Journal of Mathematical Biology (2019) 78:1389–1424
https://doi.org/10.1007/s00285-018-1313-z

Mathematical Biology

Using non-smooth models to determine thresholds for microbial pest management

Aili Wang[1] · Yanni Xiao[2] · Robert Smith?[3]

Received: 8 March 2018 / Revised: 19 August 2018 / Published online: 31 January 2019

Abstract

Releasing infectious pests could successfully control and eventually maintain the number of pests below a threshold level. To address this from a mathematical point of view, two non-smooth microbial pest-management models with threshold policy are proposed and investigated in the present paper. First, we establish an impulsive model with state-dependent control to describe the cultural control strategies, including releasing infectious pests and spraying chemical pesticide. We examine the existence and stability of an order-1 periodic solution, the existence of order-k periodic solutions and chaotic phenomena of this model by analyzing the properties of the Poincaré map. Secondly, we establish and analyze a Filippov model. By examining the sliding dynamics, we investigate the global stability of both the pseudo-equilibria and regular equilibria. The findings suggest that we can choose appropriate threshold levels and control intensity to maintain the number of pests at or below the economic threshold. The modelling and control outcomes presented here extend the results for the system with impulsive interventions at fixed moments.

Keywords Microbial pest management · Threshold policy · Impulsive differential equations · Filippov system · Global dynamics

Mathematics Subject Classification 34A37 · 34A38

✉ Yanni Xiao
yxiao@mail.xjtu.edu.cn

✉ Robert Smith?
rsmith43@uottawa.ca

Aili Wang
aily_wang83@163.com

1 School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji 721013, People's Republic of China

2 Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China

3 Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON K1N 6N5, Canada

1 Introduction

Agricultural pests—including small insects, animals and weeds—cause serious ecological and economic problems, so it remains a key issue for government and scientists to effective manage pests. Integrated pest management (IPM) is an environmentally sound long-term pest-control strategy that combines biological, cultural and chemical tactics (Lenteren 1995; Tang and Cheke 2005; Tang et al. 2015, 2010). Chemical pesticides delivered by planes, handheld units or trucks carrying spraying equipment can reduce a considerable fraction of pest population in a short period. However, chemical control is recognized as a major hazard to human health and some beneficial insects (Sasmal et al. 2016; Gao et al. 2013). In addition, overuse of pesticides has led to more than 500 species' resistance to pesticides (Liang et al. 2013).

Biological control is another effective method in the IPM (Wang et al. 2010; Tang and Liang 2013), which has a relatively low risk to human health and the environment (Lenteren and Woets 1988). The use of virus, fungi and bacteria are effective biological control methods (Cai et al. 2015; Bhattacharyya and Bhattacharya 2006). There is evidence that viral infection might accelerate the termination of susceptible prey blooms (Jacquet et al. 2002; Gastrich et al. 2004) . It suggests that disease infection accelerates to terminate pests. Infective pests are usually bred in laboratories and released with the expectation of triggering considerable infection in the pest population. For example, parasitic wasps that deliver a lethal virus to crop pests have been used in agriculture (Peng 2005). The virus is attached to the wasps' offspring when they hatch, which leads to the crop pests' infection and greatly increased death rate.

The essential target of IPM is to maintain the pest population below a threshold called the economic injury level (EIL) (Lenteren and Woets 1988; Liang et al. 2015). Based on the IPM strategy, a number of mathematical models have been proposed to evaluate the impact of biological and chemical control on pest management (Sasmal et al. 2016; Wang et al. 2010; Liang et al. 2015; Liu et al. 2015; Tang et al. 2012; Jiao et al. 2009; Kar et al. 2012). Besides the continuous control strategy (Sasmal et al. 2016; Kar et al. 2012), impulsive IPM strategies—including spraying pesticides and releasing natural enemies or infective pests at critical times—are modelled by impulsive differential equations (Wang et al. 2010; Liang et al. 2015; Jiao et al. 2009). Jiao et al. (2009) modelled an infectious disease spreading in the pest population due to the application of microbial pathogen. They investigated the effect of releasing infective pests with spraying pesticides on pest management and developed the following mathematical model with fixed-instant impulsive control measures:

$$\left.\begin{aligned} \frac{dS(t)}{dt} &= bS(t)\left(1-\frac{S(t)+\eta I(t)}{K}\right)-\frac{\beta S(t)I(t)}{1+\alpha I(t)}, \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha I(t)}-\mu I(t), \end{aligned}\right\} \quad t\neq nT, \qquad (1)$$
$$\left.\begin{aligned} \Delta S(t) &= -pS(t), \\ \Delta I(t) &= -q_1 I(t)+\tau \end{aligned}\right\} \quad t=nT, n=1,2,\ldots,$$

where $S(t)$ and $I(t)$ are the density of susceptible and infective pests, respectively; $\Delta S(t) = S(t^+) - S(t)$, $\Delta I(t) = I(t^+) - I(t)$. In their model, b stands for the intrinsic growth rate of pests; K represents the carrying capacity; η is the competing ability of infective pests with the susceptible pests; μ is the death rate of infected pests. A saturated incidence rate $\beta S(t)I(t)/(1+\alpha I(t))$ is taken in this model, which tends to the saturation level β/α when I is large. Here βI represents the infection force of the disease, and $1/(1+\alpha I)$ describes an inhibition effect as the size of infected pests increases. They denote p and q_1 as, respectively, the proportion of susceptible and infective pests removed by spraying chemical pesticides at time t; τ is the amount of infective pests released at time t. The authors show that the pests can be controlled by choosing optimal releasing periods or optimal dosage of pesticides.

A common assumption for model (1) is that the control behaviour occurs in regular pulses. However, there is an important concept in IPM, the economic threshold (ET), which guides whether the control measure is introduced or suspended. This results in a state-dependent control strategy. The first aim of our work is to improve model (1) and propose a new model with the following control strategies: when the number of susceptible pests reaches the ET, the control strategies are implemented; otherwise it is suspended. State-dependent impulsive differential equations have gained considerable attention (Xiao et al. 2013a; O'Rourke and Jones 2011; Lakshmikantham et al. 1989; Simeonov and Bainov 1988; Li and Wu 2016) and have been widely employed in variety of fields, such as neural network control (Touboul and Brette 2009; Li et al. 2018), diabetes mellitus and tumor control (Huang et al. 2012; Tang et al. 2016), HIV antiviral therapy (Smith and Wahl 2004; Lou et al. 2012; Yang et al. 2013) and pulse control in epidemics (Nie et al. 2013). To contain the number of susceptible pests below the ET, we will focus on two issues: 1. How can the IPM strategy involving releasing infectious pests and spraying chemical pesticides be applied to effectively control the pest population? 2. Can we determine the frequency of implementing control measures to effectively prevent an intolerable build-up of pests?

In state-dependent impulsive models, the core of the control is that once the number of susceptible pests reaches the ET, one implements the control measures instantaneously and immediately reduces it below the ET, which can hardly be achieved in practice. The control activities cannot be implemented instantaneously, and they always take time. The second object of this study is then to formulate a Filippov model to describe the non-instantaneous control policy: once the number of susceptible pests exceeds the ET, the control strategy is carried out; no control strategies are applied otherwise. Filippov systems have important application in many fields, including ecosystem maintenance (Křivan et al. 2016), disease control (Xiao et al. 2012, 2013b, 2015; Chong et al. 2016) and pest management (Tang et al. 2012). To explore how this type of control policy affects pest management, we would like to consider the following questions: Can this type of threshold policy contain the pest numbers below the EIL? What is the difference between implementing the impulsive control measure and the non-instantaneous control measure?

Our paper is structured as follows. First, we propose and analyze a model with state-dependent feedback control. We initially examine the analytic properties of a Poincaré map and then discuss the complexity of the domain for the existence of a positive equilibrium. We investigate the stability of an order-1 periodic solution as well as order-k periodic solutions. Second, we extend our model with impulsive control to a Filippov model. We examine the existence of sliding mode region as well as its dynamics. The global behaviour is then addressed to show the control outcome guided by the threshold level. Finally, we make some concluding remarks on the results obtained in this study, which can be used in practical decision-making.

2 Impulsive microbial pest management model with economic threshold

We will extend model (1) to an impulsive model with state-dependent feedback control measures. To this end, we initially show the dynamics of the ordinary differential equation model

$$\begin{aligned} \frac{dS(t)}{dt} &= bS(t)\left(1-\frac{S(t)+\eta I(t)}{K}\right)-\frac{\beta S(t)I(t)}{1+\alpha I(t)}, \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha I(t)}-\mu I(t). \end{aligned} \tag{2}$$

The equilibria $E_0=(0,0)$ and $E_{10}=(K,0)$ always exist for model (2) while the positive equilibrium satisfies

$$\begin{aligned} b(\alpha\mu+\beta\eta)S^2+(\beta\mu K-\alpha b\mu K-b\mu\eta)S-\mu^2K=0 \\ I=\frac{\beta S-\mu}{\alpha\mu}. \end{aligned}$$

Denote

$$f(S)=b(\alpha\mu+\beta\eta)S^2+(\beta\mu K-\alpha b\mu K-b\mu\eta)S-\mu^2K.$$

A unique positive equilibrium $E_{11}=(S_{11},I_{11})$ exists for model (2) if and only if

$$f(\mu/\beta)<0 \Longleftrightarrow R_{10}\equiv\frac{\beta K}{\mu}>1,$$

where

$$\begin{aligned} S_{11} &= \frac{\mu(b\eta+\alpha bK-\beta K)+\mu\sqrt{(\beta K-b\eta-\alpha bK)^2+4K(\alpha\mu b+b\eta\beta)}}{2(\alpha\mu b+b\eta\beta)}, \\ I_{11} &= \frac{\beta S_{11}-\mu}{\alpha\mu}. \end{aligned}$$

The equilibrium E_0 is a saddle and hence not stable. The equilibrium E_{10} is locally asymptotically stable for $R_{10}\le 1$ and unstable for $R_{10}>1$; the positive equilibrium E_{11} is locally asymptotically stable for $R_{10}>1$. The equilibrium E_{10} is a node for $R_{10}\le 1$ and a saddle for $R_{10}>1$; the equilibrium E_{11} is a node for $\delta_0\ge 0$, while it

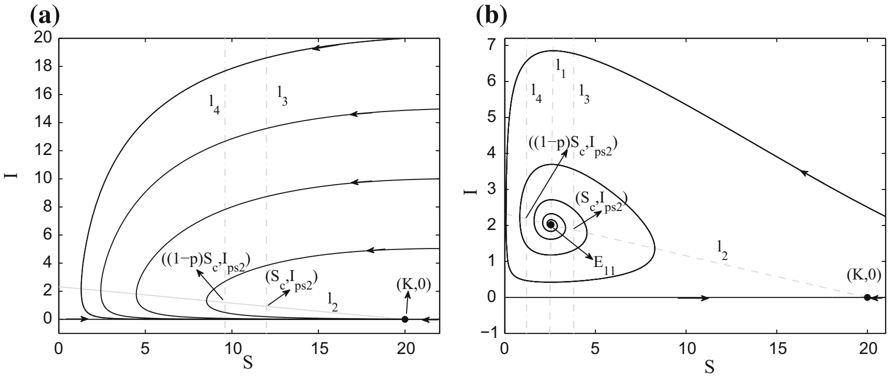

Fig. 1 The topological trajectory map of system (2). Parameters are as follows: $b = 3$; $\eta = 6$; $K = 20$; $\beta = 0.4$; $\alpha = 0.01$; and $\mu = 8.5$ (**a**), $\mu = 1$ (**b**)

is a focus for $\delta_0 < 0$, where

$$\delta_0 = \left(\frac{b}{K}S_{11} + \frac{\alpha\mu I_{11}}{1+\alpha I_{11}}\right)^2 - 4\det(J(S_{11}, I_{11}))$$

and

$$J(S, I) = \begin{pmatrix} b\left(1 - \dfrac{2S+\eta I}{K}\right) - \dfrac{\beta I}{1+\alpha I} & -\dfrac{b\eta}{K}S - \dfrac{\beta S}{(1+\alpha I)^2} \\ \dfrac{\beta I}{1+\alpha I} & \dfrac{\beta S}{(1+\alpha I)^2} - \mu \end{pmatrix}.$$

Denote

$$P(S, I) = bS\left(1 - \frac{S+\eta I}{K}\right) - \frac{\beta SI}{1+\alpha I},\ Q(S, I) = \frac{\beta SI}{1+\alpha I} - \mu I.$$

Choosing a Dulac function as $B(S, I) = 1/(SI))$, it follows that

$$\frac{\partial(BP)}{\partial S} + \frac{\partial(BQ)}{\partial I} = -\frac{b}{KI} - \frac{\alpha\beta}{(1+\alpha I)^2},$$

which indicates that no closed orbit exists for system (2); i.e., we have the following main result, as illustrated in Fig. 1.

Theorem 1 *For model* (2)*, the boundary equilibrium E_{10} is globally asymptotically stable when $R_{10} \leq 1$; while the positive equilibrium E_{11} is globally asymptotically stable when $R_{10} > 1$.*

When a combination of biological and chemical tactics are applied to reduce pests to a tolerable level, the control strategy is implemented once the number of susceptible pests reaches an ET, which is denoted by S_c in the rest of this paper, so that the economic injury level (EIL) is not exceeded (Lenteren 1995; Tang and Cheke 2005; Tang et al. 2010; Lenteren and Woets 1988). We assume that the chemical pesticides have an impact on both susceptible and infective pests while an amount of infective pests are simultaneously released into the population. Taking this threshold control strategy into account, system (2) becomes:

$$\left.\begin{aligned} \frac{dS(t)}{dt} &= bS(t)\left(1-\frac{S(t)+\eta I(t)}{K}\right)-\frac{\beta S(t)I(t)}{1+\alpha I(t)}, \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha I(t)}-\mu I(t), \end{aligned}\right\} \quad S < S_c, \\ \left.\begin{aligned} S(t^+) &= (1-p)S(t), \\ I(t^+) &= qI(t)+\tau \end{aligned}\right\} \quad S = S_c, \tag{3}$$

where $S(t^+)$ and $I(t^+)$ represent the numbers of susceptible and infective pests after the strategy (i.e., spraying pesticides and releasing infective pests) is implemented at time t. In particular, $S(0^+)$ and $I(0^+)$ represent the initial densities of susceptible and infected pests. In this section, we always assume that $S(0^+) < S_c$, $I(0^+) > 0$ and $S_c < K$. Otherwise, the initial values are chosen after the control strategy is implemented once. In model (3), $q = q_2+(1-q_1)$ and p, q_1, q_2, τ are all positive. Here, p represents the proportion of susceptible pests removed by spraying chemical pesticides at time t. $qI(t)+\tau$ stands for the number of infective pests after implementing the IPM strategy once at time t. If $\tau = 0$, then q is the increase of infective pests due to carrying out the IPM strategy once. In particular, q_1 denotes the fraction of infective pests that is removed due to the pesticides and $q_2 I(t) + \tau$ denotes the amount of infective pests after release.

In this section, we will make a rigorous study of the model to examine the complicated dynamical behaviour and suggest some interesting biological implications, based on the newly developed theory related to the density-dependent impulsive semi-dynamical system (Touboul and Brette 2009; Bainov and Simeonov 1993). To this end, we present some preliminaries in the following.

