mu + \delta + \gamma}{2\beta} < \frac{\mu + \delta + \gamma}{2\beta}$ , and there are the following two possibilities to consider:  $(B_1) \quad \frac{\mu + \delta + \gamma}{2\beta} \leq S \leq S_{c2}; \quad (B_2) \quad S_{c1} \leq S < \frac{\mu + \delta + \gamma}{2\beta}.$ 

When  $(B_1)$  holds, we have  $G_1 + G_2 + G_3 < 0$  for  $S > \frac{\mu + \delta + \gamma}{2\beta}$  by Case (A), so we only need to examine whether  $G_1 + G_2 + G_3 < 0$  for the possibility  $S = \frac{\mu + \delta + \gamma}{2\beta}$ . According to (18), we have

$$G_{1} + G_{2} + G_{3} < -fI_{c}SG_{0}\left[\beta I_{c} - \frac{\mathcal{W}_{1}}{\alpha S(1+\mathcal{W}_{1})}\right]$$
$$-(\mu+p+f)G_{0}^{2}$$
$$\leq G_{0}\left[\frac{f\mathcal{W}_{1}}{\alpha(1+\mathcal{W}_{1})} - (\mu+p)G_{0} - f(2\beta S - \mu - \delta - \gamma)I_{c}\right]$$
$$\leq 0,$$

where  $G_0 = (\beta S - \mu - \delta - \gamma)I_c$ . Hence, we have  $G_1 + G_2 + G_3 < 0$  for the possibility  $(B_1)$ .

When  $(B_2)$  holds, we only need to examine the possibility  $S_{c1} < S < \frac{\mu + \delta + \gamma}{2\beta}$  since  $G_1 + G_2 + G_3 < 0$  for  $S = S_{c1}$ . We have

$$G_{1} + G_{2} = \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha}W_{1} \right]$$

$$\times \left\{ -(\mu + p) \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha}W_{1} \right] + \frac{1}{\alpha}fW_{1} + f\beta SI_{c} - (\mu + \delta)fI_{c} \right\}$$

and

$$- (\mu + p) \left[ (\beta S - \gamma) I_c - \frac{1}{\alpha} \mathcal{W}_1 \right]$$
$$+ \frac{1}{\alpha} f \mathcal{W}_1 + f \beta S I_c - (\mu + \delta) f I_c > 0.$$

It follows that

$$G_{1} + G_{2} + G_{3} < G_{0}$$

$$\times \left\{ -(\mu + p) \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha} \mathcal{W}_{1} \right] + \frac{1}{\alpha} f \mathcal{W}_{1} + f \beta SI_{c} - (\mu + \delta) f I_{c} \right\}$$

$$- G_{0} f S \left[ \beta I_{c} - \frac{\mathcal{W}_{1}}{\alpha S(1 + \mathcal{W}_{1})} \right]$$

$$< G_{0} \left\{ -(\mu + p) \left[ (\beta S - \gamma)I_{c} - \frac{1}{\alpha} \mathcal{W}_{1} \right] + \frac{f \mathcal{W}_{1}}{\alpha(1 + \mathcal{W}_{1})} \right\}$$

$$< 0.$$

By  $(B_1)$  and  $(B_2)$ , we have  $G_1 + G_2 + G_3 < 0$  for  $\mu + \delta < \gamma$ , which demonstrates that  $\tilde{G}$  is also monotonically decreasing in this case. Concluding the above discussion for Case (A) and (B), the function  $\tilde{G}(S)$  is monotonically decreasing for  $S \in \Sigma_S$ , which demonstrates that the equilibrium of Eq. (17) is unique if it is feasible.

Next we examine the existence of roots for G(S) on  $\Sigma_S$ . According to (18), it is sufficient to solve G(S) = 0 with respect to S. Direct calculation yields

$$G(S_{c1}) = -\frac{\gamma I_c}{\beta} [\beta \mu - (\mu + p)(\mu + \delta) - \beta(\mu + \delta)I_c],$$
  

$$G(S_{c2}) = \frac{1}{\beta} \left[ (\mu + \delta)I_c - \frac{1}{\alpha} \mathcal{W}_1 \right]$$
  

$$[\beta \mu - (\mu + p + f)(\mu + \delta + \gamma) - \beta(\mu + \delta)I_c].$$

By (18), an equilibrium  $S_c$  exists for equation (17) if  $G(S_{c1}) \ge 0$  and  $G(S_{c2}) \le 0$ , which are equivalent to  $I_2 \le I_c \le I_1$  provided  $I_j$  (j = 1, 2) is positive. In terms of the monotonicity of  $\tilde{G}$ , if there exists a root  $S_c$  for  $\tilde{G}$ , it is unique. If  $R_{01} < 1$ , we have  $R_{02} < 1$  and neither of  $I_1$  and  $I_2$  is positive. Then, no equilibrium exists for (17). If  $R_{02} > 1$ , both  $I_1$  and  $I_2$  are positive. If we further have  $I_2 \le I_c \le I_1$ , there is a unique equilibrium  $S_c$  for (17), and so a unique pseudo-equilibrium  $E_S \equiv (S_c, I_c)$  exists for system (7). If  $R_{02} < 1 < R_{01}$ ,  $I_1$  is positive, while  $I_2$  is negative. If  $I_c \ge I_2$  is further true, a unique equilibrium  $S_c$  exists for (17), and so a unique pseudo-equilibrium  $E_S \equiv (S_c, I_c)$  exists for system (7).

(7). Note that  $\hat{G}(S)$  is monotonically decreasing on the sliding-mode region  $\Sigma_S$ ; we further know the unique pseudo-equilibrium  $E_S$  is locally asymptotically stable within the vicinity of  $\Sigma_S$  provided  $E_S$  exists. Therefore, we have the following result.

#### Theorem 3

(i) There is a unique pseudo-equilibrium  $E_S$  for system (7), which is locally asymptotically stable within the vicinity of  $\Sigma_{S_1}$ , if the following inequalities hold:

 $(H_1) R_{02} > 1 \text{ and } I_2 \leq I_c \leq I_1.$ 

(ii) The unique pseudo-equilibrium E<sub>S</sub> exists for system
 (7) and is locally asymptotically stable within the vicinity of Σ<sub>S2</sub> if the following inequalities hold:
 (H<sub>2</sub>) R<sub>02</sub> < 1 < R<sub>01</sub> and I<sub>c</sub> < I<sub>1</sub>.

