Simple epidemic models

• Construct ODE (Ordinary Differential Equation) models
• Relationship between the diagram and the equations
• Alter models to include other factors.
Ordinary Differential Equations (ODEs)

- ODEs deal with populations, not individuals
- We assume the population is well-mixed
- We keep track of the inflow and the outflow.

ODEs = Ordinary Differential Equations
SIS epidemic

- Susceptible $\rightarrow$ Infected $\rightarrow$ Susceptible
- You get sick, then recover, but without immunity
- E.g. the common cold.
Diagram

- Susceptibles become infected at rate $a$.
- Infecteds recover at rate $b$. 

\[ S \xrightarrow{a} I \xleftarrow{b} S \]
SIS equations

- Becoming infected depends on contact between Susceptibles and Infecteds \((aSI)\)
- Recovery is at a constant rate, proportional to number of Infecteds \((b)\).

\[
\begin{align*}
\frac{dS}{dt} &= bI - aSI \\
\frac{dI}{dt} &= aSI - bI
\end{align*}
\]

\(a = \text{infection rate}\)
\(b = \text{recovery rate}\)
Total population is constant

- Add equations together
- \( N = S + I \) (total population)
- \( dN/dt = 0 \rightarrow N \) is a constant.

\[
\frac{dS}{dt} + \frac{dI}{dt} = bI - aSI + aSI - bI
\]

\[
\frac{dN}{dt} = 0
\]

\( S' = bI - aSI \)

\( I' = aSI - bI \)

**Legend:**
- \( S = \text{Susceptible} \)
- \( I = \text{Infected} \)
Solving directly

- Since $N=S+I$, this means $S = N-I$

\[
\frac{dI}{dt} = a(N - I)I - bI
\]
\[
= (aN - b - aI)I
\]

- Let $A = aN-b$ be a constant

\[
\frac{dI}{dt} = (A - aI)I.
\]

*S = Susceptible  
I = Infected  
a = infection rate  
b = recovery rate*
Separate the variables

• Put the $I$’s on one side and the $t$’s on the other
  (including $dl$ and $dt$)

\[
\frac{dI}{(A - aI)I} = dt.
\]

\begin{itemize}
  \item $I = \text{Infected}$
  \item $a = \text{infection rate}$
  \item $A = aN - b$ (constant)
  \item $N = \text{total pop.}$
  \item $b = \text{recovery rate}$
\end{itemize}
Time series solution

• Rearrange using partial fractions
• Integrate
• Use initial condition $I(0)=I_0$

$$I = \frac{(aN - b)I_0 e^{(aN-b)t}}{(aN - b) + aI_0 [e^{(aN-b)t} - 1]}$$

$S =\text{Susceptible}$
$I = \text{Infected}$
$a = \text{infection rate}$
$N = \text{total pop.}$
$b = \text{recovery rate}$

(See Epidemic Notes)
$b = \text{recovery rate}$

For $b = 0.1$ and $b = 0.7$, the graph shows the population dynamics over time with two curves: one for Infecteds and another for Susceptibles.
Phase portraits

• Since $N = S + I$, $I = -S + N$
• This is a straight line in $I$ and $S$
• Time is implicit.

$S = $ Susceptible
$I = $ Infected
$N = $ total pop.
Equilibrium points

• Equilibria occur when derivatives are zero:

\[(b - aS)I = 0 \quad \text{when } S = \frac{b}{a} \quad \text{or when } I = 0\]

\[(aS - b)I = 0 \quad \text{when } S = \frac{b}{a} \quad \text{or when } I = 0\]

\[(\text{We’ll call } \frac{b}{a} ‘p’.)\]

S = Susceptible
I = Infected
a = infection rate
b = recovery rate

S' = bl-aSl
I' = aSl-bl
Two equilibria

- Thus our equilibrium points are

\[(\bar{S}, \bar{I}) = (p, N - p)\]

\[\text{or} \quad (\bar{S}, \bar{I}) = (N, 0)\]