We will briefly introduce some notations and definitions related to planar impulsive semi-dynamical systems with state-dependent feedback control. The system can be described as follows

$$\begin{aligned} &\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y),\ (x,y) \notin M \\ &\Delta x = \psi_1(x,y),\ \Delta y = \psi_2(x,y),\ (x,y) \in M, \end{aligned} \tag{4}$$

where $(x, y) \in \mathcal{R}^2$, $\Delta x = x^+ - x$, $\Delta y = y^+ - y$, M denotes the impulsive set and P, Q, ψ_1, ψ_2 are continuous functions from $\mathcal{R}^2$ into $\mathcal{R}$. The point $z^+ = (x^+, y^+)$ is the impulsive image of $z = (x, y)$, and $\psi(x, y) = (\psi_1(x, y), \psi_2(x, y))$ defines the impulsive function.

Let $N \equiv \psi(M) = \{z^+ | z^+ = \psi(z), z \in M\}$ be the phase set and $N \cap M = \phi$. Let $(X, \Pi, \mathcal{R}_+)$ be a semi-dynamical system (Lakshmikantham et al. 1989) with X a metric space and $\mathcal{R}_+$ the set of nonnegative numbers. The function $\Pi_z = \Pi(z, t)$:

$X \times \mathcal{R}_+ \to X$ satisfies $\Pi(z, 0) = z$. We have $\Pi(\Pi(z, t), s) = \Pi(z, t+s)$ for all $z \in X$ and $t, s \in \mathcal{R}_+$, so $\Pi(z, t)$ is a continuous map. The set

$$C^+(z) = \{\Pi(z, t)|t \in \mathcal{R}_+\}$$

is called the positive orbit of z. For any subset $M \subset X$ and $z \in X$, define

$$M^+(z) = C^+(z) \cap M - \{z\}, M^-(z) = G(z) \cap M - \{z\},$$

where

$$G(z) = \cup\{G(z, t)|t \in \mathcal{R}_+\}, \quad G(z, t) = \{\omega|\Pi(\omega, t) = z\}.$$

We set $M(z) = M^+(z) \cup M^-(z)$.

It is worth noting that three possible properties exist for the trajectory of any point $z_0 \in R^2$ for model (4) (Bainov and Simeonov 1993).

1. The trajectory of z_0 has infinitely many discontinuities and experiences infinitely many impulses if z_k^+ is defined well and $M^+(z_k^+) \neq \emptyset$ for any $k \geq 1$.
2. If there is an integer $k_0 > 0$ such that z_k is defined well for $k = 1, 2 \ldots, k_0$, $M^+(z_k) \neq \emptyset$ for $k < k_0$ and $M^+(z_{k_0}) = \emptyset$, the orbit of z_0 has finite discontinuities and experiences finitely many impulses.
3. If $M^+(z) = M^+(z_0) = \emptyset$ for any $z \in \Pi_{z_0}$, no discontinuities exist for the trajectory of z_0; i.e., the trajectory experiences no impulses in such case.

Definition 1 A planar impulsive semi-dynamical system $(\mathcal{R}^2, \Pi; M, \psi)$ refers to a semi-dynamic system $(\mathcal{R}^2, \Pi)$ with a nonempty closed subset $M \subset \mathcal{R}^2$ and a continuous function $\psi : M \to \mathcal{R}^2$ such that, for any $z \in M$, there is a $\delta_z > 0$ such that

$$G(z, (0, \delta_z)) \cap M = \emptyset, \quad \Pi(z, (0, \delta_z)) \cap M = \emptyset.$$

We denote the points of discontinuity of Π_z by $\{z_n^+\}$ and call z_n^+ the impulsive image of z_n throughout this paper. A function ι is defined from $\mathcal{R}^2$ into $\mathcal{R}_+ \cup \{\infty\}$ as follows: for any $z \in X$, set $\iota(z) = s$ if s satisfies $\Pi(z, s) \in M$, $\Pi(z, t) \notin M$ for $0 < t < s$, and $\iota(z) = \infty$ if $M^+(z) = \emptyset$.

Definition 2 Let Π_z be a trajectory in $(\mathcal{R}^2, \Pi; M, \psi)$. If there are non-negative integers m and k such that k is the smallest integer satisfying $z_m^+ = z_{m+k}^+$, Π_z is said to be periodic of period T_k and order k with $T_k = \sum_{i=m}^{m+k-1} \iota(z_i) = \sum_{i=m}^{m+k-1} s_i$.

For simplification, a periodic trajectory of period T_k and order k is called an order-k periodic solution in the rest of this paper. If an order-k periodic solution is isolated, it is said to be an order-k limit cycle. Finally, we provide the following analogue of the Poincaré criterion to determine the local stability of order-k periodic solutions (Simeonov and Bainov 1988).

Lemma 1 *Let $\chi(x, y)$ be a sufficiently smooth function such that $\chi(x, y) = 0$ if and only if $(x, y) \in M$. The order-k periodic solution $(\xi(t), \eta(t))$ of model (4) is orbitally asymptotically stable and enjoys the property of asymptotic phase if the multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where*

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left[\int_0^T \Big(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\Big)dt\right]$$

$$\Delta_k = \frac{P_+\Big(\frac{\partial \psi_2}{\partial y}\frac{\partial \chi}{\partial x} - \frac{\partial \psi_2}{\partial x}\frac{\partial \chi}{\partial y} + \frac{\partial \chi}{\partial x}\Big) + Q_+\Big(\frac{\partial \psi_1}{\partial x}\frac{\partial \chi}{\partial y} - \frac{\partial \psi_1}{\partial y}\frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial y}\Big)}{P\frac{\partial \chi}{\partial x} + Q\frac{\partial \chi}{\partial y}},$$

where $P, Q, \frac{\partial \psi_1}{\partial x}, \frac{\partial \psi_1}{\partial y}, \frac{\partial \psi_2}{\partial x}, \frac{\partial \psi_2}{\partial y}, \frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial y}$ are calculated at the point $(\xi(t_k), \eta(t_k))$, $P_+ = P(\xi(t_k^+), \eta(t_k^+))$ and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$ with $t_k^+ = \iota(\xi(t_{k-1}^+), \eta(t_{k-1}^+))$.

2.1 Properties of the Poincaré map for impulsive system (3)

In this subsection, we will construct a Poincaré map based on impulsive points in phase set, which plays an important role in addressing the dynamics of system (4). We initially address the isoclines for system (2), which can be defined as

$$l_1 = \big\{(S, I) \in \mathcal{R}_+^2 : I = g_1(S)\big\}, \quad l_2 = \big\{(S, I) \in \mathcal{R}_+^2 : I = g_2(S)\big\},$$

where

$$g_1(S) = \frac{\beta}{\alpha\mu}S - \frac{1}{\alpha}$$

$$g_2(S) = \frac{\alpha b(K - S) - b\eta - \beta S + \sqrt{[(\beta + \alpha b)S + b\eta - \alpha bK]^2 + 4\alpha b^2\eta(K - S)}}{2\alpha b\eta}.$$

Define

$$l_3 = \Big\{(S, I) \in R_+^2 : S = S_c\Big\}, \quad l_4 = \Big\{(S, I) \in R_+^2 : S = (1 - p)S_c\Big\}.$$

The intersection points of the curves l_2 and l_i $(i = 3, 4)$ are (S_c, I_{s2}) and $((1 - p)S_c, I_{ps2})$, respectively, where $I_{s2} = g_2(S_c)$, $I_{ps2} = g_2((1 - p)S_c)$, as shown in Fig. 1. Similarly, the intersection points of curves l_1 and l_i $(i = 3, 4)$ are (S_c, I_{s1}) and $((1 - p)S_c, I_{ps1})$, respectively, where $I_{s1} = g_1(S_c)$, $I_{ps1} = g_1((1 - p)S_c)$. Denote the region in the first quadrant to the left of l_3 as $\Omega_1 = \{(S, I) \in R_+^2 : S < S_c\}$.

To clearly address the properties of Poincaré map and the global behaviour of system (3), we consider the following two cases: (C_1) $R_{10} < 1$ and (C_2) $R_{10} > 1$.

We initially examine the properties of Poincaré map for case (C_1). In this scenario, no positive equilibrium exists for system (2), and the equilibrium $(K, 0)$ is globally asymptotically stable. This implies that every trajectory initiating from Ω_1 of system

(2) will ultimately reach the line l_3, so the impulsive set of system (3) in such scenario takes the form

$$M = \left\{ (S_c, I) \in R_+^2 : 0 \le I \le I_{s2} \right\}. \tag{5}$$

The impulsive function for model (3) reads

$$\psi(S, I) = (\psi_1(S, I), \psi_2(S, I)) = \big(-pS_c, (q-1)I(t) + \tau \big)$$

and so the phase set takes the form

$$N = \left\{ ((1-p)S_c, I) \in R_+^2 : \tau \le I \le qI_{s2} + \tau \right\}. \tag{6}$$

Let $\Pi(t, S_0, I_0) = (S(t, S_0, I_0), I(t, S_0, I_0))^T$ be the solution of system (2) satisfying $S(t_0, S_0, I_0) = S_0, I(t_0, S_0, I_0) = I_0$. Then $(R_+^2, \Pi; M, \psi)$ is an impulsive semi-dynamical system. We assume the initial point $Z_0 = (S_0, I_0) \in N$ and denote the impulsive image of Z_k as Z_k^+ in the rest of this paper unless specified otherwise. Denote

$$X_{psc} = \left\{ ((1-p)S_c, I) \in R_+^2 : I \ge 0 \right\}, \quad X_{sc} = \left\{ (S_c, I) \in R_+^2 : I \ge 0 \right\}.$$

For any point $Z_0^+((1-p)S_c, I_0) \in X_{psc}$, the trajectory $\Pi(t, (1-p)S_c, I_0)$ must reach the impulsive set in a finite time, denoted by t_1. Then we have $S(t_1, (1-p)S_c, I_0) = S_c$. Denote the intersection point as $Z_1 = (S_1, I_1)$. Then

$$\begin{aligned} S_1 &= S_c, & I_1 &= I(t_1, (1-p)S_c, I_0), \\ S_1^+ &= (1-p)S_c, & I_1^+ &= qI_1 + \tau. \end{aligned}$$

Performing the above procedure repeatedly yields a sequence $\{I_k^+\}$, where

$$I_{k+1}^+ = qI_{k+1} + \tau, \quad I_{k+1} = I\left(t_{k+1}, (1-p)S_c, I_k^+\right) \doteq \mathcal{F}\left(I_k^+\right), k = 0, 1, \ldots$$

Select X_{psc} as the Poincaré section, then the Poincaré map ϕ can be defined as

$$I_{k+1}^+ = q\mathcal{F}\left(I_k^+\right) + \tau \equiv \phi\left(I_k^+\right). \tag{7}$$

Theorem 2 *Assume that $R_0 < 1$ and $S_c < K$. Then the Poincaré map ϕ satisfies the following properties:*

(i) *The domain and range of ϕ are $[0, +\infty)$ and $[\tau, qI_{s2} + \tau)$, respectively. The Poincaré map ϕ is increasing on $[0, I_{ps2}]$ and decreasing on $(I_{ps2}, +\infty)$.*
(ii) *ϕ is continuously differentiable.*
(iii) *ϕ is concave down on $[0, I_{ps2})$.*
(iv) *A unique fixed point exists for ϕ.*
(v) *There is a horizontal asymptote for ϕ as $I_k^+ \to +\infty$.*

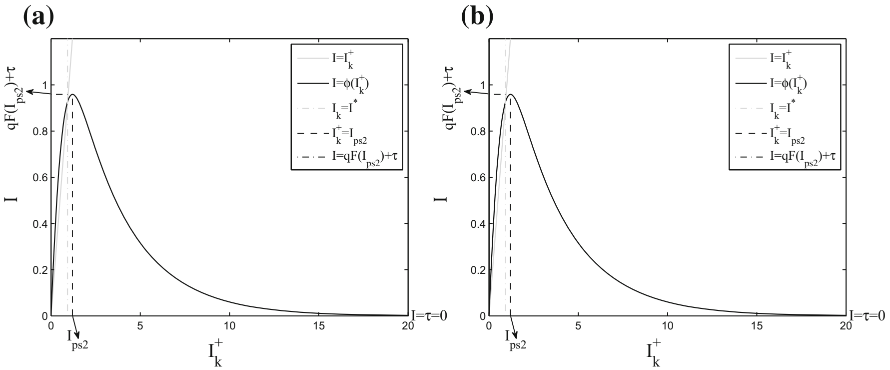

Fig. 2 The Poincaré map ϕ and ϕ^2. The parameter values are $b = 3, \eta = 6, K = 20, \beta = 0.4, \alpha = 0.01, \mu = 8.5, S_c = 12, p = 0.2, \tau = 0$ and $q = 6$ (**a**), $q = 12$ (**b**)

Proof (i) Based on the vector field of system (2), the domain of ϕ is $[0, +\infty)$. For any $I_1, I_2 \in [0, I_{ps2}]$ with $I_1 < I_2$, since $dS/dt > 0$ below the isocline l_2, as shown in Fig. 1a, and there is no intersection between the trajectories $\Pi(t, (1-p)S_c, I_1)$ and $\Pi(t, (1-p)S_c, I_2)$, we have $\mathcal{F}(I_1) < \mathcal{F}(I_2)$. Thus $\phi(I_1) < \phi(I_2)$.

For $I_k \in (I_{ps2}, +\infty)(k = 1, 2)$ with $I_1 < I_2$, since $dS/dt < 0$ above the isocline l_2 and $dS/dt > 0$ below the isocline l_2, the orbit $\Pi(t, (1-p)S_c, I_k)$ $(k = 1, 2)$ will cross the line $S = (1-p)S_c$ once before it hits the line $S = S_c$. Denote the vertical coordinate of the intersection point between the trajectory $\Pi(t, (1-p)S_c, I_k)$ and line l_4 as I_{k1}. Then the order of the two new positions I_{11} and I_{21} is inverted (i.e., $I_{11} > I_{21}$). A similar process to the previous case yields

$$\phi(I_1) = \phi(I_{11}) > \phi(I_{21}) = \phi(I_2).$$

Hence, ϕ is increasing on $[0, I_{ps2}]$ and decreasing on $(I_{ps2}, +\infty)$; the range of ϕ takes the form $[\tau, q\mathcal{F}(I_{ps2}) + \tau]$, as shown in Fig. 2.

(ii) It follows from the theorem of regularity of the solution of an ordinary differential equation with respect to its initial condition that every solution of system (2) initiating from the first quadrant is continuous and differentiable since both functions $P(S, I)$ and $Q(S, I)$ are continuous and differentiable in the first quadrant. Hence, $\mathcal{F}(I_k^+)$ is continuous and differentiable with respect to I_k^+, which yields that the Poincaré map ϕ in (7) is continuously differentiable. It is easy to show that ϕ is regular.