#### Global dynamics of the targeted model

In this section, we will examine the long-term behavior of system (7). To this end, we initially conclude the dynamics of subsystem  $S_i$  as follows: For subsystems  $S_1$  and  $S_2$ , by the "Media impact switching policy" section, there is a generalized equilibrium  $E_1$ , which is locally asymptotically stable within the region  $G_1 \cup G_2$ ; for subsystems  $S_3$  and  $S_4$ , we similarly derive that there is another generalized equilibrium  $E_2$ , which is locally asymptotically stable within the region  $G_3 \cup G_4$ .

It is worth mentioning that there are three switching boundaries,  $\Sigma_1, \Sigma_2$  and  $\Sigma_3$ . Different locations of the equilibria  $E_i(i = 1, 2)$  between or on these switching boundaries will lead to different results. We first consider the location of equilibrium  $E_1$  (resp.  $E_2$ ) on the switching boundary  $\Sigma_1$ (resp.  $\Sigma_2$ ); since it is exactly on the switching boundary, it is not a regular equilibrium but a generalized equilibrium. Second, we analyze the location of the equilibrium  $E_1$  (or  $E_2$ ) compared to the switching boundary  $\Sigma_3$ , for which there are two cases: (a)  $E_1$  (resp.  $E_2$ ) is below  $\Sigma_3$  (resp. above  $\Sigma_3$ ); i.e., it is in its own region  $G_1 \cup G_2$  (resp.  $G_3 \cup G_4$ ); (b)  $E_1$ (resp.  $E_2$ ) is above  $\Sigma_3$  (resp. below  $\Sigma_3$ ); i.e., it is in the opposite region  $G_3 \cup G_4$  (resp.  $G_1 \cup G_2$ ). If (a) is true, the equilibrium  $E_i$  (j = 1, 2) is not only generalized but also real. To highlight this nature and distinguish it from the generalized equilibrium of a non-smooth continuous system with one threshold and distinguish it from the real equilibrium of a Filippov system with one threshold, we denote it by  $E_i^{\text{gr}}$ . If (b) is true,  $E_j$  is virtual, denoted by  $E_i^{gv}$ . Only the real generalized equilibria can be attractors for system (7).

To further address the global stability of the equilibria for system (7), we next explore the nonexistence of limit cycles.

#### Nonexistence of limit cycles

In this section, we shall establish several lemmas to preclude the existence of limit cycles for system (7). If there exist limit cycles for system (7), they could take one of the following forms:

 $(L_1)$  Regular cycles, those lying totally in one of the regions  $G_j$  (j = 1, 2, 3, 4), as shown in Fig. 2a.

( $L_2$ ) Crossing cycles (I) that do not surround the sliding segment, which are composed of pieces of trajectories of the subsystems  $S_j$  and  $S_{j+1}$ , where j = 1 or j = 3, as shown in Fig. 2b.

(*L*<sub>3</sub>) Crossing cycles (II) that do not surround the sliding segment, which are composed of pieces of trajectories of the subsystems  $S_j$  and  $S_{j+2}$ , and points of the crossing region  $\Sigma_3 \setminus \Sigma_S$ , where j = 1 or j = 2, as shown in Fig. 2c.

 $(L_4)$  Canard cycles, which are composed of pieces of the trajectories of subsystems  $S_j$  and  $S_{j+1}$ , and pieces of the trajectory of sliding dynamics with j = 1 or j = 3. These types of cycles are tangent to or contain part of the sliding segment, as shown in Fig. 2d.

(*L*<sub>5</sub>) Crossing cycles surrounding the sliding segment, which are composed of pieces of orbits of subsystems  $S_j$  (j = 1, 2, 3, 4), and points of the crossing region  $\Sigma_3 \setminus \Sigma_S$ , as shown in Fig. 2e.

According to the "Media impact switching policy" section, there are no limit cycle totally in subregion  $G_1$  and

no limit cycle totally in subregion  $G_2$  for system (7), which can be seen by defining the Dulac functions  $B_1$  and  $B_2$ . We can easily get the nonexistence of limit cycles totally contained in subregion  $G_3$  for system (7) by adopting the Dulac function  $B_1$ . Define the Dulac function

$$B_3 = \frac{\alpha\beta}{\text{Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta + \gamma)I))},$$

and we can rule out the existence of a limit cycle totally in subregion  $G_4$  for system (7). Thus, no limit cycle of the form  $(L_1)$  exists, as shown in Fig. 2a, and we get the following conclusion.

#### Lemma 4 There is no regular cycle for system (7).

For those crossing cycles of the form  $(L_2)$ , Lemma 1 excludes the existence of those cycles totally in the region  $G_1 \bigcup \Sigma_1 \bigcup G_2$  (i.e., the limit cycle  $\gamma_1$  shown in Fig. 2b) for system (7). Let

$$B_{M_1} = \begin{cases} \frac{1}{SI}, & S \leq S_{c2}, \\ \\ \frac{\alpha\beta}{\text{Lambert } W(\alpha\beta SI \exp(\alpha(\mu + \delta + \gamma)I))}, & S > S_{c2}. \end{cases}$$

Performing a similar analysis to Lemma 1, we derive that there is no crossing cycle totally in the region  $G_3 \bigcup \Sigma_3 \bigcup G_4$ , which rules out the existence of the limit cycle  $\gamma_2$  shown in Fig. 2).



**Fig. 2** Phase plane of model (7) demonstrating the form of possible limit cycles. The thick gray solid lines represent sliding segments, the thin gray dashed lines represent crossing regions, and  $\gamma_i$  (j = 1, 2, 3, 4) denotes the possible limit cycles

For those crossing cycles of the form  $(L_3)$ —i.e., the limit cycles  $\gamma_1$  and  $\gamma_2$  shown in Fig. 2c—we shall explore their existence by analyzing the vector field of system (7). Without loss of generality, we carry out the analysis for the case when  $E_1^{\text{gr}}$  and  $E_2^{gv}$  coexist for system (7). The vector field to the left of the null isocline  $g_I^1$  and  $g_I^2$  points downward, and that to the right of  $g_I^1$  and  $g_I^2$  points upward, as shown in Fig. 3, so no crossing cycle of the form  $(L_3)$  exists for system (7). Now we have ruled out the existence of all possible crossing cycles that do not surround the sliding segment; i.e., the cycles of the form  $(L_2)$  and  $(L_3)$  as shown in Fig. 2b, c.