- The latter always exists, the former is only biologically reasonable if \(p < N\).

\[S = \text{Susceptible} \quad I = \text{Infected}\]
\[N = \text{total pop.} \quad p = \frac{b}{a}\]
\(b = \text{recovery rate} \quad a = \text{infection rate}\]
$N = \text{total pop.}$

$p = \frac{b}{a}$

$b = \text{recovery rate}$

$a = \text{infection rate}$
Stability

\[ S' = a(p-S)I \]
\[ I' = a(S-p)I \]

- \( S < p \) \( \Rightarrow \) \( S' > 0 \), \( I' < 0 \)
- \( S > p \) \( \Rightarrow \) \( S' < 0 \), \( I' > 0 \)

\( S = \text{Susceptible} \quad I = \text{Infected} \)
\( N = \text{total pop.} \quad p = \frac{b}{a} \)
\( b = \text{recovery rate} \quad a = \text{infection rate} \)
Case I: \( p < N \)

- \( N \) = total pop.
- \( p = \frac{b}{a} \)
- \( b = \) recovery rate
- \( a = \) infection rate
Stability implications

- When \( p = b/a > N \), the recovery rate is high, so infecteds recover quickly and the population moves to a population of susceptibles.
- When \( p = b/a < N \), the infection rate is high and the infection stabilises at an endemic equilibrium.

\( N = \text{total pop.} \)
\( b = \text{recovery rate} \)
\( a = \text{infection rate} \)
SIR epidemics

- Susceptible $\rightarrow$ Infected $\rightarrow$ Removed
- Removed can be recovered, immune, or dead.

$a =$ infection rate  
$b =$ recovery rate
SIR equations

- Becoming infected depends on contact between Susceptibles and Infecteds \((aSI)\)
- Recovery is at a constant rate, proportional to number of Infecteds \((b)\).

\[
\begin{align*}
\frac{dS}{dt} &= -aSI \\
\frac{dI}{dt} &= aSI - bI \\
\frac{dR}{dt} &= bI
\end{align*}
\]

- \(a = \) infection rate
- \(b = \) recovery rate
SIR with vaccination

• A vaccine sends some Susceptibles directly to the Recovered (immune) state
• $N=S+I+R$.

$S = \text{Susceptible} \quad I = \text{Infected} \quad R = \text{Recovered} \quad a = \text{infection rate} \quad b = \text{recovery rate} \quad c = \text{vaccination rate}$
Vaccination equations

- Vaccination is assumed to be a fixed number of shots per time period \((c)\).

\[
\begin{align*}
\frac{dS}{dt} &= -aSI - c \\
\frac{dI}{dt} &= aSI - bI \\
\frac{dR}{dt} &= bI + c
\end{align*}
\]

- **S** = Susceptible
- **I** = Infected
- **R** = Recovered
- **a** = infection rate
- **b** = recovery rate
- **c** = vaccination rate
SIR with mutation

- If the virus mutates, Recovereds lose their immunity.

\[ S \rightarrow a \rightarrow I \rightarrow b \rightarrow R \]

\[ S = \text{Susceptible} \quad I = \text{Infected} \quad R = \text{Recovered} \quad a = \text{infection rate} \quad b = \text{recovery rate} \quad e = \text{mutation rate} \]
Mutation equations

- A time-delay $T$ allows a ‘grace period’ before people are susceptible again.
- They become susceptible at a rate $(e)$ depending on their status at time $t-T$.

\[
\begin{align*}
\frac{dS(t)}{dt} &= -aS(t)I(t) + eR(t - T) \\
\frac{dI(t)}{dt} &= aS(t)I(t) - bI(t) \\
\frac{dR(t)}{dt} &= bI(t) - eR(t - T)
\end{align*}
\]

$S = \text{Susceptible}$
$I = \text{Infected}$
$R = \text{Recovered}$

$a = \text{infection rate}$
$b = \text{recovery rate}$
Delay Differential Equations

- These are called delay-differential equations
- They are harder to analyse than ordinary differential equations, but are often more realistic.