(iii) System (3) can be rewritten as the following form in the phase plane

$$\begin{aligned} \frac{dI}{dS} &= \frac{\frac{\beta SI}{1+\alpha I} - \mu I}{bS\left(1 - \frac{S+\eta I}{K}\right) - \frac{\beta SI}{1+\alpha I}} \equiv g(S, I) \\ & I((1-p)S_c) = I_0. \end{aligned} \tag{8}$$

For model (8), we only focus on the region

$$\Omega_2 = \left\{(S, I) \in R_+^2 : I < g_2(S)\right\}.$$

The function $g(S, I)$ is continuously differentiable in Ω_2, and the solution of (8) is

$$I(S, I_0) = I_0 - \int_S^{(1-p)S_c} g(S, I(S, I_0))dS.$$

For convenience, we take x instead of I_0, which results in the solution

$$I(S, x) = x - \int_S^{(1-p)S_c} g(S, I(S, x))dS. \tag{9}$$

Hence the Poincaré map takes the form

$$\phi(x) = qI(S_c, x) + \tau. \tag{10}$$

Note that

$$\begin{aligned}
\frac{\partial g}{\partial I} &= \frac{\left[\frac{\beta S}{(1+\alpha I)^2} - \mu\right] P(S, I) + \left[\frac{\beta S}{(1+\alpha I)^2} + \frac{b\eta S}{K}\right] Q(S, I)}{\left[bS\left(1 - \frac{S+\eta I}{K}\right) - \frac{\beta SI}{1+\alpha I}\right]^2} \\
\frac{\partial^2 g}{\partial I^2} &= \frac{\frac{2\alpha\beta S}{(1+\alpha I)^3}\left[\mu I - bS\left(1 - \frac{S+\eta I}{K}\right)\right] P(S, I) + 2\left[\frac{b\eta S}{K} + \frac{\beta S}{(1+\alpha I)^2}\right] P^2(S, I)\frac{\partial g(S,I)}{\partial I}}{\left[bS\left(1 - \frac{S+\eta I}{K}\right) - \frac{\beta SI}{1+\alpha I}\right]^3}.
\end{aligned} \tag{11}$$

Next, we will determine the sign of formulae (11), which will be useful in the discussion of concavity of the Poincaré map ϕ. Since $S \le S_c < K$ and $I < I_{ps2}$, we obtain $P(S, I) > 0$, $Q(S, I) < 0$ and

$$\frac{\beta S}{(1+\alpha I)^2} - \mu < \frac{1}{(1+\alpha I)I} Q(S, I) < 0,$$

so $\dfrac{\partial g}{\partial I} < 0$.

By $P(S, I) > 0$, $Q(S, I) < 0$, the second equation of (11) can be written as

$$\begin{aligned}
\frac{\partial^2 g}{\partial I^2} &< \frac{\left\{\frac{2\alpha\beta S}{(1+\alpha I)^3}\left[\mu I - bS\left(1-\frac{S+\eta I}{K}\right)\right] + 2\left[\frac{b\eta S}{K} + \frac{\beta S}{(1+\alpha I)^2}\right]\left[\frac{\beta S}{(1+\alpha I)^2} - \mu\right]\right\} P(S, I)}{\left[bS\left(1-\frac{S+\eta I}{K}\right) - \frac{\beta S I}{1+\alpha I}\right]^3} \\
&< \frac{\left\{\frac{2\alpha\beta S}{(1+\alpha I)^3}\left[\mu I - bS\left(1-\frac{S+\eta I}{K}\right)\right] + \frac{2\beta S(1+\alpha I)}{(1+\alpha I)^3}\left[\frac{\beta S I}{(1+\alpha I)^2} - \mu I\right]\right\} P(S, I)}{\left[bS\left(1-\frac{S+\eta I}{K}\right) - \frac{\beta S I}{1+\alpha I}\right]^3} \\
&< \frac{\frac{2\alpha\beta S}{(1+\alpha I)^3}\left[\frac{\beta S I}{(1+\alpha I)^2} - bS\left(1-\frac{S+\eta I}{K}\right)\right] P(S, I)}{\left[bS\left(1-\frac{S+\eta I}{K}\right) - \frac{\beta S I}{1+\alpha I}\right]^3} \\
&< 0.
\end{aligned}$$

It follows from the theorem of Cauchy and Lipschitz that

$$\begin{aligned}
\frac{\partial I(S, x)}{\partial x} &= \exp\left(\int_{(1-p)S_c}^{S} \frac{\partial}{\partial I}\left(\frac{Q(u, I(u, x))}{P(u, I(u, x))}\right) du\right) > 0 \\
\frac{\partial^2 I(S, x)}{\partial x^2} &= \frac{\partial I(S, x)}{\partial x} \int_{(1-p)S_c}^{S} \frac{\partial^2}{\partial I^2}\left(\frac{Q(u, I(u, x))}{P(u, I(u, x))}\right) \frac{\partial I(u, x)}{\partial x} du.
\end{aligned} \tag{12}$$

Further calculation yields that $\frac{\partial^2 I(S, x)}{\partial x^2} < 0$, so the Poincaré map is increasing and concave down for $I < I_{ps2}$, as shown in Fig. 2.

(iv) Note that ϕ is decreasing on $(I_{ps2}, +\infty)$, so there exists $\tilde{I} \in (I_{ps2}, +\infty)$ such that $\phi(\tilde{I}) < \tilde{I}$. Further, since $\phi(0) = \tau > 0$, there exists $I^* \in (0, \tilde{I})$ such that $\phi(I^*) = I^*$; that is, there is a fixed point on $[0, +\infty)$ for ϕ.

If $\phi(I_{ps2}) < I_{ps2}$, then the fixed point is $I^* \in (0, I_{ps2})$, as shown in Fig. 2a. On the one hand, since ϕ is decreasing on $(I_{ps2}, +\infty)$, we have $\phi(I) < \phi(I_{ps2}) < I_{ps2}$ for $I \in (I_{ps2}, +\infty)$, which indicates no fixed point exists for ϕ on $(I_{ps2}, +\infty)$. On the other hand, ϕ is concave down on $[0, I_{ps2}]$, so a unique fixed point exists for ϕ on $[0, I_{ps2}]$.

If $\phi(I_{ps2}) > I_{ps2}$, no fixed point exists for ϕ on $[0, I_{ps2}]$ due to the concavity and $\phi(0) > 0$; on the other hand, since ϕ is decreasing on $(I_{ps2}, +\infty)$, there is a unique fixed point for ϕ on $(I_{ps2}, +\infty)$, as shown in Fig. 2b.

(v) Let

$$\overline{\Omega}_2 = \{(S, I) : I \leq g_2(S), S \geq 0, I \geq 0\}.$$

We will prove that $\overline{\Omega}_2$ is an invariant set of system (2). The set defined by $\mathcal{R}_+^2 = \{(S, I) : S, I \geq 0\}$ is an invariant set of system (2), since $\left.\frac{dS}{dt}\right|_{S=0} = 0$, $\left.\frac{dI}{dt}\right|_{I=0} = 0$. To examine the invariability of set $\overline{\Omega}_2$, it is necessary to show every trajectory initiating from some point out of $\overline{\Omega}_2$ will flow into it ultimately. This is confirmed by

$$\begin{aligned}
&(P(S,I),Q(S,I))\cdot\left(g_2'(S),-1\right)\Big|_{I=g_2(S)}\\
&=P(S,g_2(S))\frac{(-\alpha b-\beta)\sqrt{C_0^2+4\alpha b^2\eta(K-S)}-2\alpha b^2\eta+(\beta+\alpha b)C_0}{2\alpha b\eta\sqrt{[(\beta+\alpha b)S+b\eta-\alpha bK]^2+4\alpha b^2\eta(K-S)}}\\
&\quad-Q(S,g_2(S))\\
&=-Q(S,g_2(S))>0,
\end{aligned}$$

where $C_0=(\beta+\alpha b)S+b\eta-\alpha bK$ and $\cdot$ represents the scalar product of two vectors. Thus, the trajectory $\Pi(t,(1-p)S_c,x)$ will enter into the invariant set $\overline{\Omega}_2$ and approach the unique equilibrium $(K,0)$, which implies that the trajectory $\Pi(t,(1-p)S_c,x)$ intersects with the line l_3 for any $x\in[0,+\infty)$. We claim that $\lim_{x\to+\infty}I(T,(1-p)S_c,x)=0$, where T satisfies $\lim_{x\to+\infty}S(T,(1-p)S_c,x)=S_c$. Otherwise, if $\lim_{x\to+\infty}I(T,(1-p)S_c,x)=C_1>0$, the solution of the backward system of (3), denoted by $\left(X_1(\overline{T},S_c,\frac{C_1}{2}),X_2(\overline{T},S_c,\frac{C_1}{2})\right)$, satisfies

$$X_1\left(\overline{T},S_c,\frac{C_1}{2}\right)=(1-p)S_c,\ \ X_2\left(\overline{T},S_c,\frac{C_1}{2}\right)>+\infty.$$

This is impossible and so

$$\lim_{x\to+\infty}\phi(x)=\lim_{x\to+\infty}I(T,(1-p)S_c,x)+\tau=\tau,$$

which indicates that $\phi(x)$ is bounded and converges to a finite value τ as $x\to+\infty$. Hence, there does exist a horizontal asymptote $I=\tau$ for the Poincaré map $I=\phi(x)$ as shown in Fig. 2. This completes the proof.

Now we turn to case (C_2). In this scenario, we have $R_{10}>1$, and there is a positive equilibrium $E_{11}=(S_{11},I_{11})$ for model (2), which is globally asymptotically stable. We will study different cases where there are no impulses, finite impulses or infinite impulses for the solutions starting from $((1-p)S_c,I_0^+)$. The domain of ϕ may vary as the variation of impulses. There are two cases to consider according to whether or not the threshold level S_c is greater than the critical value S_{11}.

We first consider the case $S_c\le S_{11}$. It follows that $(1-p)S_c<S_{11}$ and any solution starting from the point $((1-p)S_c,I_0^+)$ will experience infinite impulses, which results in the domain of the Poincaré map ϕ being $[0,+\infty)$. Further, the Poincaré map ϕ has the same property as in Theorem 2.

Now, we focus on the case where $S_c>S_{11}$. In this scenario, there is a trajectory tangent to the impulsive set M (or line l_3) at the point (S_c,I_{s2}), denoted by L. Denote the smallest abscissa of all the intersection points between L and the isocline l_2 as $S_{\min}$. If $(1-p)S_c\le S_{\min}$, the Poincaré map ϕ is defined well and has the same characteristics as described in Theorem 1, so we only examine the property of ϕ for $(1-p)S_c>S_{\min}$ in the next. Denote the largest coordinate of the intersection points of lines l_4 and L as $I_{\max}$ and the smallest one as $I_{\min}$. Any trajectory initiating from $(I_{\min}, I_{\max})$ cannot hit the impulsive set $S = S_c$, so it is free from impulses, and $M^+(I_0^+) = \emptyset$ for $I_0^+ \in (I_{\min}, I_{\max})$. Otherwise, any orbit starting from $[0, I_{\min}]$ will reach the impulsive set $S = S_c$ in finite time, and any orbit initiating from $[I_{\max}, +\infty)$ will cross the line l_4 once and then hits the impulsive set. We summarize the above result as follows.

Lemma 2 *If $S_c \leq S_{11}$ or $S_c > S_{11}$ and $(1-p)S_c \leq S_{\min}$, the domain of the Poincaré map ϕ is $[0, +\infty)$ and all properties listed in* Theorem 2 *are true. If $S_c > S_{11}$ and $(1-p)S_c > S_{\min}$, the definition of Poincaré map ϕ is $D_\phi = D_{\phi 1} \cup D_{\phi 2}$, where $D_{\phi 1} = [0, I_{\min}]$, $D_{\phi 2} = [I_{\max}, +\infty)$.*

To address the properties of the Poincaré map ϕ in detail, we distinguish the possibilities where $\phi(I_{\min}) \leq I_{\min}$, $\phi(I_{\min}) \geq I_{\max}$ and $I_{\min} < \phi(I_{\min}) < I_{\max}$. The main result is as follows.

Theorem 3 (i) *If $\phi(I_{\min}) \leq I_{\min}$, any trajectory of model* (3) *initiating from D_ϕ experiences infinite impulses, and a stable order-1 periodic solution exists for model* (3).

(ii) *If $\phi(I_{\min}) \geq I_{\max}$, any trajectory of model* (3) *initiating from D_ϕ experiences either finite or infinite impulses, and a fixed point $\overline{I} \in D_{\phi 2}$ may exist for the Poincaré map.*

(iii) *If $I_{\min} < \phi(I_{\min}) < I_{\max}$, the orbit of model* (3) *experiences only finite impulses.*

Proof (i) It follows from the direction of the vector field of model (2) that $\phi(D_{\phi 2}) \subset \phi(D_{\phi 1}) \subset D_{\phi 1}$. Since $I_{\min} < I_{ps2}$ and $\phi(I_{\min}) \leq I_{\min}$, any trajectory of model (3) initiating from $D_{\phi 1}$ experiences infinitely many impulses. If $\phi(I_{\min}) = I_{\min}$ or $\phi(I_{\min}) < I_{\min}, \phi(I_0^+) > I_0^+$, then $I_k^+ = \phi^k(I_0^+)$ is monotonically increasing for $k = 1, 2, \ldots$ If $\phi(I_{\min}) < I_{\min}$ and $\phi(I_0^+) < I_0^+$, then $I_k^+ = \phi^k(I_0^+)$ is monotonically decreasing for $k = 1, 2, \ldots$ Consequently, $\{I_k^+\}$ is convergent; i.e., there is a stable order-1 periodic solution for model (3) in this case.

(ii) If $\phi(I_{\min}) \geq I_{\max}$, we distinguish three possibilities: $\phi(0) \geq I_{\max}$, $I_{\min} \leq \phi(0) < I_{\max}$ and $\phi(0) < I_{\min}$.

If the first case is true, $\phi(D_{\phi 2}) \subset \phi(D_{\phi 1}) \subset D_{\phi 2}$. For any $I_0^+ \in D_\phi$, we have $M^+(I_0^+) \neq \emptyset$ and $M^+(I_k^+) \neq \emptyset$ with $I_k^+ = \phi^k(I_0^+)$, so any trajectory of model (3) initiating from D_ϕ experiences infinitely many impulses. By Theorem 1, there is a unique fixed point $\overline{I} \in D_{\phi 2}$ for ϕ in such scenario, so a stable order-1 periodic solution exists for model (3).

If the second case holds, the continuity of the Poincaré map ϕ on $D_{\phi 1}$ together with $\phi(I_{\min}) \geq I_{\max}, \phi(0) < I_{\max}$ leads to the existence of $0 < I_c \leq I_{\min}$ such that $\phi(I_c) = I_{\max}$. In such case, any orbit starting from the point $((1-p)S_c, I_0^+)$ with $I_0^+ \in (0, I_c)$ is free from impulse after one impulse, while those orbits initiating from $((1-p)S_c, I_0^+)$ with $I_0^+ \in (I_c, I_{\min})$ experience infinite impulses.

For the third case, similar discussion yields two points $I_{c1}, I_{c2} \in D_{\phi 1}$ satisfying $\phi(I_{c1}) = I_{\min}, \phi(I_{c2}) = I_{\max}$. Then any trajectory initiating from the point $((1-p)S_c, I_0^+)$ with $I_{c1} < I_0^+ < I_{c2}$ is also free from impulsive effect after one impulse, and other trajectories with $I_0^+ \in (0, I_{c1}) \cup (I_{c2}, I_{\min})$ experience infinite impulses.