# **Lemma 5** There is no crossing cycle that does not surround the sliding segment for system (7).

We now rule out the existence of the canard cycles; i.e., the limit cycles of the form  $(L_4)$ , which are tangent to or contain part of the sliding segment as shown in Fig. 2d.

#### **Lemma 6** There are no canard cycles for system (7).

**Proof** It follows from the "Sliding dynamics of the targeted model" section that if there exists a pseudo-equilibrium  $E_s$  for system (7), it is stable within the vicinity of the sliding segment  $\Sigma_3$ . No canard cycles exist in this scenario, and it is sufficient to preclude the existence of the canard cycles when  $E_j^{\text{gr}}$  and  $E_l^{gv}$   $(j, l = 1, 2 \text{ and } j \neq l)$ coexist for system (7). Without loss of generality, we implement the analysis for the case when  $E_1^{\text{gr}}$  and  $E_2^{gv}$ coexist, as shown in Fig. 5. By Sect. 3.1,  $E_1^{\text{gr}}$  is locally



**Fig. 3** Phase plane of model (7) indicating the vector field in different subregions divided by the switching boundaries  $S = S_{c1}, S = S_{c2}, I = I_c$ , the horizontal null isocline  $g_I^1$  (resp.  $g_I^2$ ) of the subsystems  $F_1$  and  $F_2$  (resp. subsystems  $F_3$  and  $F_4$ ), and the vertical null isoclines  $g_S^i$  (j = 1, 2, 3) of the subsystems  $F_j$  (j = 1, 2, 3). Parameter values are:  $\mu = 0.5, \alpha = 0.8, \beta = 1.8, \delta = 0.2, \gamma = 0.5, p = 0.1, f = 0.1, \tau = 0.5$ .

stable within the region  $G_1 \cup G_2$ . We claim that the trajectory initiating from the point  $T_1$ , which we denote as  $l_1$ for convenience, will not reach the switching boundary  $\Sigma_3$ again. In fact,  $l_1$  approaches  $E_1^{\text{gr}}$  directly or spirally depending on whether  $E_1^{\text{gr}}$  is a node or a focus. If the former case is true,  $l_1$  cannot reach  $\Sigma_3$ . If the latter case is true,  $l_1$ must intersect with the line segment  $\overline{T_1}E_1^{\text{gr}}$  at some point *P*. Since  $\overline{T_1}E_1^{\text{gr}}$  is a non-tangent segment, the point *P* is between  $T_1$  and  $E_1^{\text{gr}}$ . This suggests the trajectory  $l_1$  cannot reach  $\Sigma_3$  again. Therefore, no limit cycle is tangent to or contains part of the sliding segment  $\Sigma_3$  for system (7).  $\Box$ 

Finally, we shall prove there is no crossing cycle surrounding the sliding segment.

# **Lemma 7** There is no limit cycle surrounding the sliding segment for system (7).

**Proof** Suppose there is a limit cycle  $\Gamma$  surrounding the sliding segment  $\Sigma_S$ for system (7). Let  $\Gamma_i = \Gamma \cap G_i$  (j = 1, 2, 3, 4), as shown in Fig. 4. Denote the intersection points of  $\Gamma$  and the switching boundaries  $\Sigma_i$  (*j* = 1, 2, 3) by  $D_l$  (*l* = 1, 2, 3, 4); the intersection points of  $\Gamma$  and the two horizontal auxiliary lines  $I = I_c - s$ , I = $I_c + s$  by  $M_{11}, M_{12}, M_{21}, M_{22}$ ; the intersection points of  $\Gamma$ and the four vertical auxiliary lines  $S = S_{c1} - s$ ,  $S = S_{c1} + s$  $s, S = S_{c2} - s, S = S_{c2} + s$  by  $M_{j1}$  (j = 3, 4, 5, 6); and the intersection points of the two horizontally auxiliary lines and the four vertically auxiliary lines by  $M_{j2}$  (j = 3, 4, 5, 6), where s is any sufficiently small positive number. Let  $D_{\Gamma_1}$ (resp.  $D_{\Gamma_2}$ ) be the bounded region that is delimited by  $\Gamma_1$ (resp.  $\Gamma_2$ ) and the auxiliary lines  $I = I_c - s$  and  $S = S_{c1} - s$ (resp.  $S = S_{c1} + s$ ); and  $D_{\Gamma_3}$  (resp.  $D_{\Gamma_4}$ ) be the region delimited by  $\Gamma_3$  (resp.  $\Gamma_4$ ) and the auxiliary lines  $I = I_c + s$ and  $S = S_{c2} - s$  (resp.  $S = S_{c2} + s$ ). Denote the boundary of  $D_{\Gamma_i} j = 1, 2, 3, 4$  by  $X_{\Gamma_i}$ .