(iii) If $I_{\min} < \phi(I_{\min}) < I_{\max}$, performing a similar process to Theorem 1 yields that ϕ is continuous, increasing and concave down on $D_{\phi 1}$. Since $\phi(I_{\min}) > I_{\min}$, no fixed point exists for ϕ on $D_{\phi 1}$. For any $I_0^+ \in D_{\phi 1}$, there exist a positive integer k such that $I_j^+ = \phi^j(I_0^+) \in D_{\phi 1}$, $j = 1, 2, \ldots, k-1$ and $I_k^+ = \phi^k(I_0^+) \in (I_{\min}, I_{\max})$. This indicates that any trajectory initiating from $((1-p)S_c, I_0^+)$ will be free from impulses after finite impulses, where $I_0^+ \in D_{\phi 1}$, which is also true for any orbit initiating from $((1-p)S_c, I_0^+)$ with $I_0^+ \in D_{\phi 2}$ due to $\phi(D_{\phi 2}) \subset \phi(D_{\phi 1})$. This completes the proof. □

It is important to emphasize that the fixed point of the Poincaré map ϕ corresponds to an order-one periodic solution of system (4), which we address in detail in the next section. The order-k periodic solutions will be examined in the following section.

2.2 Dynamic properties of impulsive system (3)

In this subsection, we will focus on the existence and stability of periodic solutions for Case (C_1). For Case (C_2), according to the previous subsection, the domain of the Poincaré map and impulses will result in complicated dynamic behaviour, which can be examined by implementing a similar procedure. We omit it here.

According to Sect. 2.1, there is an infinite sequence $\{I_n^+\}$ for any $I_0 \in [0, +\infty)$ for Case (C_1), where $I_n^+ = \phi^n(I_0)$. In this subsection, we examine the globally asymptotic stability of $\{I_n^+\}$, which refers to the order-k $(k \geq 1)$ periodic solutions of system (3). We initially investigate the global stability of order-1 periodic solution.

Theorem 4 *If $R_{10} < 1$, $S_c < K$ and $\phi(I_{ps2}) \leq I_{ps2}$, there is a unique order-1 periodic solution of system (3), and it is globally asymptotically stable.*

Proof By Sect. 2.1, there is a unique fixed point for the Poincaré map ϕ, which we denote by $\overline{I}$. This suggests that a unique order-1 periodic solution with initial condition $S(t_0) = (1-p)S_c$, $I(t_0) = \overline{I}$ exists for system (4), denoted by $(\xi(t), \eta(t))$.

For any $I \in [0, I_{ps2}]$, we have $\phi(I) \in [0, I_{ps2}]$, which is equivalent to $\phi([0, I_{ps2}]) \subset [0, I_{ps2}]$, so the fixed point theorem guarantees $\overline{I} \in [0, I_{ps2}]$. Moreover, ϕ is decreasing on $(I_{ps2}, +\infty)$, and so one obtains $\phi(I) \leq \phi(I_{ps2}) \leq I_{ps2}$ for $I \in (I_{ps2}, +\infty)$. Thus $\phi((I_{ps2}, +\infty)) \subset [0, I_{ps2}]$. Therefore, it is sufficient to prove that the sequence $\{I_n^+\}$ converges on $[0, I_{ps2}]$.

Next, we will examine the global stability of the solution $(\xi(t), \eta(t))$. We consider the following two cases.

(i) For $I_0^+ \in [0, I_{ps2}]$, we have $I_1^+ = \phi(I_0^+)$. The result is obvious for $I_1^+ = I_0^+$, so there are the following two possibilities to consider: (a) $I_1^+ > I_0^+$ and (b) $I_1^+ < I_0^+$. For possibility (a), it follows from the increasing property of ϕ on $[0, I_{ps2}]$ that $I_2^+ = \phi(I_1^+) > \phi(I_0^+) = I_1^+$. A repeated procedure shows that $I_n^+ = \phi^n(I_0^+) (n \geq 1)$ is monotonically increasing and bounded, so the sequence $\{I_n^+\}$ converges to the fixed point $\overline{I}$; i.e.,

$$\lim_{n \to +\infty} I_n^+ = \lim_{n \to +\infty} \phi^n\left(I_0^+\right) = \overline{I},$$

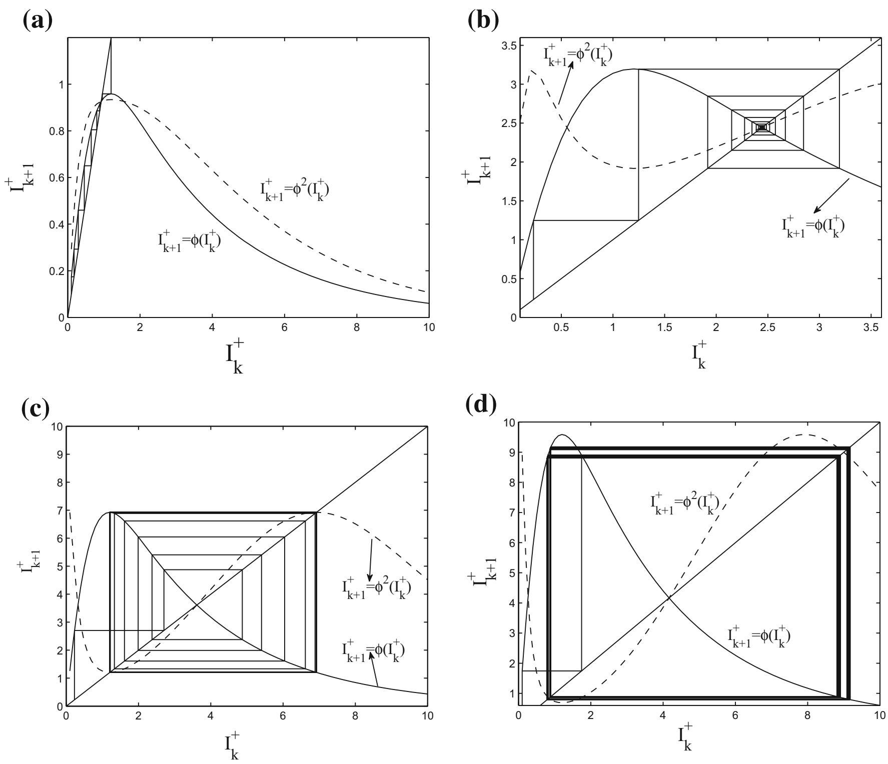

Fig. 3 The Poincaré maps ϕ and ϕ^2 with the stability of order-1 periodic solutions (shown in **a** and **b**), order-2 periodic solution (shown in **c**) and order-4 periodic solution (shown in **d**). The parameter values are $b = 3, \eta = 6, K = 20, \beta = 0.4, \alpha = 0.01, \mu = 8.5, S_c = 12, p = 0.2, \tau = 0$ and $q = 3.6$ (**a**), $q = 12$ (**b**), $q = 26$ (**c**), $q = 36$ (**d**)

as shown in Fig. 3a. For possibility (b), we get $I_2^+ = \phi(I_1^+) < \phi(I_0^+) = I_1^+$ since ϕ is increasing. A similar analysis to (a) yields that the sequence $\{I_n^+\}$ converges to $\overline{I}$. Hence, the order-1 periodic solution $(\xi(t), \eta(t))$ is globally asymptotically stable in this case.

(ii) For $I_0^+ \in (I_{ps2}, +\infty)$, we have $I_1^+ \in [0, I_{ps2}]$, so case (i) yields that I_n^+ also converges to $\overline{I}$ and the globally asymptotic stability of the solution $(\xi(t), \eta(t))$ is derived. This completes the proof.

Remark 1 For the critical case $\phi(I_{ps2}) = I_{ps2}$, I_{ps2} is the unique fixed point of the Poincaré map ϕ. This indicates a unique order-1 periodic solution exists for model (3), which is globally asymptotically stable in this scenario.

Theorem 5 *If $R_{10} < 1, S_c < K$ and $\phi(I_{ps2}) > I_{ps2}$, then the order-1 periodic solution for model (3) is globally asymptotically stable if and only if $\phi^2(I^+) > I^+$ for all I^+ with $I_{ps2} \le I^+ < \overline{I}$, where $\overline{I}$ represents the fixed point of the Poincaré map ϕ.*

Proof By Theorem 2, a unique fixed point $\overline{I} > I_{ps2}$ exists for the Poincaré map ϕ if $\phi(I_{ps2}) > I_{ps2}$, where ϕ has the following properties:

- $\phi(I^+) > I^+$ for $0 \le I^+ < \overline{I}$ and $\phi(I^+) < I^+$ for $I^+ > \overline{I}$;
- $\phi(I^+)$ is increasing for $0 \le I^+ \le I_{ps2}$ and decreasing for $I^+ > I_{ps2}$.

(Sufficiency.) We focus our discussion on the following three possibilities: (a) $I_{ps2} \le I^+ < \overline{I}$, (b) $0 \le I^+ < I_{ps2}$ and (c) $I^+ > \overline{I}$.

(a) For any $I_{ps2} < I_0^+ < \overline{I}$, the decreasing property of ϕ on $[I_{ps2}, +\infty)$ yields $I_1^+ = \phi(I_0^+) > \overline{I}$. It follows that $I_2^+ = \phi(I_1^+) < \overline{I}$ and $I_2^+ = \phi^2(I_0^+) > I_0^+$, which implies that

$$I_0^+ < I_2^+ < \overline{I}.$$

Further, we have $I_3^+ = \phi(I_2^+) > \overline{I}$ and $I_3^+ = \phi(I_2^+) < \phi(I_0^+) = I_1^+$, so

$$\overline{I} < I_3^+ < I_1^+.$$

Performing the above procedure repeatedly, we see that $\{I_{2k}^+\}$ is increasing ,$\{I_{2k+1}^+\}$ is decreasing and

$$I_{ps2} < I_{2k}^+ < \overline{I} < I_{2k+1}^+ < I_1^+, \ \ k = 1, 2, \ldots$$

Denote

$$\lim_{k\to+\infty} I_{2k}^+ = I_*, \quad \lim_{k\to+\infty} I_{2k+1}^+ = I^*.$$

We claim $I_* = I^* = \overline{I}$. In fact, if $I_* < \overline{I}$, we easily get

$$\overline{I_*} \equiv \phi(I_*) > \phi(\overline{I}) = \overline{I} \Longleftrightarrow \overline{I_*} > \overline{I},$$

since ϕ is decreasing on $(I_{ps2}, +\infty)$. It follows that $\overline{\overline{I_*}} \equiv \phi(\overline{I_*}) < \overline{I}$. Then there are two possibilities to consider: (a) $\overline{\overline{I_*}} > I_*$; (b) $\overline{\overline{I_*}} \le I_*$; If possibility (a) holds, we then have $\phi^2(I_*) \ne I_*$, which contradicts the assumption that $I_* = \lim_{k\to+\infty} I_{2k}^+$. If possibility (b) holds, since we have $I_{ps2} < I_* < \overline{I}$ in this scenario, it contradicts with the condition that $\phi^2(I^+) > I^+$ for all $I_{ps2} < I^+ < \overline{I}$. Hence, $I_* = \overline{I}$ holds.

If $I^* > \overline{I}$, we get

$$\overline{I^*} \equiv \phi(I^*) < \phi(\overline{I}) = \overline{I} \Longleftrightarrow \overline{I^*} < \overline{I}$$

on one hand. On the other hand, since $\overline{I} < I^* < I_{2k+1}$ and $I_{ps2} < I_{2k+2} = \phi(I_{2k+1}) < \overline{I}$ for $k = 1, 2, \ldots$, we have $\overline{I^*} \equiv \phi(I^*) > I_{2k+2} > I_{ps2}$. As a result, we get

$$I_{ps2} < \overline{I^*} < \overline{I}. \tag{13}$$

Note that $I^* = \lim_{k\to+\infty} I_{2k+1}^+$, so

$$\overline{\overline{I^*}} \equiv \phi(\overline{I^*}) = \phi^2(I^*) = I^*,$$

which leads to

$$\phi^2(\overline{I^*}) = \phi(I^*) = \overline{I^*}. \tag{14}$$

Thus another contradiction occurs between (13)–(14) and the condition that $\phi(I^+) > I^+$ for all $I_{ps2} < I^+ < \overline{I}$. Hence $I^* = \overline{I}$ in this scenario.

Therefore we have $\lim_{k\to+\infty} I^+_{2k} = \lim_{k\to+\infty} I^+_{2k+1} = \overline{I}$ in such case.

(b) For any $0 \le I_0^+ < I_{ps2}$, there is a positive integer k_1 such that $I^+_{k_1} = \phi^{k_1}(I_0^+) \in [I_{ps2}, \overline{I})$ or $I^+_{k_1} > \overline{I}$. If the former case is true, it follows from case (a) that $\{I^+_{k_1+k}\}$ converges to $\overline{I}$; i.e., $\lim_{k\to+\infty} I^+_{k_1+k} = \overline{I}$. If the latter case holds, Theorem 2 implies that there exists $\tilde{I} \in [I_{ps2}, \overline{I})$ such that $\phi(\tilde{I}) = \phi^{k_1}(I_0^+)$, as shown in Fig. 3b. Again from case (a), $\{I^+_{k_1+k}\}$ converges to $\overline{I}$.

(c) For every $I_0^+ > \overline{I}$, we have $I_1^+ = \phi(I_0^+) < \overline{I}$ since $\phi(I^+)$ is decreasing on $(\overline{I}, +\infty)$, so $I_1^+ \in [0, I_{ps2})$ or $I_1^+ \in [I_{ps2}, \overline{I})$. Thus it follows from case (b) or case (a) that $\{I^+_{1+k}\}$ converges to $\overline{I}$.

(Necessity.) We need to prove $\phi^2(I_0^+) > I_0^+$ for every $I_0^+ \in [I_{ps2}, \overline{I})$. Assume that there is an $\check{I}_0 \in [I_{ps2}, \overline{I})$ such that $\phi^2(\check{I}_0) < \check{I}_0$. Since $\{I_k^+\}$ converges to $\overline{I}$ with $I_k^+ = \phi^k(I_0^+)$, there is an $\hat{I} \in (\overline{I} - \epsilon, \overline{I} + \epsilon)$ such that $\phi^2(\hat{I}) > \hat{I}$ for a sufficiently small positive number ϵ. Thus the differentiability of the Poincaré map ϕ leads to the existence of $\tilde{I}$ between $\check{I}$ and $\hat{I}$ with $\phi^2(\tilde{I}) = \tilde{I}$, which suggests the existence of an order-2 periodic solution initiating from $((1-p)S_c, \tilde{I})$ for model (3). That contradicts the global stability of order-1 periodic solution. This completes the proof.

Theorem 6 *If $R_{10} < 1$, $S_c < K$ and $\tau < I_{ps2}$, there is a threshold value μ_c for μ such that the unique order-1 periodic solution is globally asymptotically stable for $\mu > \mu_c$.*

Proof It is worth noting that $R_{10} < 1$ if and only if $\mu > \beta K$, which implies that there exists a threshold value $\mu_{c1} \equiv \beta K$ such that $R_{10} < 1$ for $\mu > \mu_{c1}$. According to Theorem 4, it is enough to seek a threshold value μ_c such that the Poincaré map $\phi(I)$ satisfies $\phi(I_{ps2}) \le I_{ps2}$ for $\mu > \mu_c$. Since $P(S, I) > 0$ and $Q(S, I) < 0$ in region Ω_2, system (8) can be rewritten as

$$\frac{dI}{dS} = \frac{\frac{\beta SI}{1+\alpha I} - \mu I}{bS\left(1 - \frac{S+\eta I}{K}\right) - \frac{\beta SI}{1+\alpha I}} < \frac{\beta SI - \mu I}{bS\left(1 - \frac{S}{K}\right)}.$$

Consider the following system

$$\begin{cases} \dfrac{dV}{dU} = \dfrac{\beta UV - \mu V}{bU\left(1 - \frac{U}{K}\right)} \\ V((1-p)S_c) = I_{ps2}. \end{cases} \tag{15}$$

Solving (15) with respect to V yields that

$$V(U) = I_{ps2}\left[\frac{(1-p)S_c}{U}\right]^{\frac{\mu}{b}}\left[\frac{K-U}{K-(1-p)S_c}\right]^{\frac{\mu-\beta K}{b}}, \ U \in [(1-p)S_c, S_c],$$

so we obtain

$$V(S_c) = I_{ps2}(1-p)^{\frac{\mu}{b}} \left[\frac{K - S_c}{K-(1-p)S_c}\right]^{\frac{\mu-\beta K}{b}}.$$

By the comparison theorem, the solution of system (8) satisfies

$$I(S_c; I_{ps2}) \leq I_{ps2}(1-p)^{\frac{\mu}{b}} \left[\frac{K - S_c}{K-(1-p)S_c}\right]^{\frac{\mu-\beta K}{b}}.$$

It follows that

$$\lim_{\mu\to+\infty} I(S_c; I_{ps2}) \leq 0,$$

so there exists $\mu_c \geq \mu_{c1}$ such that

$$\phi(I_{ps2}) = I(S_c; I_{ps2}) + \tau \leq I_{ps2}$$

for $\mu \geq \mu_c$. This completes the proof.

Theorem 7 *If $R_{10} < 1, S_c < K, \phi(I_{ps2}) > I_{ps2}$ and $\phi^2(I_{ps2}) \geq I_{ps2}$, there is a stable order-1 or order-2 periodic solution of system* (3).

Proof We claim that, for $I_0^+ \in [0, +\infty)$, there is an integer k such that $I_k^+ = \phi^k(I_0^+) \in [I_{ps2}, \phi(I_{ps2})]$. In fact, for $I_0^+ \in [0, I_{ps2}]$, no fixed point exists on $[0, I_{ps2}]$, and ϕ is increasing on $[0, I_{ps2}]$, so there is an integer k such that $I_{k-1}^+ = \phi^{k-1}(I_0^+) < I_{ps2}$ and $I_k^+ = \phi^k(I_0^+) \geq I_{ps2}$. It follows that $I_k^+ = \phi(I_{k-1}^+) \leq \phi(I_{ps2})$ and so $I_k^+ \in [I_{ps2}, \phi(I_{ps2})]$. For $I_0^+ \in (I_{ps2}, +\infty)$, since ϕ is decreasing on $(I_{ps2}, +\infty)$, $I_1^+ = \phi(I_0^+) \leq \phi(I_{ps2})$. If, additionally, $I_1^+ \geq I_{ps2}$, set $k = 1$ and $I_k^+ \in [I_{ps2}, \phi(I_{ps2})]$. Otherwise, if $I_1^+ < I_{ps2}$, following the above discussion, there is an integer $k > 1$ such that $I_k^+ = \phi^k(I_0^+) \in [I_{ps2}, \phi(I_{ps2})]$.

Since ϕ is decreasing on $[I_{ps2}, \phi(I_{ps2})]$, ϕ^2 is increasing on $[I_{ps2}, \phi(I_{ps2})]$ and

$$\phi\left(\left[I_{ps2}, \phi(I_{ps2})\right]\right) = \left[\phi^2(I_{ps2}), \phi(I_{ps2})\right] \subset \left[I_{ps2}, \phi(I_{ps2})\right].$$

For $I_0^+ \in [I_{ps2}, \phi(I_{ps2})]$, denote $I_n^+ = \phi^n(I_0^+)$. Suppose neither an order-1 nor an order-2 periodic solution exists for model (3) in such scenario. It follows that $I_1^+ \neq I_0^+$ and $I_2^+ \neq I_0^+$. If $I_1^+ > I_0^+$, we then have

$$I_2^+ = \phi\left(I_1^+\right) < \phi\left(I_0^+\right) = I_1^+,$$

due to the decreasing nature of ϕ on the interval $[I_{ps2}, \phi(I_{ps2})]$. On the other hand, we have

$$I_2^+ \equiv \phi^2\left(I_0^+\right) > \phi^2(I_{ps2}) \geq I_{ps2}$$

due to the increasing nature of ϕ^2 on the interval $[I_{ps2}, \phi(I_{ps2})]$. So there are two possibilities: (i) $I_1^+ > I_0^+ > I_2^+ \geq I_{ps2}$ and (ii) $I_1^+ > I_2^+ > I_0^+$.

If $I_1^+ < I_0^+$, it follows from the monotonicity of ϕ and ϕ^2 on $[I_{ps2}, \phi(I_{ps2})]$ that

$$I_2^+ = \phi\left(I_1^+\right) > \phi\left(I_0^+\right) = I_1^+$$
$$I_2^+ \equiv \phi^2\left(I_0^+\right) > \phi^2(I_{ps2}) \leq \phi(I_{ps2}),$$

so there are another two possibilities to consider: (iii) $I_1^+ < I_2^+ < I_0^+$; (iv) $I_1^+ < I_0^+ < I_2^+$.

Concluding the above discussion, there are four possibilities to consider.

(i) $I_1^+ > I_0^+ > I_2^+$. We have $I_3^+ = \phi(I_2^+) > \phi(I_0^+) = I_1^+$ and $I_4^+ = \phi(I_3^+) < \phi(I_1^+) = I_2^+$, so $I_3^+ > I_1^+ > I_0^+ > I_2^+ > I_4^+$. Repeating the above procedure yields

$$\cdots > I_{2n+1}^+ > I_{2n-1}^+ > \cdots > I_3^+ > I_1^+ > I_0^+ > I_2^+ > I_4^+ > \cdots > I_{2n}^+ > I_{2n+2}^+ > \cdots \quad (16)$$

(ii) $I_1^+ > I_2^+ > I_0^+$. In this case, we get that $\phi(I_1^+) = I_2^+ < I_3^+ = \phi(I_2^+) < \phi(I_0^+) = I_1^+$ and $\phi(I_2^+) = I_3^+ > I_4^+ = \phi(I_3^+) > \phi(I_1^+) = I_2^+$, which suggests $I_1^+ > I_3^+ > I_4^+ > I_2^+ > I_0^+$. Again, we derive the inequalities by induction

$$I_1^+ > I_3^+ > \cdots > I_{2n-1}^+ > I_{2n+1}^+ > \cdots > I_{2n+2}^+ > I_{2n}^+ > \cdots > I_4^+ > I_2^+ > I_0^+. \quad (17)$$

(iii) $I_1^+ < I_2^+ < I_0^+$. Implementing a similar process to (ii) gives

$$I_1^+ < I_3^+ < \cdots < I_{2n-1}^+ < I_{2n+1}^+ < \cdots < I_{2n+2}^+ < I_{2n}^+ < \cdots < I_4^+ < I_2^+ < I_0^+. \quad (18)$$

(iv) $I_1^+ < I_0^+ < I_2^+$. Performing a similar process to (i), we derive

$$\cdots < I_{2n+1}^+ < I_{2n-1}^+ < \cdots < I_3^+ < I_1^+ < I_0^+ < I_2^+ < I_4^+ < \cdots < I_{2n}^+ < I_{2n+2}^+ < \cdots \quad (19)$$

For cases (ii) and (iii), there are two possibilities:

(a) there exists $\overline{I} \in [I_{ps2}, \phi(I_{ps2})]$ such that $\{I_n^+\}$ converges to $\overline{I}$, as shown in Fig. 3b;
(b) there are two distinct numbers $\overline{I}_1 < \overline{I}_2$ such that the sequence $\{I_{2n+1}^+\}$ converges to $\overline{I}_1$ while the sequence $\{I_{2n}^+\}$ converges to $\overline{I}_2$, as shown in Fig. 3c.

For cases (i) and (iv), only the latter holds true. Hence, there is an order-1 or order-2 stable periodic solution for system (3), as shown in Fig. 4a–b. This completes the proof. □

Remark 2 It follows from Theorem 7 that if $\phi(I_{ps2}) > I_{ps2}$, $\phi^2(I_{ps2}) \geq I_{ps2}$, then a stable order-1 or order-2 periodic solution exists for system (3). If we strengthen the second inequality of Theorem 7 as $\phi^2(I^+) \geq I^+$, $I_{ps2} \leq I^+ \leq \overline{I}$, then the global stability of the order-1 periodic solution is feasible, as shown in Theorem 5. In fact, the condition required in Theorem 5 is more rigid than the one in Theorem 7, so the globally stable order-1 periodic solution is available in Theorem 5 but it is not available in Theorem 7. Theorem 7 is more general than Theorem 5.

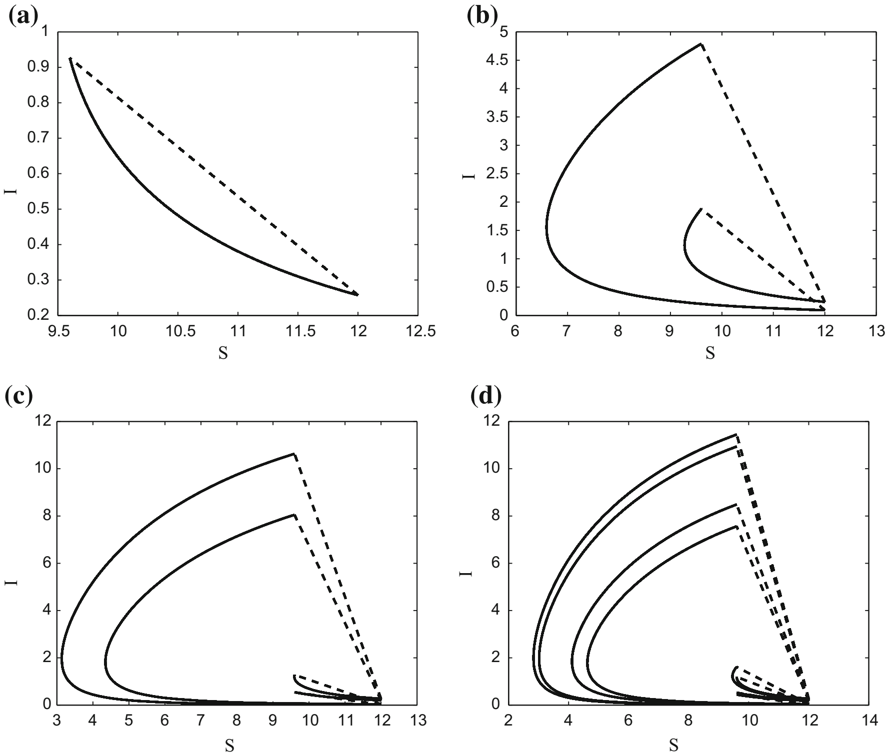

Fig. 4 Existence of an order-1 periodic solution for $q = 3.6$ (**a**), an order-2 periodic solution $q = 20$ (**b**), an order-4 periodic solution $q = 40$ (**c**) and an order-8 periodic solution $q = 43$ (**d**). The other parameter values are $b = 3$, $\eta = 6$, $K = 20$, $\beta = 0.4$, $\alpha = 0.01$, $\mu = 8.5$, $S_c = 12$, $p = 0.2$, $\tau = 0$

Theorem 8 *If* $R_{10} < 1$, $S_c < K$, $\phi(I_{ps2}) > I_{ps2}$, $\phi^2(I_{ps2}) < I_{c1}$ *with* $I_{c1} = \min\{I : \phi(I) = I_{ps2}\}$, *a nontrivial order-3 periodic solution exists for system* (3)*, and so there are nontrivial order-k*$(k \geq 3)$ *periodic solutions for system* (3).

Proof It follows from $\phi(I_{ps2}) > I_{ps2}$ that a unique fixed point $\overline{I}$ exists for the Poincaré map ϕ, where $\overline{I}$ satisfies $\overline{I} > I_{ps2}$.

Define $G(I) = \phi^3(I) - I$. Then $G(I)$ is continuous on $[0, +\infty)$ and

$$G(0) = \phi^3(0) - 0 = \phi^2(\tau) > 0$$
$$G(I_{c1}) = \phi^3(I_{c1}) - I_{c1} = \phi^2(\phi(I_{c1})) - I_{c1} = \phi^2(I_{ps2}) - I_{c1} < 0,$$

so there exists a number $\tilde{I} \in (0, I_{c1})$ satisfying $G(\tilde{I}) = 0$; i.e., $\phi^3(\tilde{I}) = \tilde{I}$.

We now examine the equality of $\overline{I}$ and $\tilde{I}$. We claim that $\overline{I} \neq \tilde{I}$. In fact, we have $I_{c1} < I_{ps2}$ according to the definition of I_{c1}. On the other hand, the above discussion yields $\overline{I} > I_{ps2}$. Hence, the two fixed points of the Poincaré map ($\overline{I}$ and $\tilde{I}$) are distinct. Therefore, initiating from $((1-p)S_c, \tilde{I})$, a nontrivial order-3 periodic solution exists for system (3). By Sarkovskii's theorem (Devaney 2003), an order-k periodic solution

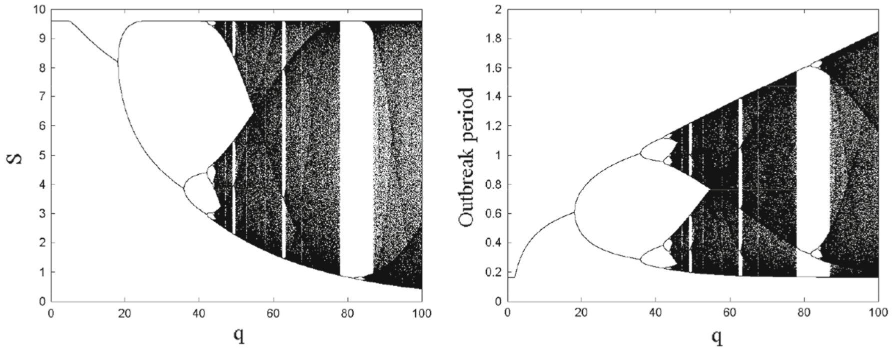

Fig. 5 Bifurcation diagrams with respect to q for the case of nonexistence of an interior equilibrium. The parameter values are $b = 3$, $\eta = 6$, $K = 20$, $\beta = 0.4$, $\alpha = 0.01$, $\mu = 8.5$, $S_c = 12$, $p = 0.2$, $\tau = 0$

exists for system (4) for any integer $k \geq 3$, as shown in Figs. 3d and 4c–d. This completes the proof. □

According to the above discussion, for any positive integer k, the possible order-k periodic solutions exist for model (3). To further illustrate the complexity, we choose the parameter q (i.e., the increasing proportion of infective pests when $\tau = 0$ due to carrying out the IPM strategy once) as the bifurcation parameter and fix all other parameters. Figure 5 shows how the susceptible pest sizes and the impulsive outbreak periods vary as q varies. The bifurcation diagrams with respect to q suggest the complexity of the dynamics of model (3). It follows from Fig. 5 that there is a stable order-1 periodic solution for relatively small q, as shown in Fig. 4a. A stable order-2 periodic solution occurs as q increases, which is also illustrated in Fig. 4b. Stable order-4 and order-8 periodic solutions appear as q increases further, which can also be seen from Fig. 4c, d. Moreover, the order-3 periodic solution exists for a certain range of q, which separates the window of chaos. It is worth emphasizing that the periodic solutions relate to the regular variation of the pest population, while the chaotic behaviour relates to their irregular variation. For the existence of stable periodic solutions, the IPM can be implemented at every time $\overline{T}$ without evaluating the pest amount, where $\overline{T}$ stands for the period of periodic solutions. However, for the existence of chaotic solutions, evaluating pest quantities is difficult, so initiating the IPM strategy is complicated.