Let the Dulac functions  $B_1, B_2, B_3$  be defined as before. By Green's theorem, we derive

$$\int \int_{D_{\Gamma_1}} \left[ \frac{\partial (B_1 F_{11})}{\partial S} + \frac{\partial (B_1 F_{12})}{\partial I} \right] dS dI$$

$$= \oint_{X_{\Gamma_1}} B_1 (F_{11} dI - F_{12} dS)$$

$$= \int_{\overline{M_{31}M_{32}}} B_1 F_{11} dI - \int_{\overline{M_{32}M_{11}}} B_1 F_{12} dS$$

$$\int \int_{D_{\Gamma_2}} \left[ \frac{\partial (B_2 F_{21})}{\partial S} + \frac{\partial (B_2 F_{22})}{\partial I} \right] dS dI$$

$$= \oint_{X_{\Gamma_2}} B_2 (F_{21} dI - F_{22} dS)$$

$$= \int_{\overline{M_{42}M_{41}}} B_2 F_{21} dI - \int_{\overline{M_{12}M_{42}}} B_2 F_{22} dS$$



**Fig. 4** Schematic diagram illustrating the nonexistence of limit cycles surrounding the sliding segment for model (7). The gray lines represent the switching boundaries  $\Sigma_j$  (j = 1, 2, 3), the solid one of which represents the sliding segment  $\Sigma_s$ . The segments  $\Gamma_j$  (j = 1, 2, 3, 4) are pieces of the limit cycle  $\Gamma$  split by the switching boundary

Suppose abscissae the of the points  $x_1, x_2, x_1 + s_1(s), x_1 +$  $D_1, D_2, M_{11}, M_{21}, M_{12}, M_{22}$ are  $\tilde{s}_1(s), x_2 - s_2(s), x_2 - \tilde{s}_2(s)$  and the ordinates of the points  $D_3, D_4, M_{31}, M_{41}, M_{51}, M_{61}$ are  $y_1, y_2, y_1 + s_3(s), y_1 -$  $\tilde{s}_3(s), y_2 + s_4(s)$  and  $y_2 - \tilde{s}_4(s)$ , where  $s_i(s)$  (j = 1, 2, 3, 4)and  $\tilde{s}_i(s)$  are continuously dependent on s and satisfy  $\lim_{s\to 0} s_j(s) = \lim_{s\to 0} \tilde{s}_j(s) = 0, s_j(0) = \tilde{s}_j(0) = 0.$  Then we have

$$\begin{split} & \int_{\overline{M_{31}M_{32}}} B_1 F_{11} dI + \int_{\overline{M_{42}M_{41}}} B_2 F_{21} dI \\ &= \int_{y_3 + s_3(s)}^{I_c - s} B_1 F_{11} dI + \int_{I_c - s}^{y_3 + \tilde{s}_3(s)} B_2 F_{21} dI \\ &= \int_{y_3 + s_3(s)}^{I_c - s} \left[ -\beta + \frac{\mu}{SI} - \frac{\mu + p}{I} \right] dI \\ &+ \int_{I_c - s}^{y_3 + \tilde{s}_3(s)} \left[ -\beta + \frac{\alpha\beta\mu}{W_1} - \frac{\alpha\beta(\mu + p)S}{W_1} \right] dI. \end{split}$$

By the properties of the Lambert W function, we derive

$$\lim_{s \to 0} \left[ \frac{\mu}{SI} - \frac{\alpha \beta \mu}{\mathcal{W}_1} \right]$$
  
=  $\frac{\beta \mu}{\alpha (\mu + \delta)^2 I^2} [$ Lambert  $W(\alpha (\mu + \delta)I \exp(\alpha (\mu + \delta)I))$   
-  $\alpha (\mu + \delta)I ] = 0.$ 

We similarly have

$$\lim_{s \to 0} \left[ \frac{\alpha \beta(\mu + p)S}{\mathcal{W}_1} - \frac{\mu + p}{I} \right] = 0,$$

so

$$\lim_{s \to 0} \left\{ \int_{\overline{M_{31}M_{32}}} B_1 F_{11} dI + \int_{\overline{M_{42}M_{41}}} B_2 F_{21} dI \right\} = 0$$

Applying Green's theorem on  $D_{\Gamma_3}$  and  $D_{\Gamma_4}$  yields

$$\int \int_{D_{\Gamma_3}} \left[ \frac{\partial (B_1 F_{31})}{\partial S} + \frac{\partial (B_1 F_{32})}{\partial I} \right] dS dI$$
  
=  $\oint_{X_{\Gamma_3}} B_1(F_{31} dI - F_{32} dS)$   
=  $\int_{\overline{M_{52}M_{51}}} B_1 F_{31} dI - \int_{\overline{M_{21}M_{52}}} B_1 F_{32} dS,$   
 $\int \int_{D_{\Gamma_4}} \left[ \frac{\partial (B_3 F_{41})}{\partial S} + \frac{\partial (B_3 F_{42})}{\partial I} \right] dS dI$   
=  $\oint_{X_{\Gamma_4}} B_3(F_{41} dI - F_{42} dS)$   
=  $\int_{\overline{M_{61}M_{62}}} B_3 F_{41} dI - \int_{\overline{M_{62}M_{22}}} B_3 F_{42} dS.$ 

Then we have

$$\begin{split} &\lim_{s \to 0} \left\{ \int_{\overline{M_{52}M_{51}}} B_1 F_{31} dI + \int_{\overline{M_{61}M_{62}}} B_3 F_{41} dI \right\} \\ &= \lim_{s \to 0} \left\{ \int_{I_c + s}^{y_4 + s_4(s)} B_1 F_{31} dI + \int_{y_4 - \bar{s}_4(s)}^{I_c + s} B_3 F_{41} dI \right\} \\ &= \lim_{s \to 0} \left\{ \int_{I_c + s}^{y_4 + s_4(s)} \left[ -\beta + \frac{\gamma}{S} + \frac{\mu}{SI} - \frac{\mu + p + f}{I} \right] dI \\ &+ \int_{y_4 - \bar{s}_4(s)}^{I_c + s} \left[ -\beta + \frac{\alpha\beta\mu}{W_1} - \frac{\alpha\beta(\mu + p + f)S}{W_1} + \frac{\alpha\beta\gamma I}{W_1} \right] dI \right\} \\ &= 0. \end{split}$$