3 Filippov microbial pest management model with economic threshold

In this section, we further extend the state-dependent impulsive control measures to the following non-instantaneous control policy: the control measure is implemented only when the amount of susceptible pests exceed the threshold level S_c; if this amount is less than S_c, the control measure is suspended. We use Filippov systems to characterize this type of non-instantaneous control strategy. The model takes the form

$$\begin{aligned} \frac{dS}{dt} &= bS(t)\left(1 - \frac{S(t)+\eta I(t)}{K}\right) - \frac{\beta S(t)I(t)}{1+\alpha I(t)} - \psi(S)pS(t) \\ \frac{dI}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu I(t) + \psi(S)\left(q_2 I(t) - q_1 I(t)\right), \end{aligned} \tag{20}$$

with

$$\psi(S) = \begin{cases} 0, & \text{if } S < S_c \\ 1, & \text{if } S > S_c, \end{cases} \tag{21}$$

where all the parameters take the same meaning as in model (4). We assume $b > p, q_2 > q_1$ and $\mu + q_1 > q_2$ in the rest of this study. Denote $H(S) = S - S_c$ and $X = (S, I)$, where $H(S)$ is a smooth scale function. Thus system (20)–(21) can be rewritten as

$$\frac{dX}{dt} = \begin{cases} F_{S_1}(X), & H(S) < 0 \\ F_{S_2}(X), & H(S) > 0, \end{cases} \tag{22}$$

with

$$\begin{aligned} F_{S_1}(X) &= \left(bS(t)\left(1 - \frac{S(t)+\eta I(t)}{K}\right) - \frac{\beta S(t)I(t)}{1+\alpha I(t)}, \frac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu I(t)\right)^T \\ F_{S_2}(X) &= \left(bS(t)\left(1 - \frac{S(t)+\eta I(t)}{K}\right) - \frac{\beta S(t)I(t)}{1+\alpha I(t)} - pS(t)\right., \\ &\qquad \left.\frac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu I(t) - q_1 I(t) + q_2 I(t)\right)^T. \end{aligned}$$

Before examining the dynamics of system (20)–(21), we introduce some technological definitions. Denote

$$\Sigma = \left\{(S, I) \in R_+^2 : S = S_c\right\},$$

which is indeed the switching boundary of system (22) and splits R_+^2 into two parts:

$$G_1 = \left\{(S, I) \in R_+^2 : S < S_c\right\}, \qquad G_2 = \left\{(S, I) \in R_+^2 : S > S_c\right\}.$$

For convenience, we call the subsystem defined on the subregion G_i $(i = 1, 2)$ system S_{G_i}. Denote $X_c = (S_c, I)$, and we distinguish the following three regions on Σ:

- sliding region

$$\Sigma_s = \left\{X_c \in \Sigma : \langle H_X(X_c), F_{S_1}(X_c)\rangle \geq 0, \langle H_X(X_c), F_{S_2}(X_c)\rangle \leq 0\right\};$$

- crossing region

$$\Sigma_c = \left\{X_c \in \Sigma : \langle H_X(X_c), F_{S_1}(X_c)\rangle\langle H_X(X_c), F_{S_2}(X_c)\rangle > 0\right\};$$

- escaping region

$$\Sigma_e = \left\{X_c \in \Sigma : \langle H_X(X_c), F_{S_1}(X_c)\rangle \le 0, \langle H_X(X_c), F_{S_2}(X_c)\rangle \ge 0\right\},$$

where $H_X(X_c) = \left(\frac{\partial H}{\partial S}, \frac{\partial H}{\partial I}\right)$.

Definition 3 (i) We call point X_* a *real equilibrium* of system (22) if it satisfies $F_{S_1}(X_*) = 0$ $(F_{S_2}(X_*) = 0)$ and $H(X_*) < 0(H(X_*) > 0)$, which is denoted by X_*^r.
(ii) We can a point X_* a *virtual equilibrium* of system (22) if it satisfies $F_{S_1}(X_*) = 0(F_{S_2}(X_*) = 0)$ and $H(X_*) > 0(H(X_*) < 0)$, which is denoted by X_*^v.
Both the real equilibria and virtual equilibria are called *regular equilibria*.

Definition 4 If there is a point $X_* \in \Sigma$ satisfying $\lambda F_{S_1}(X_*) + (1-\lambda)F_{S_2}(X_*) = 0$, $H(Z_*) = 0$ with

$$\lambda = \frac{\langle H_X(X^*), F_{S_1}(X^*)\rangle}{\langle H_X(X^*), F_{S_1}(X^*) - F_{S_2}(X^*)\rangle},$$

then X_* is called a *pseudo-equilibrium* of system (22).

The dynamics of subsystem S_{G_1} are examined in Sect. 2. We now address the dynamics of subsystem S_{G_2}. Denote $\overline{q} = q_2 - q_1$, $\overline{\mu} = \mu - \overline{q}$ and

$$R_{20} = \frac{\beta K(b-p)}{b\overline{\mu}}.$$

Theorem 9 *For system S_{G_2}, the boundary equilibrium $E_{20} = ((b-p)K/b, 0)$ is globally asymptotically stable when $R_{20} \le 1$. The positive equilibrium $E_{21} = (S_{21}, I_{21})$ is globally asymptotically stable when $R_{20} > 1$, where $S_{21} =$*

$$\frac{\overline{\mu}(b\eta + \alpha bK - \beta K - Kp\alpha) + \overline{\mu}\sqrt{(b\eta + \alpha bK - \beta K - Kp\alpha)^2 + 4bK(\overline{\mu}\alpha + \beta\eta)}}{2b(\overline{\mu}\alpha + \beta\eta)},$$

$$I_{21} = \frac{\beta S_{22} - \overline{\mu}}{\overline{\mu}\alpha}.$$

3.1 Sliding dynamics of the Filippov system (22)

We now explore the existence of sliding mode region Σ_s for system (22). Since

$$\langle H_X, F_{S_1}\rangle \ge 0 \Longleftrightarrow \alpha b\eta I^2 + [\beta K + b\eta - \alpha b(K - S_c)]\, I - b(K - S_c) \le 0,$$

solving the last inequality with respect to I yields

$$I \ge \frac{\alpha b(K - S_c) - \beta K - b\eta + \sqrt{[\alpha b(K - S_c) - \beta K - b\eta]^2 + 4\alpha b^2\eta(K - S_c)}}{2\alpha b\eta}$$

$$\equiv I_{c1}.$$

Similarly, we have

$$\begin{aligned} &\langle H_X, F_{S_2}\rangle \leq 0 \Longleftrightarrow \\ &\quad \alpha b\eta I^2 + [\beta K + b\eta - \alpha b(K - S_c) + \alpha pK] I - b(K - S_c) + pK \leq 0. \end{aligned} \tag{23}$$

There are two possibilities to consider: (A_1) $b(K - S_c) - pK \leq 0$ and (A_2) $b(K - S_c) - pK > 0$. If (A_1) holds, then $S_c \geq (b - p)K/b$ and so (23) is assured for $I \geq 0$. If (A_2) holds, then $S_c < (b - p)K/b$ and and so (23) is true for $I \geq I_{c2}$, where

$$I_{c2} = \frac{\alpha b(K - S_c) - \beta K - b\eta - \alpha pK + \sqrt{[\alpha b(K - S_c) - \beta K - b\eta - \alpha pK]^2 + 4\alpha b^2\eta(K - S_c) - pK}}{2\alpha b\eta}.$$

Therefore, the sliding mode takes the form

$$\Sigma_{s1} \equiv \Big\{(S_c, I) : 0 \leq I \leq I_{c1}\Big\}$$

for $S_c \geq (b - p)K/b$ and takes the form

$$\Sigma_{s2} \equiv \Big\{(S_c, I) : I_{c2} \leq I \leq I_{c1}\Big\}$$

for $S_c < (b - p)K/b$.

Direct calculation gives $\langle H_X, F_{S_1}\rangle \geq \langle H_X, F_{S_2}\rangle$, so there is no escaping region for the Filippov system (22).

Next we examine the sliding mode dynamics of system (22) on Σ_{s1} or Σ_{s2} by using the equivalent control method. Since

$$\frac{dH}{dt} = \frac{dS}{dt} = bS(t)\left(1 - \frac{S(t) + \eta I(t)}{K}\right) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - \psi(S)pS(t),$$

solving $dH/dt = 0$ with respect to $\psi(S)$ yields

$$\psi(S) = \frac{b}{p}\left(1 - \frac{S + \eta I}{K}\right) - \frac{\beta I}{p(1 + \alpha I)}.$$

Substituting $S = S_c$ and $\psi(S)$ into the second equation of system (20) gives

$$\frac{dI}{dt} = \frac{\beta S_c I}{1 + \alpha I} - \mu I + \overline{q}I\left[\frac{b(K - S_c - \eta I)}{pK} - \frac{\beta I}{p(1 + \alpha I)}\right], \tag{24}$$

where $I \in \Sigma_{s1}$ or $I \in \Sigma_{s2}$. Equation (24) is the sliding mode dynamics for system (20).

There always exists a root $I = 0$ for Eq. (24), so a pseudo-equilibrium $E_{s0}(S_c, 0)$ always exists for the Filippov system (22). If $I \neq 0$, letting $dI/dt = 0$ in Eq. (24) gives

$$\begin{aligned} &\alpha b\overline{q}\eta I^2 + [\alpha\mu pK + \overline{q}\beta K - \alpha\overline{q}b(K - S_c) + \overline{q}b\eta]\, I \\ &\quad + [\mu pK - \beta pKS_c - \overline{q}b(K - S_c)] = 0. \end{aligned} \tag{25}$$

Denote

$$\Upsilon_1 = \alpha\mu pK + \overline{q}\beta K - \alpha\overline{q}b(K - S_c) + \overline{q}b\eta, \quad \Upsilon_2 = \mu pK - \beta pKS_c - \overline{q}b(K - S_c).$$

Note that $\Upsilon_2 \geq 0 \Longrightarrow \Upsilon_1 > 0$, so no positive root exists for (25). Therefore no positive pseudo-equilibrium exists for system (22). If $\Upsilon_2 < 0$, there is a unique positive root for (24) and so a pseudo-equilibrium exists for the Filippov system (22). Further analysis gives $\Upsilon_2 < 0$ if one of the following conditions hold:

(C_1) $\mu p < \overline{q}b < \beta pK$;

(C_2) $\min\left\{\mu p, \beta pK\right\} > \overline{q}b, \quad S_c > \dfrac{(\mu p - \overline{q}b)K}{\beta pK - \overline{q}b}$;

(C_3) $\max\left\{\mu p, \beta pK\right\} < \overline{q}b, \quad S_c < \dfrac{(\overline{q}b - \mu p)K}{\overline{q}b - \beta pK}$.

Therefore, if (C_1) or (C_2) or (C_3) holds, there is a unique nontrivial pseudo-equilibrium $E_s = (S_c, I_c)$ for the Filippov system (22) with

$$I_c = \frac{-\Upsilon_1 + \sqrt{\Upsilon^2 - 4\alpha b\overline{q}\eta\Upsilon_2}}{2\alpha b\overline{q}\eta}.$$

Next, we examine whether $E_s \in \Sigma_{si}$ $(i = 1, 2)$. Note that

$$\begin{aligned} &\mathrm{sgn}(I_{c1} - I_c) = \mathcal{A}_{11} + \mathcal{A}_{12} \\ &\qquad \mathcal{A}_{11} = \mu pK\sqrt{[\alpha b(K - S_c) - \beta K - b\eta]^2 + 4\alpha b^2\eta(K - S_c)} \\ &\qquad \mathcal{A}_{12} = \mu pK[\alpha b(K - S_c) - \beta K - b\eta] - 2b\eta[\beta pKS_c - \mu pK]. \end{aligned}$$

We easily get that $\mathcal{A}_{11} > 0$; while $\mathcal{A}_{12} \geq 0$ if

$$S_c \leq \frac{\mu K\alpha b + b\mu\eta - \mu\beta K}{2b\eta\beta + \alpha b\mu}. \tag{26}$$

Denote

$$\mathcal{A}_{13} \equiv \frac{\mu K\alpha b + b\mu\eta - \mu\beta K}{2b\eta\beta + \alpha b\mu}.$$

Then we have

$$\mathrm{sgn}\left\{S_{11} - \mathcal{A}_{13}\right\}$$

$$
\begin{aligned}
&= \operatorname{sgn}\Bigg\{ - \alpha b[\mu(b\eta + \alpha bK - \beta K)] \\
&\quad + (2b\eta\beta + \alpha b\mu)\sqrt{(\beta K - b\eta - \alpha bK)^2 + 4K(\alpha\mu b + b\eta\beta)}\Bigg\} \\
&= \operatorname{sgn}\Bigg\{ - (\alpha b\mu)^2(b\eta + \alpha bK - \beta K) \\
&\quad + (2b\eta\beta + \alpha b\mu)^2\Big[(\beta K - b\eta - \alpha bK)^2 + 4K(\alpha\mu b + b\eta\beta)\Big]\Bigg\} \\
&= 1.
\end{aligned}
$$

It follows that $\mathcal{A}_{13} < S_{11}$. Similarly, we have $\mathcal{A}_{12} < 0$ for $S_c > \mathcal{A}_{13}$. In this case, we have

$$
\begin{aligned}
&\operatorname{sgn}\left(\mathcal{A}_{11}^2 - \mathcal{A}_{12}^2\right) \\
&\quad = \operatorname{sgn}\left\{\alpha b\mu^2(K - S_c) - b\eta(\beta S_c - \mu)^2 + \mu(\beta S_c - \mu)[(\alpha bK - \beta K - b\eta) - \alpha bS_c]\right\} \\
&\quad = \operatorname{sgn}\left\{-(b\beta\eta + \alpha b\mu)S_c^2 - (\beta\mu K - \alpha b\mu K - b\mu K)S_c + \mu^2 K\right\}.
\end{aligned}
$$

It is easy to get that

$$
(b\beta\eta + \alpha b\mu)S_c^2 + (\beta\mu K - \alpha b\mu K - b\mu K)S_c - \mu^2 K < 0
$$

for $S_c < S_{11}$. Then we have $I_{c1} > I_c$ if $\mathcal{A}_{13} < S_c < S_{11}$. Concluding the above discussion, we get $I_c < I_{c1}$ for $S_c < S_{11}$.

Similarly, we have

$$
\begin{aligned}
\operatorname{sgn}(I_c - I_{c2}) &= \mathcal{A}_{21} + \mathcal{A}_{22} \\
\mathcal{A}_{21} &= 2b\eta\beta S_c - \overline{\mu}[\alpha b(K - S_c) - \beta K + b\eta - \alpha pK] \\
\mathcal{A}_{22} &= -\overline{\mu}\sqrt{[\alpha b(K - S_c) - \beta K - b\eta - \alpha pK]^2 + 4\alpha b\eta[b(K - S_c) - pK]}.
\end{aligned}
$$

Since

$$
S_c > \frac{\overline{\mu}(\alpha bK - \beta K + b\eta - \alpha pK)}{\alpha b\overline{\mu} + 2b\eta\beta} \Longrightarrow \mathcal{A}_{21} > 0
$$

and

$$
S_{21} > \frac{\overline{\mu}(\alpha bK - \beta K + b\eta - \alpha pK)}{\alpha b\overline{\mu} + 2b\eta\beta},
$$

we get $I_c > I_{c2}$ for $S_c > S_{21}$.

Concluding the above analysis, the pseudo-equilibrium E_s is well defined and $E_s \in \Sigma_{s1}$ if the inequalities (C_1) or (C_2) or (C_3) hold and

$$\frac{(b-p)K}{b} < S_c < S_{11},$$

while $E_s \in \Sigma_{s2}$ if the condition (C_i) $(i = 1$ or 2 or $3)$ holds and

$$S_{21} < S_c < \min\left\{\frac{(b-p)K}{b},\ S_{11}\right\}.$$

3.2 Global dynamics of the Filippov system (22)

Direct calculation yields that

$$R_{20} > R_{10} \Longleftrightarrow \frac{\overline{q}}{p} > \frac{\mu}{b}.$$

If $R_{20} > R_{10}$ (i.e., $\overline{q}/p > \mu/b$), only conditions (C_1) and (C_3) can be true. Then the pseudo-equilibrium E_s exists for the Filippov system (22). If $R_{20} < R_{10}$ (i.e., $\overline{q}/p < \mu/b$), only condition (C_2) can be true, and so E_s also exists for (22). We focus our attention on the case $\overline{q}/p > \mu/b$ in the following and omit the dynamic analysis for the case $\overline{q}/p < \mu/b$, which can be done similarly. (We ignore the case $\overline{q}/p = \mu/b$.) Denote

$$\mathcal{B} = \frac{\overline{q}b - \mu p}{\overline{q}b - \beta pK}K$$

and we get that $(b-p)K/b > \mathcal{B}$ for $R_{20} < 1$ and $(b-p)K/b < \mathcal{B}$ for $R_{20} > 1$. After some algebra, we have $S_{21} < (b-p)K/b$, while $S_{11} > (b-p)K/b$ for $R_{10} < 1$. It follows from $R_{20} < 1$ that $b\overline{\mu} > \beta pK$, so

$$S_{21} > \frac{\overline{\mu}(b\eta + \alpha bK - \alpha pK)}{b(\overline{\mu}\alpha + \beta\eta)}$$

and

$$\begin{aligned}
&\operatorname{sgn}\left\{\frac{\overline{\mu}(b\eta + \alpha bK - \alpha pK)}{b(\overline{\mu}\alpha + \beta\eta)} - \frac{(\overline{q}b - \mu p)K}{\overline{q}b - \beta pK}\right\} \\
&\quad = \operatorname{sgn}\Big\{\overline{\mu}(\overline{q}b - \beta pK)\big[b\eta + \alpha K(b-p)\big] + bK(\overline{\mu}\alpha + \beta\eta)\big[\overline{\mu}p - (b-p)\overline{q}\big]\Big\} \\
&\quad = 1.
\end{aligned}$$

Therefore, we have $S_{21} > \mathcal{B}$ for $R_{20} < 1$.

We will now examine the global dynamics of the Filippov system (22). To this end, we initially address the existence of the limit cycles. For system (22), there are the following four types of possible limit cycles.

- Standard limit cycles; i.e., limit cycles entirely in the subregion G_1 or G_2. We have already excluded this class of limit cycles by using the Dulac function in Sects. 2.1 and 3.1.
- Crossing cycles without surrounding any sliding mode region; i.e., closed orbits containing no sliding mode region in their interiors. These types of cycles are composed of pieces of the orbits of subsystems S_{G_1}, S_{G_2} and pieces of the crossing region. They can be easily precluded by analyzing the vector field for the system.
- Crossing cycles surrounding one sliding mode region; i.e., crossing cycles enclosed one sliding mode region in its interior. The existence of these type of limit cycles can be ruled out by using Green's Formula and some analytic techniques. See Wang and Xiao (2014) and Wang (2006) for a similar analysis.
- Canard cycles; i.e., limit cycles containing one point or part of the sliding mode region. These type of limit cycles can be precluded by analyzing the vector field, especially the flow direction along the sliding mode region. See Wang and Xiao (2014) for similar details about this technique.

In conclusion, no limit cycles exist for the Filippov system (22). For the global behaviour, we consider the following three cases according to relationship between R_{10}, R_{20} and the unit 1.

Case 1. $R_{10} < R_{20} < 1$.

According to the above analysis, we have $\mathcal{B} < S_{21} < (b-p)K/b < S_{11} < K$ in this case. Considering the biological meaning, we only consider those thresholds S_c with $S_c < K$. It is easy to get that only condition (C_3) can be true to ensure the existence of the pseudo-equilibrium E_s. But it can be shown that $E_s \notin \Sigma_{s2}$, so there are three possible disease-free equilibria (E_{10}, E_{20} and E_{s0}) for the Filippov system (22). For $S_c > (b-p)K/b$, the sliding mode region is Σ_{s1}, and both regular disease-free equilibria E_{10} and E_{20} are virtual (denoted by E_{10}^v and E_{20}^v), so only the pseudo-equilibrium E_{s0} is locally stable. It follows from the above analysis that there are no limit cycles, so the pseudo-equilibrium E_{s0} is globally asymptotically stable in this scenario, as shown in Fig. 6a. For $S_c < (b-p)K/b$, the sliding mode region turns out to be Σ_{s2}; no pseudo-equilibrium exists, and the disease-free equilibrium E_{10} is virtual; E_{20} is real and locally stable, denote by E_{20}^r. Hence, the nonexistence of limit cycles leads to the global stability of E_{20}^r, as shown in Fig. 6b.

In Fig. 6, the thick grey solid lines represent the sliding mode region; the thin grey dashed lines denote the crossing region; the solid (hollow) diamond points denote the real (virtual) disease-free equilibria; the square points denote the pseudo-equilibria; the circles are the endemic equilibria. The black solid lines show asymptotic stability.

According to the above discussion, the pseudo-equilibrium E_s can act as an attractor provided it is defined on one of the sliding mode regions Σ_{s1} and Σ_{s2}; conversely, only real and locally stable regular equilibria can act as the attractors for the Filippov system (22). So we only focus on the existence of possible attractors in the following.

Case 2. $R_{10} < 1 < R_{20}$.

In this case, we have $S_{21} < (b-p)K/b < \min\left\{\mathcal{B}, S_{11}\right\} < K$. For $\min\{S_{11}, \mathcal{B}\} < S_c < K$, the sliding mode region is Σ_{s1}, and the pseudo-equilibrium $E_{s0} \in \Sigma_{s1}$ is globally asymptotically stable. For $(b-p)K/b < S_c < \min\{S_{11}, \mathcal{B}\}$, the sliding mode region is also Σ_{s1} and two pseudo-equilibria E_{s0} and E_s coexist in this scenario.

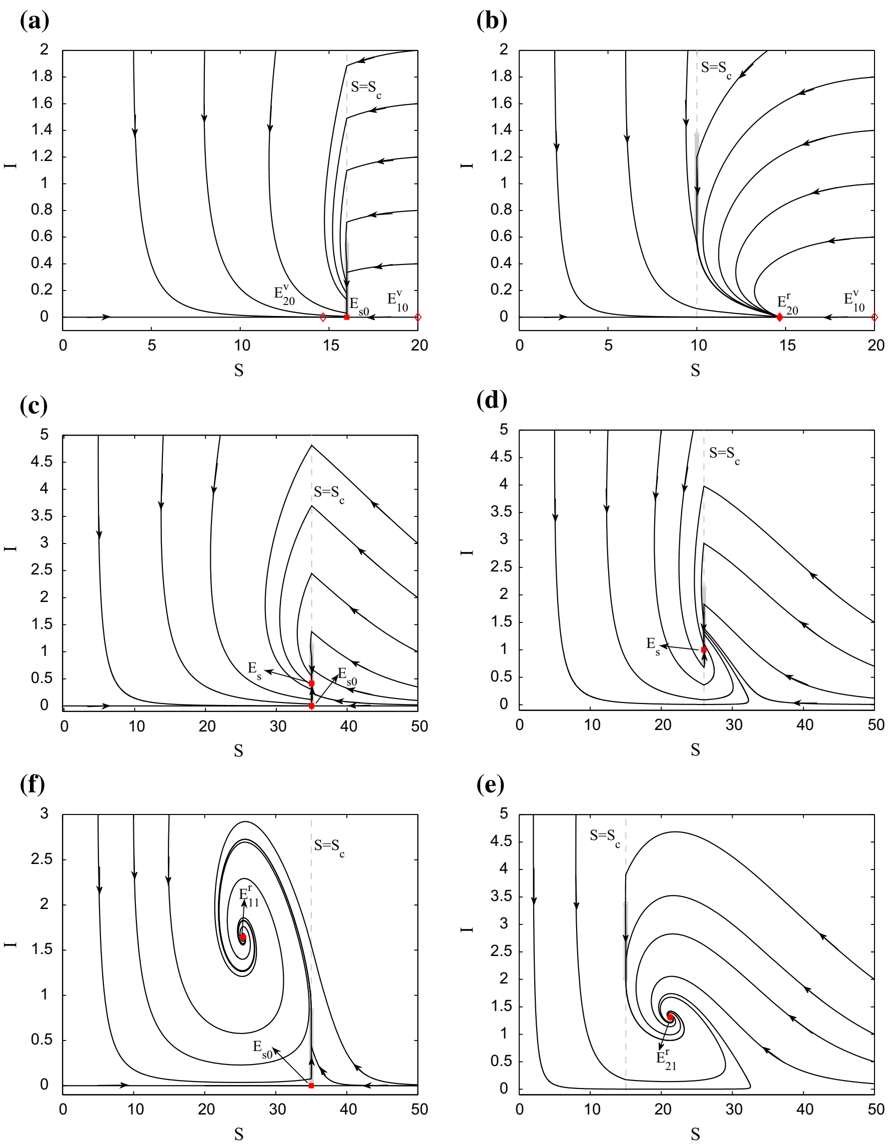

Fig. 6 Phase plane of the Filippov system (22), showing the sliding mode region and asymptotically equilibria in parameter space. The parameter values are $b = 3$, $\eta = 6$, $\alpha = 0.01$, $\mu = 10$, $p = 0.8$, $q = 5.8$ $\beta = 0.2$, $K = 20$, $S_c = 16$ (**a**), $\beta = 0.2$, $K = 20$, $S_c = 10$ (**b**), $\beta = 0.2$, $K = 45$, $S_c = 35$ (**c**) $\beta = 0.2$, $K = 45$, $S_c = 26$ (**d**). $\beta = 0.2$, $K = 45$, $S_c = 15$ (**e**) and $\beta = 0.4$, $K = 45$, $S_c = 35$ (**f**)

However, only the pseudo-equilibrium E_s is globally asymptotically stable and E_{s0} is unstable, as shown in Fig. 6c. For $S_{21} < S_c < (b-p)K/b$, the sliding mode region turns out to be Σ_{s2} and the pseudo-equilibrium $E_s \in \Sigma_{s2}$ is globally asymptotically stable, as shown in Fig. 6d. For $S_c < S_{21}$, the sliding mode region also appears as Σ_{s2}; while the regular endemic equilibrium E_{21} is real and globally asymptotically stable (denoted by E_{21}^r), as shown in Fig. 6e.

Case 3. $R_{20} > R_{10} > 1$.

In this case, we have $S_{21} < \min\left\{S_{11}, (b-p)K/b, \mathcal{B}\right\} < K$. If we further have $S_{11} > (b-p)K/b$, then we get $S_{21} < (b-p)K/b < \min\left\{S_{11}, \mathcal{B}\right\} < K$. It follows that for $S_c > S_{11}$, the sliding mode region is Σ_{s1} and the endemic equilibrium E_{11} is real and globally asymptotically stable, denoted by E_{11}^r; and the pseudo-equilibrium E_s is globally asymptotically stable for $(b-p)K/b < S_c < S_{11}$; the sliding mode region turns out to be Σ_{s2} and E_s is globally asymptotically stable for $S_{21} < S_c < (b-p)K/b$; the sliding mode region is also Σ_{s2} and the real endemic equilibrium E_{21}^r is globally asymptotically stable. However, if the inequality $S_{11} < (b-p)K/b$ holds instead, we have $S_{21} < S_{11} < (b-p)K/b < \mathcal{B} < K$. In this case, the sliding mode region is Σ_{s1} and the regular endemic equilibrium E_{11} is real and globally asymptotically stable for $S_c > (b-p)K/b$ (also denoted by E_{11}^r), as shown in Fig. 6f; the sliding mode region turns out to be Σ_{s2}, and the real endemic equilibrium E_{11}^r remains as the global attractor for $S_{11} < S_c < (b-p)K/b$; for $S_{21} < S_c < S_{11}$, the sliding mode region remains as Σ_{s2}, and the pseudo-equilibrium E_s is globally asymptotically stable; for $S_c < S_{21}$, the sliding mode is also Σ_{s2}, and the regular equilibrium E_{21} is real and globally asymptotically stable, denoted by E_{21}^r.

Concluding the above result, we can get the following theorem.

Theorem 10 *If $\overline{q}/p > \mu/b$, there are different attractors in different parameter spaces.*

(i) *For $R_{10} < R_{20} < 1$, the disease cannot spread in the pests. In particular, the pseudo-equilibrium E_{s0} is globally asymptotically stable for $S_c > (b-p)K/b$; while the regular equilibrium E_{20}^r is globally asymptotically stable for $S_c < (b-p)K/b$.*

(ii) *For $R_{10} < 1 < R_{20}$, the disease spreads in the pest population when $S_c < \min\left\{S_{11}, \mathcal{B}\right\}$; it cannot spread when $\min\left\{S_{11}, \mathcal{B}\right\} < S_c < K$. In particular, the pseudo-equilibrium E_{s0} (or E_s) is globally asymptotically stable for $\min\left\{S_{11}, \mathcal{B}\right\} < S_c < K$ (or $S_{21} < S_c < \min\left\{S_{11}, \mathcal{B}\right\}$); while the endemic equilibrium E_{21}^r, which is real, is globally asymptotically stable for $S_c < S_{21}$.*

(iii) *For $R_{20} > R_{10} > 1$, the disease always spreads in the pests. In particular, the regular equilibrium E_{11}^r (or E_{21}^r) is globally asymptotically stable for $S_c > S_{11}$ (or $S_c < S_{21}$); the pseudo-equilibrium E_s is globally asymptotically stable for $S_{21} < S_c < S_{11}$.*

For clarity, we list the results in Theorem 10 in Table 1.