It follows that

$$\begin{split} \int \int_{D_{\Gamma_1}} \left[ \frac{\partial(B_1F_{11})}{\partial S} + \frac{\partial(B_1F_{12})}{\partial I} \right] \mathrm{d}S\mathrm{d}I \\ &+ \int \int_{D_{\Gamma_2}} \left[ \frac{\partial(B_2F_{21})}{\partial S} + \frac{\partial(B_2F_{22})}{\partial I} \right] \mathrm{d}S\mathrm{d}I \\ &+ \int \int_{D_{\Gamma_3}} \left[ \frac{\partial(B_1F_{31})}{\partial S} + \frac{\partial(B_1F_{32})}{\partial I} \right] \mathrm{d}S\mathrm{d}I \\ &+ \int \int_{D_{\Gamma_4}} \left[ \frac{\partial(B_3F_{41})}{\partial S} + \frac{\partial(B_3F_{42})}{\partial I} \right] \mathrm{d}S\mathrm{d}I \\ &= -\int_{M_{32}M_{11}} B_1F_{12}\mathrm{d}S, -\int_{M_{12}M_{42}} B_2F_{22}\mathrm{d}S \\ &- \int_{M_{21}M_{52}} B_1F_{32}\mathrm{d}S \\ &- \int_{M_{62}M_{22}} B_3F_{42}\mathrm{d}S \\ &= \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\mu + \delta}{S} \right] \mathrm{d}S \\ &+ \int_{S_{c1}+s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}+s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta + \gamma)I}{W_2} \right] \mathrm{d}S \\ &= \int_{S_{c2}+s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta + \gamma)I}{W_2} \right] \mathrm{d}S \\ &+ \int_{S_{c2}+s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &+ \int_{S_{c2}+s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &+ \int_{S_{c2}-s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &+ \int_{S_{c2}-s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}+s}^{S_{c2}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c2}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu + \delta)I}{W_1} \right] \mathrm{d}S \\ &- \int_{S_{c1}-s}^{S_{c1}-s} \left[ \beta - \frac{\alpha\beta(\mu +$$

where

 $W_2$  = Lambert  $W(\alpha\beta SI \exp(\alpha(\mu + \delta + \gamma)I))$ .

Direct calculation gives

$$\begin{split} &\lim_{s\to 0} \left\{ \int_{x_1+s_1(s)}^{S_{c1}-s} \left[ \beta - \frac{\mu+\delta}{S} \right] \mathrm{d}S - \int_{x_1+\tilde{s}_1(s)}^{S_{c1}} \left[ \beta - \frac{\mu+\delta+\gamma}{S} \right] \mathrm{d}S \right\} \\ &= \gamma \ln \frac{S_{c1}}{x_1}. \end{split}$$

By the properties of the Lambert W function, we have

$$\begin{split} &\lim_{s\to 0} \left\{ \int_{S_{c2}}^{x_2-s_2(s)} \left[ \beta - \frac{\alpha\beta(\mu+\delta)I}{W_1} \right] \mathrm{d}S - \int_{S_{c2}+s}^{x_2-\tilde{s}_2(s)} \left[ \beta - \frac{\alpha\beta(\mu+\delta+\gamma)I}{W_2} \right] \mathrm{d}S \right\} \\ &> \frac{\alpha\beta I}{\alpha(\mu+\delta+\gamma)I\overline{W_1}} \left[ (\mu+\delta+\gamma)\overline{W_1} - \alpha(\mu+\delta)(\mu+\delta+\gamma)I \right] \\ &> 0, \end{split}$$

where

 $\overline{W}_1$  = Lambert  $W(\alpha(\mu + \delta + \gamma)I\exp(\alpha(\mu + \delta)I))$ .

Similarly, we get

$$\lim_{s\to 0}\left\{\int_{S_{c1}+s}^{S_{c2}}\left[\beta-\frac{\alpha\beta(\mu+\delta)I}{\mathcal{W}_1}\right]\mathrm{d}S-\int_{S_{c1}}^{S_{c2}-s}\left[\beta-\frac{(\mu+\delta+\gamma)}{S}\right]\mathrm{d}S\right\}=0.$$

Hence, we have

$$\begin{split} &\lim_{s \to 0} \left\{ \int \int_{D_{\Gamma_1}} \left[ \frac{\partial (B_1 F_{11})}{\partial S} + \frac{\partial (B_1 F_{12})}{\partial I} \right] \mathrm{d}S \mathrm{d}I + \int \int_{D_{\Gamma_2}} \left[ \frac{\partial (B_2 F_{21})}{\partial S} + \frac{\partial (B_2 F_{22})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \\ & \int \int_{D_{\Gamma_3}} \left[ \frac{\partial (B_1 F_{31})}{\partial S} + \frac{\partial (B_1 F_{32})}{\partial I} \right] \mathrm{d}S \mathrm{d}I + \int \int_{D_{\Gamma_4}} \left[ \frac{\partial (B_3 F_{41})}{\partial S} + \frac{\partial (B_3 F_{42})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \right\} \\ & \geq \gamma \ln \frac{S_{c_1}}{x_1} \\ &> 0. \end{split}$$

$$(19)$$

However, according to the "Media impact switching policy" section, we have

$$\begin{split} &\int \int_{D_{\Gamma_1}} \left[ \frac{\partial (B_1 F_{11})}{\partial S} + \frac{\partial (B_1 F_{12})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \\ &+ \int \int_{D_{\Gamma_2}} \left[ \frac{\partial (B_2 F_{21})}{\partial S} + \frac{\partial (B_2 F_{22})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \\ &\int \int_{D_{\Gamma_3}} \left[ \frac{\partial (B_1 F_{31})}{\partial S} + \frac{\partial (B_1 F_{32})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \\ &+ \int \int_{D_{\Gamma_4}} \left[ \frac{\partial (B_3 F_{41})}{\partial S} + \frac{\partial (B_3 F_{42})}{\partial I} \right] \mathrm{d}S \mathrm{d}I \\ &< 0, \end{split}$$

which contradicts (19). This excludes the existence of limit cycles surrounding the sliding segment.  $\Box$ 

### **Global dynamics**

In this subsection, we study the dynamics of system (7). We have  $R_{01} > R_{02}$ , so there are three cases to consider.

Case  $(C_1) R_{01} > R_{02} > 1$ .