Table 1 Main results of the Filippov system

Values of R_{10}, R_{20}	Conditions	Global attractors
$1 > R_{20} > R_{10}$	$S_c > \frac{(b-p)K}{b}$	E_{s0}
	$S_c < \frac{(b-p)K}{b}$	E_{20}
$R_{20} > 1 > R_{10}$	$\min\{\mathcal{B}, S_{11}\} < S_c < K$	E_{s0}
	$S_{21} < S_c < \min\{\mathcal{B}, S_{11}\}$	E_s
	$S_c < S_{21}$	E_{21}
$R_{20} > R_{10} > 1$	$S_c > S_{11}$	E_{11}
	$S_{21} < S_c < S_{11}$	E_s
	$S_c < S_{21}$	E_{21}

4 Conclusion and biological implications

Pest control is still a worldwide problem for agricultural management due to the high loss of agriculture and increasing demands for food and energy with the increasing population (Kar et al. 2012). The main methods to control pests are application of chemical pesticides, releasing natural enemies and introducing microbial pathogens to the pests. However, the harmful effects of pesticides, high costs of microbial pathogens or unavailability of natural enemies are barriers to effective control (Gao et al. 2013; Liu et al. 2015; Jiao et al. 2009). This resulted in the development of integrated pest management (IPM). We extend the model developed by Jiao et al. (2009) by considering the threshold policy and establish two types of non-smooth models: a state-dependent impulsive model and a Filippov model. We determined threshold values such that implementing the combined control strategy we can effectively control the pest population.

By modifying the impulsive human intervention with fixed instants to the one with non-fixed instants, we have established our non-smooth microbial model, which is a state-dependent impulsive model. Theoretical analysis of the proposed model reveals rich dynamics. For the model without control measure, there are two possible cases: without positive equilibria or with a unique positive equilibrium. For the first case, the Poincaré map of the phase set possesses several important properties, including monotonicity, differentiability and concavity. When a positive equilibrium is possible for the model without control, model (3) exhibits more complicated dynamic behaviour. In this case, variation of the threshold value S_c leads to a variable domain and range of the Poincaré map. For a sufficiently large threshold level S_c and sufficiently small dosage of pesticide, the domain of the Poincaré map consists of two intervals. The solution initiating from the phase set can experience infinite impulses or experience finite impulses or be free from impulsive actions. All these characteristics indicate that the dynamic behaviour of model (3) can exhibit various phenomena including the existence of order-1 or order-2 or any order periodic solutions.

The regularity of the Poincaré map also allows us to provide a sharp condition for global stability of the order-1 periodic solution. In particular, we derived the critical value of the parameter μ to ensure the existence of a globally stable order-1 periodic solution. This indicates that, when the disease-induced mortality rate exceeds a critical level, the final size of pests population varies periodically. The existence of an order-3 periodic solution confirms that any order-k periodic solutions can occur. Moreover, chaotic behavior occurs as the parameter q varies, which suggests that the final size of the pest population becomes unpredictable as the artificial net increment rate of infective pests varies. All of these situations can guide different control strategies. The existence of the periodic solution indicates that periodic control could be possible, where the control period can be determined analytically. For chaotic solutions, the irregularity and unpredictability of variation of pest size causes difficulty in quantifying when to initiate the IPM strategy. In particular, if the order-1 periodic solution is globally asymptotically stable, the pests can be maintained below the threshold level by periodically implementing the control strategies. Hence the density-dependent impulsive control regime is converted into a fixed-time pulse-like control.

Taking the process of implementing the human intervention into account, we further modify the instantaneous intervention into a non-instantaneous one and establish a Filippov microbial pest model. We aim to describe the following control strategy: once the amount of pests exceed the critical value, the combined control measures are conducted; otherwise, they are suspended. We have examined the dynamics when the ratio of the net increment rate of infective pests to the killing rate on susceptible pests caused by humans is larger than the ratio of the death rate to the birth rate (i.e., $\overline{q}/p > \mu/b$). The main results demonstrate that when the basic reproduction number for the case without human interventions—i.e., the average amount of new infections produced by an infective pest in the early stage—is greater than 1 (i.e., $R_{10} > 1$), the disease can spread in the pests. In particular, the regular equilibrium E_{11} or E_{21} or pseudo-equilibrium E_s is globally asymptotically stable for $S_c > S_{11}$ or $S_c < S_{21}$ or $S_{21} < S_c < S_{11}$. This suggests that if we carry out the interventions later (i.e., $S_c > S_{11}$) or earlier (i.e. $S_c < S_{21}$), the pest will be contained at the relatively high level S_{11} or low level S_{21}; while if we implement the interventions at the right time (i.e., $S_{21} < S_c < S_{11}$), the pest can be contained at the previously given level. When the basic reproduction number for the case with human interventions is greater than 1 but is less than 1 without human interventions (i.e., $R_{20} > 1 > R_{10}$), the disease can spread for $S_c < \min\left\{\mathcal{B}, S_{11}\right\}$, but it cannot spread for $S_c > \min\left\{\mathcal{B}, S_{11}\right\}$. If we further have $S_c < S_{21}$, the final size of pests is relatively small (i.e., S_{21}); if we have $S_c > S_{21}$, the pests can be contained at the given level S_c. When the basic reproduction number either with human interventions or without interventions is less than 1, the disease cannot spread in the pests. This is not biologically or economically desirable.

It is worth mentioning that the Filippov model in this work is formulated with a control threshold value for the susceptible pest population. In practice, it is more realistic to set the total number of susceptible and infective pests as the threshold value. However, the analysis of such a model is difficult and would likely result in rich dynamic phenomena; we will leave this for future work.

The interventions in the density-dependent impulsive modeling are assumed to be implemented instantaneously; conversely, in the Filippov system, the control measure is triggered once the amount of susceptible pests exceeds a critical value, and the interventions last for a duration until the next switch. For the density-dependent impulsive model, a globally asymptotically stable order-1 periodic solution, order-k ($k \geq 2$) periodic solution or chaos may occur under appropriate conditions; however, a regular equilibrium or pseudo-equilibrium is globally asymptotically stable for different control parameters for the Filippov model. It follows that the Filippov model has advantages over the impulsive model from the point of view of mathematical modeling; from the point of view of kinetics, their dynamics are qualitatively different. More importantly, although both impulsive control measures and switching measures could maintain the amount of pests below the economic threshold, there still exists obvious difference in the final size of pests. Moreover, there is a large difference in the final size of pests when implementing these two control measures. In particular, if $R_{10} < 1$, the infectious disease cannot spread in the pests, and the carrying capacity K is the final size of the pest population when no control measure is adopted. When implementing the impulsive or switching control measures, the disease can successfully spread in the pests, and its final size varies periodically below the scheduled level for the former measure; while it stabilizes at the previously given level S_c for the latter measure. If $R_{10} > 1$, the disease will spread in the pests if no control measures are used, but the final size of pests (i.e., S_{11}) is comparatively high. However, the pest population varies periodically and is less than S_{11} if implementing a certain impulsive control measure; if implementing a proper switching control measure, the final size of the pest population can be the previously given level S_c or a lower level S_{12}.

In this work, we focused on two non-smooth models with thresholds for pest control. Note that the control strategies focus on how the pest population can be curbed in applying microbial pathogens, so the modeling framework in this study is built on the basis of general epidemic system. Our main results demonstrate that it is essential to carefully choose the threshold level of susceptible pests before initiating the control measures.

Acknowledgements The authors are grateful to two anonymous reviewers, whose comments greatly improved the manuscript. AW was supported by the National Natural Science Foundation of China (NSFC, 11801013), the Shaanxi Education Department (16JK1047) and funding from the Baoji University of Arts and Sciences (ZK16048). YX was supported by the National Natural Science Foundation of China (NSFC, 11571273 and 11631012) and Fundamental Research Funds for the Central Universities (GK 08143042). RS? was supported by an NSERC Discovery Grant. For citation purposes, note that the question mark in "Smith?" is part of his name.

References

Bainov D, Simeonov P (1993) Impulsive differential equations: periodic solutions and applications. CRC Press, Boca Raton

Bhattacharyya S, Bhattacharya D (2006) Pest control through viral disease: mathematical modeling and analysis. J Theor Biol 238:177–197

Cai Y, Zhao C, Wang W, Wang J (2015) Dynamics of a Leslie–Gower predator–prey model with additive allee effect. Appl Math Model 39:2092–2106

Chong N, Dionne B, Smith R? (2016) An avian-only Filippov model incorporating culling of both susceptible and infected birds in combating avian influenza. J Math Biol 73:751–784

Devaney R (2003) An introduction to chaotic dynamical systems. Westview Press, Boulder

Gao S, He Y, Chen L (2013) An epidemic model with pulses for pest management. Appl Math Comput 219:4308–4321

Gastrich M, Leigh-Bell J, Gobler C, Anderson O, Wilhelm S (2004) Viruses as potential regulators of regional brown tide blooms caused by the alga *Aureococcus anophagefferens*. Estuaries 27:112–119

Huang M, Li J, Song X, Guo H (2012) Modeling impulsive injections of insulin: towards artificial pancreas. SIAM J Appl Math 72:1524–1548

Jacquet S, Heldal M, Iglesias-Rodriguez D, Larsen A, Wilson W, Brabak G (2002) Flow cytometric analysis of an *Emiliana huxleyi* bloom terminated by viral infection. Aquat Microb Ecol 27:111–124

Jiao J, Chen L, Cai S (2009) Impulsive control strategy of a pest management SI model with nonlinear incidence rate. Appl Math Model 33:555–563

Kar T, Ghorai A, Jana S (2012) Dynamics of pest and its predator model with disease in the pest and optimal use of pesticide. J Theor Biol 310:187–198

Křivan V, Lewis M, Bentz B, Bewick S, Lenhart S, Liebhold A (2016) A dynamical model for bark beetle outbreaks. J Theor Biol 407:25–37

Lakshmikantham V, Bainov D, Simeonov P (1989) Theory of impulsive differential equations. World Scientific, Singapore

Lenteren JV (1995) Integrated pest management in protected crops. In: Dent D (ed) Integrated pest management. Chapman & Hall, London, pp 311–320

Lenteren JV, Woets J (1988) Biological and integrated pest control in greenhouses. Annu Rev Entomol 33:239–250

Li X, Wu J (2016) Stability of nonlinear differential systems with state-dependent delayed impulses. Automatica 64:63–69

Li X, Song S, Wu J (2018) Impulsive control of unstable neural networks with unbounded time-varying delays. Sci China Inf Sci 61:012203

Liang J, Tang S, Nieto J, Cheke R (2013) Analytical methods for detecting pesticide switches with evolution of pesticide resistance. Math Biosci 245:249–257

Liang J, Tang S, Cheke R, Wu J (2015) Models for determining how many natural enemies to release inoculatively in combinations of biological and chemical control with pesticide resistance. J Math Anal Appl 422:1479–1503

Liu B, Liu W, Tao F, Cong J (2015) A dynamical analysis of a piecewise smooth pest control SI model. Int J Bifurc Chaos 25:1550068

Lou J, Lou Y, Wu J (2012) Threshold virus dynamics with impulsive antiretroviral drug effects. J Math Biol 65:623–652

Nie L, Teng Z, Guo B (2013) A state dependent pulse control strategy for a SIRS epidemic system. Bull Math Biol 75:1697–1715

O'Rourke M, Jones L (2011) Analysis of landscape-scale insect pest dynamics and pesticide use: an empirical and modeling study. Ecol Appl 21:3199–3210

Peng H (2005) Wasps deliver deadly virus to crop pests. Virus Res 114:80–81

Sasmal S, Bhowmick A, Al-Khaled K, Bhattacharya S, Chattopadhyay J (2016) Interplay of functional responses and weak allee effect on pest control via viral infection or natural predator: an eco-epidemiological study. Differ Equ Dyn Syst 24:21–50

Simeonov P, Bainov D (1988) Orbital stability of periodic solutions of autonomous systems with impulse effect. Int J Syst Sci 19:2561–2585

Smith R, Wahl L (2004) Distinct effects of protease and reverse transcriptase inhibition in an immunological model of HIV-1 infection with impulsive drug effects. Bull Math Biol 66:1259–1283

Tang S, Cheke R (2005) State-dependent impulsive models of integrated pest management (ipm) strategies and their dynamics consequences. J Math Biol 50:257–292

Tang S, Liang J (2013) Global qualitative analysis of a non-smooth Gause predator–prey model with a refuge. Nonlinear Anal Theor Methods Appl 76:165–180

Tang S, Xiao Y, Cheke R (2010) Dynamical analysis of plant disease models with cultural control strategies and economic thresholds. Math Comput Simul 80:894–921

Tang S, Liang J, Xiao Y, Cheke R (2012) Sliding bifurcations of Filippov two stage pest control models with economic thresholds. SIAM J Appl Math 72:1061–1080

Tang S, Pang W, Wu J, Cheke R (2015) Global dynamics of a state-dependent feedback control system. Adv Differ Equ 2015:322

Tang B, Xiao Y, Tang S, Cheke R (2016) A feedback control model of comprehensive therapy for treating immunogenic tumours. Int J Bifurc Chaos 26:1650039

Touboul J, Brette R (2009) Spiking dynamics of bidimensional integrate-and-fire neurons. SIAM J Appl Dyn Syst 8:1462–1506

Wang W (2006) Backward bifurcation of an epidemic model with treatment. Math Biosci 201:58–71

Wang A, Xiao Y (2014) A Filippov system describing media effects on the spread of infectious diseases. Nonlinear Anal Hybrid Syst 11:84–97

Wang L, Chen L, Nieto J (2010) The dynamics of an epidemic model for pest control with impulsive effect. Nonlinear Anal Real World Appl 11:1374–1386

Xiao Y, Xu X, Tang S (2012) Sliding mode control of outbreaks of emerging infectious diseases. Bull Math Biol 74:2403–2422

Xiao Y, Miao H, Tang S, Wu H (2013a) Modeling antiretroviral drug responses for HIV-1 infected patients using differential equation models. Adv Drug Deliv Rev 65:940–953

Xiao Y, Zhao T, Tang S (2013b) Dynamics of an infectious diseases with media/psychology induced non-smooth incidence. Math Biosci Eng 10:445–461

Xiao Y, Tang S, Wu J (2015) Media impact switching surface during an infectious disease outbreak. Sci Rep 5:7838

Yang Y, Xiao Y, Wu J (2013) Pulse HIV vaccination: feasibility for virus eradication and optimal vaccination schedule. Bull Math Biol 75:725–751