In this case, the sliding segment is  $\Sigma_{S_1}$ . To determine the location of the sliding segment  $\Sigma_{S_1}$ , we need to compare the

relationship of its endpoints  $T_i$  (i = 1, 2) and the attraction region  $\Omega$ . Solving the equations

$$\frac{\mu+\delta}{\beta} = 1 - I_c$$
 and  $\frac{\mu+\delta+\gamma}{\beta} = 1 - I_c$ 

with respect to  $I_c$  gives

$$I_c = rac{eta - \mu - \delta}{eta} \equiv I_{c_{\min}} \quad ext{ and } \quad I_c = rac{eta - \mu - \delta - \gamma}{eta} \equiv I_{c_{\max}}$$

Then we know that the sliding segment  $\Sigma_{S_1}$  lies entirely in the attraction region  $\Omega$  for  $I_c < I_{c_{\min}}$ , partly in the attraction region  $\Omega$  for  $I_{c < I_c} < I_{c_{\max}}$  and entirely out of the attraction region  $\Omega$  for  $I_c > I_{c_{\max}}$ . Direct calculation gives  $I_{c_{\max}} > I_1, I_{c_{\min}} > I_2$  and  $I_{c_{\max}} > I_{c_{\min}}$ .

In this scenario, there are two disease-free equilibria  $(E_{01} \text{ and } E_{02})$  and two generalized equilibria  $(E_1 \text{ and } E_2)$  for system (7). If the inequalities  $(H_1)$  in Theorem 3 are further true, there exists a pseudo-equilibrium  $E_S$ . The disease-free equilibria  $E_{01}$  and  $E_{02}$  cannot be attractors of system (7) since  $E_{02}$  is virtual and  $E_{01}$  is unstable, although it is real. Hence, one of the regular endemic equilibria  $E_1$  and  $E_2$  or the pseudo-equilibrium  $E_S$  acts as the attractor of system (7). There are further three possibilities to consider:  $(C_{11}) I_c > I_1; (C_{12}) I_c < I_2; (C_{13}) I_2 < I_c < I_1.$ 

If the case  $(C_{11})$  is true, the generalized endemic equilibrium  $E_1$  is real and denoted by  $E_1^{\rm gr}$ , while the generalized endemic equilibrium  $E_2$  is virtual and denoted by  $E_2^{gv}$ . No pseudo-equilibrium exists in this scenario. If we further have  $I_c < I_{c_{\min}}$  or  $I_{c_{\min}} < I_c < I_{c_{\max}}$ , the sliding segment  $\Sigma_{S_1}$ lies totally or partly in the attraction region  $\Omega$ ; if we have  $I_c > I_{c_{\max}}$ , the sliding segment  $\Sigma_{S_1}$  lies totally outside  $\Omega$ . It follows from Lemmas 4-7 that there is no limit cycle for system (7), so the generalized equilibrium  $E_1^{\rm gr}$  is globally asymptotically stable, as shown in Fig. 5. In Fig. 5, the circular points represent the endemic equilibria, and the square points represent the disease-free equilibria. The dashed dotted lines stand for the trajectories of subsystem  $S_1$ , which is the absence of media effect, enhanced vaccination or treatment; the thin solid lines stand for the trajectories of subsystem  $S_2$ , which includes media effect but not enhanced vaccination or treatment; the dashed lines stand for the trajectories of subsystem  $S_3$ , which includes enhanced vaccination and treatment but excludes media effects; the thick solid lines stand for the trajectories of the subsystem  $S_4$ , which includes media effects, enhanced vaccination and treatment. We use these notations throughout the rest of this paper.

If case  $(C_{12})$  is true, we similarly get that the generalized endemic equilibrium  $E_2$  is real and  $E_1$  is virtual, so they are denoted by  $E_2^{\text{gr}}$  and  $E_1^{gv}$ . There is no pseudoequilibrium for system (7) in this case. We easily get  $I_c < I_{c_{\min}}$ , so the sliding segment  $\Sigma_{S_1}$  is totally in the attraction region  $\Omega$ . By Lemmas 4–7, no limit cycle exists for system (7). It follows that the generalized equilibrium  $E_2^{\text{gr}}$  is globally asymptotically stable in this scenario, as shown in Fig. 6a.

For case  $(C_{13})$ , both the generalized endemic equilibria  $E_1$  and  $E_2$  are virtual, which are denoted by  $E_1^{gv}$  and  $E_2^{gv}$ . In this scenario, the pseudo-equilibrium  $E_S$  exists for the Filippov system (7). According to Theorem 3,  $E_S$  is locally asymptotically stable. If we further have  $I_c < I_{c_{\min}}$  or  $I_{c_{\min}} < I_c < I_{c_{\max}}$ , the sliding segment  $\Sigma_{S_1}$  is totally or partly in the attraction region  $\Omega$ . By Lemmas 4–7, no limit cycle exists and so the pseudo-equilibrium  $E_S$  is globally asymptotically stable for system (7), as shown in Fig. 6b.

Case  $(C_2) R_{01} > 1 > R_{02}$ .

In this case, the generalized endemic equilibrium  $E_1$ coexists with the two disease-free equilibria  $E_{01}$  and  $E_{02}$ . The pseudo-equilibrium  $E_S$  exists if the inequalities  $(H_2)$  in Theorem 3 are true. The disease-free equilibrium  $E_{01}$  is not stable, since  $R_{01} > 1$ . The disease-free equilibrium  $E_{02}$  is stable for the subsystems  $S_3$  and  $S_4$ , but it cannot act as the attractor of system (7) since it is virtual in this scenario. Both  $(\mu + \delta + \gamma)/\beta \le 1$  and  $(\mu + \delta)/\beta \le 1$  may be true due to  $R_{02} < 1$ . For the former case, the sliding segment is  $\Sigma_{S_1}$ , while the sliding segment takes the form  $\Sigma_{S_2}$  for the latter case. There are two possibilities to consider in this case:  $(C_{21}) I_c > I_1$  and  $(C_{22}) I_c < I_1$ .

For the case  $(C_{21})$ , the generalized endemic equilibrium  $E_1$  is real, which is denoted by  $E_1^{\rm gr}$ , and no pseudo-equilibrium exists for system (7). If  $(\mu + \delta)/\beta \le 1 < (\mu + \delta + \gamma)/\beta$ , the sliding segment is  $\Sigma_{S_2}$ , which is partly in the attraction region  $\Omega$ . If  $(\mu + \delta + \gamma)/\beta \le 1$ , the sliding segment is  $\Sigma_{S_1}$ . If we further have  $I_c > I_{c_{\min}}$  or  $I_{c_{\min}} < I_c < I_{c_{\max}}$ , the sliding segment  $\Sigma_{S_1}$  is entirely or partly in the attraction region  $\Omega$ ; if we have  $I_c > I_{c_{\max}}$ ,  $\Sigma_{S_1}$  is entirely out of  $\Omega$ . Since the generalized endemic equilibrium  $E_1^{\rm gr}$  is locally asymptotically stable and Lemmas 4-7 exclude the existence of limit cycles for system (7),  $E_1^r$  is globally asymptotically stable, as shown in Fig. 7a.

For case  $(C_{22})$ , the endemic equilibrium  $E_1$  is virtual and the pseudo-equilibrium  $E_S$  exists for system (7). Similarly, we know that the sliding segment takes the form  $\Sigma_{S_2}$ , which is partly out of the attraction region  $\Omega$ , for  $(\mu + \delta)/\beta \le 1 < (\mu + \delta + \gamma)/\beta$ , while for  $(\mu + \delta + \gamma)/\beta < 1$ , the sliding segment is  $\Sigma_{S_1}$ , which is entirely or partly out the attraction region  $\Omega$  for  $I_c < I_{c_{\min}}$  or  $I_{c_{\min}} < I_c < I_{c_{\max}}$ . By the local stability of the pseudo-equilibrium  $E_S$  in the vicinity of the sliding segment  $\Sigma_S$  and Lemmas 4–7, the pseudo-equilibrium  $E_S$  is globally asymptotically stable, as shown in Fig. 7b.

Case  $(C_3)$   $R_{02} < R_{01} < 1$ .



**Fig. 5** Phase plane for the targeted model (7), showing the asymptotic equilibria (small circles  $E_1^{gr}, E_2^{gv}$  and small squares:  $E_{01}^r, E_{02}^v$  for Case ( $C_{11}$ )). Parameters values are  $\mu = 0.5, \alpha = 0.8, \beta = 1.8, \delta = 0.2, \gamma = 0.5, p = 0.1, f = 0.1, I_c = 0.5$ 

In this case, no endemic equilibrium or pseudo-equilibrium exists for system (7), since  $R_{0i} < 1$  (i = 1, 2). The disease-free equilibrium  $E_{02}$  is virtual, so only the disease-free equilibrium  $E_{01}$  can act as the attractor of system (7). Since no limit cycle exists by Lemmas 4–7,  $E_{01}$  is globally asymptotically stable in this scenario, as shown in Fig. 8.

Concluding the above discussion, we get the following result.

#### Theorem 8

- (i) The disease becomes endemic for  $R_{01} > R_{02} > 1$ . In particular, one of the two generalized equilibria  $(E_1^{\text{gr}} \text{ or } E_2^{\text{gr}})$  or the pseudo-equilibrium  $(E_S)$  is globally asymptotically stable if we further have  $I_c > I_1$  or  $I_c < I_2$  or  $I_2 < I_c < I_1$ .
- (ii) The disease also becomes endemic for  $R_{01} > 1 > R_{02}$ . In particular, the generalized equilibrium  $E_1^{gr}$  or the pseudo-equilibrium  $E_s$  is globally asymptotically stable for  $I_c > I_1$  or  $I_c < I_1$ .
- (iii) The disease can be eradicated from the population for  $R_{02} < R_{01} < 1$ . In particular, the disease-free equilibrium  $E_{01}$  is globally asymptotically stable in this case.

# Discussion

It is widely acknowledged that media coverage plays a key role in influencing both public behavior and government control strategies toward epidemics. The public will take various precautions (e.g., hand washing, social distancing, wearing face masks), which can help them reduce the chance of being infected. That can influence the pattern of disease transmission and lower the rate of infection. In this work, we adopted a negative exponential function dependent on the derivative of the case number to represent the reduction effect of media coverage. We also considered vaccination and treatment policies driven by the case number in order to determine a threshold policy. The resulting system was an implicitly defined and non-smooth one [i.e., system (1)-(3)], for which there exist some difficulties in analyzing its dynamics.

By applying the properties of the Lambert *W* function, the implicitly defined system was converted into an explicit one [i.e., the targeted model, system (7)], which is a nonsmooth system with three thresholds ( $I_c$ ,  $S_{c1}$ ,  $S_{c2}$ ). Once the case number exceeded the threshold level  $I_c$  (i.e.,  $I > I_c$ ), the enhanced vaccination and treatment measures were implemented; otherwise, only the general vaccination measure (and no treatment) was carried out. For  $I > I_c$  (or  $I < I_c$ ), there was further a threshold level  $S_{c2}$  (or  $S_{c1}$ ), which governs whether the media coverage is effective. In particular, the classic epidemic model applies if the number of susceptible individuals is less than the threshold value  $S_{c2}$  (or  $S_{c1}$ ); otherwise, the reduction factor in the incidence rate induced by the media coverage is incorporated into the classic model, as defined in system (7).

We initially considered the limiting cases  $I_c = +\infty$  (or  $I_c = 0$ ), in which both the enhanced vaccination and treatment measures were always suspended (or always implemented). They ultimately led to two non-smooth but continuous systems; i.e., model (10) and model (11). By employing the generalized Jacobian and distribution theory, we examined the global stability of all the equilibria, including the regular equilibria  $(E_{01} \text{ and } E_{02})$  and generalized equilibria ( $E_1$  and  $E_2$ ). For model (10), the diseasefree equilibrium  $E_{01}$  (or the generalized endemic equilibrium  $E_1$ ) was globally asymptotically stable when the basic reproduction number  $R_{01} < 1$  (or  $R_{01} > 1$ ). This suggests that the disease can be eradicated for  $R_{01} < 1$ , while it becomes endemic for  $R_{01} > 1$ . These results have demonstrated that, although media coverage could not eradicate the disease, it nevertheless postponed the arrival of the infection peak, as shown in Fig. 9. It is also shown in Fig. 9 that media coverage diminishes the outbreak size. Similar results were derived for system (11).

The main purpose of this work is to establish all possible dynamic behaviors that our targeted model—i.e., model (7)—can exhibit. The present reduction factor to the infection rate triggers two switching boundaries ( $\Sigma_1$  and  $\Sigma_2$ ) besides the switching boundary  $\Sigma_3$ . The threshold policy with three thresholds ( $S_{c1}, S_{c2}, I_c$ ) results in a variable structure system with four distinct structures (i.e.,



**Fig. 6** Phase plane for the targeted model (7), showing the asymptotic equilibria (small circle  $E_i$  (i = 1, 2), small square  $E_{0i}$  (i = 1, 2) and small diamond  $E_S$ ) for different parameter sets. The isoclinic lines  $g_S^1$  (resp.  $g_S^2$ ) and  $S = S_{c1}$  are plotted for the subsystem  $S_{V_1}$  ( $S_{V_2}$ ). The



**Fig. 7** Phase plane for the non-smooth model (7), showing the asymptotic equilibria (small circle  $E_i$  (i = 1, 2), small square  $E_{0i}$  (i = 1, 2) and small diamond  $E_S$ ) for different parameter sets. The isoclinic lines  $g_S^1$  (resp.  $g_S^2$ ) and  $S = S_{c1}$  are plotted for the subsystem  $S_1$  (resp.

subsystems  $S_1, S_2, S_3$  and  $S_4$ ). Since only the crossing region is available on the switching boundaries  $\Sigma_1$  and  $\Sigma_2$ , the sliding dynamics are obvious; for the switching boundary  $\Sigma_3$ , the sliding dynamics are determined by the scalar equation (17). It is interesting to note that there does not exist any regular endemic equilibrium for system (7). In fact, the three endemic equilibria  $(E_1, E_2 \text{ and } E_5)$  lie on the switching boundaries  $\Sigma_i$  (i = 1, 2, 3), respectively. The two endemic equilibria  $E_1$  and  $E_2$  are generalized



curves represent the orbits in the phase plane indicating the asymptotic equilibria. Parameters values are  $\mu = 0.5$ ,  $\alpha = 0.8$ ,  $\beta = 1.8$ ,  $\delta = 0.2$ , p = 0.1, f = 0.1 with (**a**)  $\gamma = 0.35$ ,  $I_c = 0.08$  and (**b**)  $\gamma = 0.5$ ,  $I_c = 0.3$ 



*S*<sub>2</sub>). The curves represent the orbits in the phase plane indicating the asymptotic equilibria. Parameters values are  $\mu = 0.5$ ,  $\alpha = 0.8$ ,  $\beta = 1.5$ ,  $\delta = 0.2$ ,  $\gamma = 0.8$ , p = 0.1, f = 0.1 with (a)  $I_c = 0.4$  and (b)  $I_c = 0.2$ 

equilibria, while  $E_s$  is a pseudo-equilibrium. The global stability of these endemic equilibria is addressed by excluding all the possible limit cycles, including general cycles, crossing cycles without surrounding the sliding segment, crossing cycles surrounding the sliding segment and canard cycles, as shown in Fig. 4. The results suggest that the real endemic equilibria  $(E_1^{gr} \text{ and } E_2^{gr})$ , pseudo-equilibrium  $(E_s)$  and the disease-free equilibrium  $(E_{01}^{r})$  can act as the attractors of system (7). In particular, the disease-



**Fig. 8** Phase plane for the non-smooth model (7), showing the asymptotic equilibria (i.e., small squares  $E_{01}^r$  and  $E_{02}^{\nu}$ ) for different parameter sets. The curves represent the orbits in the phase plane. Parameters values are  $\mu = 0.5, \alpha = 0.8, \beta = 0.8, \delta = 0.2, \gamma = 0.5, p = 0.1, f = 0.1, I_c = 0.5$ 

free equilibrium  $E_{01}$  is globally asymptotically stable if  $R_{02} < R_{01} < 1$ ; the endemic equilibrium  $E_1^{\text{gr}}$  or pseudoequilibrium  $E_S$  is globally asymptotically stable if  $R_{02} < 1 < R_{01}$ ; and the endemic equilibria  $E_1^{\text{gr}}$  or  $E_2^{\text{gr}}$  or the pseudo-equilibrium  $E_S$  is globally asymptotically stable if  $1 < R_{02} < R_{01}$ .

The main results indicate that the enhanced vaccination policy, the treatment policy or the media impact cannot drive an epidemic extinct. However, if we choose a relatively small threshold level such that the enhanced vaccination policy and treatment policy are carried out earlier



**Fig. 9** Time series of the infected individuals I(t) for model (10) and the counterpart without media impact. Parameters values are  $\mu = 0, \alpha = 1.2, \beta = 3, \delta = 1, p = 0.1$ 

(i.e.,  $I_c < I_1$ ), the number of infected individuals can be contained at a priori level ( $I_c$ ) for  $R_{02} < 1 < R_{01}$ , while it can be curbed at a relatively low level  $(I_2)$  for  $R_{01} > R_{02} > 1$ . If we choose an appropriate threshold level to implement the enhanced vaccination and treatment policies (i.e.,  $I_2 < I_c < I_1$ ), the number of infected individuals can also be controlled at the previously given level  $I_c$ . If we select a relatively large threshold level (i.e.,  $I_c > I_1$ ), then the case number ultimately stabilizes at a high level  $(I_1)$  for  $R_{01} > 1$ . In this case, the media coverage significantly reduces the epidemic size and contributes to diminishing the disease spread, as shown in Fig. 9, although it has no effect in destabilizing the endemic steady state. Under certain conditions, different initial data can cause different disease transmission dynamics. For instance, Fig. 5 shows that the ultimate endemic state is reached via the following distinct processes: (a) with or without media effect and free from enhanced vaccination and treatment; (b) with media effect and free from enhanced vaccination and treatment; then alternation of common and enhanced vaccination, alternation of no treatment and treatment, and alternation of no media effect and media effect; (c) with media effect, treatment and enhanced vaccination; then the above three types of alternations; and finally no media effect or treatment or enhanced vaccination.

In this study, we have explored the impact of media coverage, enhanced vaccination and treatment on disease spread by proposing a non-smooth model with three thresholds. The main results obtained in this work demonstrate that the infection size can be contained either at an a priori level or at a relatively low/high level depending on the threshold level, if the disease cannot be eradicated. The media coverage significantly reduces the outbreak size and delays the epidemic peak. This will be beneficial for policymakers to determine appropriate control strategies.

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